HOMEWORK 10 FOR 18.100B/C, FALL 2010
DUE THURSDAY 2 DECEMBER

As usual, homework is due in 2-108 by 11AM on Thursday 2 December, or by email before 5PM on the same day. This is the last homework for this course!

(1) Let E be a set and let Y be a metric space. Consider all the bounded maps $f : E \longrightarrow Y$ – so for each such map there is a constant M and a point $y \in Y$ with the property that $d_Y(f(e), y) \leq M$ for all $e \in E$. Let $\mathcal{B}(E;Y)$ be the set of these bounded maps and define for each f, $g \in \mathcal{B}(E;Y)$

$$d(f,g) = \sup_{e \in E} d_Y(f(e), g(e)).$$

Show that this is a metric on $\mathcal{B}(E;Y)$ and that if Y is complete so is $\mathcal{B}(E;Y)$.

(2) Show that if X is a metric space and Y is a (complete if you want because of the previous question but it is not needed here) metric space then the set $\mathcal{C}(X;Y)$ of continuous and bounded maps $f : X \longrightarrow Y$ is a closed subset of $\mathcal{B}(X;Y)$ defined in the previous question.

(3) Rudin Chap 7 No 24. Let X be a metric space with metric d and suppose $a \in X$ is a fixed point. Assign to each $p \in X$ the function $f_p : X \longrightarrow \mathbb{R}$ where $f_p(x) = d(p,x) - d(a,x)$ for each $x \in X$. Show that $f_p \in \mathcal{C}(X)$, i.e. it is a continuous bounded function, and so this construction defines a map $\Phi : X \longrightarrow \mathcal{C}(X)$. Prove that

$$\sup_{x \in X} |f_p(x) - f_q(x)| = d(p,q) \ \forall \ p, q \in X.$$

Hence conclude that Φ is an *isometry* (a map between metric spaces preserving the distance) which maps X 1-1 onto a subset of $\mathcal{C}(X)$. Since the latter is complete deduce that the closure of the range of Φ is a metric completion of X – is a complete metric space which has X (represented as the range of Φ) as a dense subset and which has a distance which restricts to the distance on X.

(4) Let $f_n : [0,1] \longrightarrow [0,\infty)$ be a sequence of continuous functions which is pointwise decreasing with $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

(5) Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is a Riemann integrable function. Show that, as a consequence of the Stone-Weierstrass theorem, there is a sequence of polynomials p_n such that

$$\lim_{n\to\infty} \int_0^1 |f(x) - p_n(x)|dx = 0.$$

Hint: Use a problem from earlier homework about Riemann integrable functions.