HOMEWORK 10 FOR 18.100B/C, FALL 2010
SOLUTIONS

As usual, homework is due in 2-108 by 11AM on Thursday 2 December, or by email before 5PM on the same day. This is the last homework for this course!

(1) Let E be a set and let Y be a metric space. Consider all the bounded maps $f : E \longrightarrow Y$ – so for each such map there is a constant M and a point $y \in Y$ with the property that $d_Y(f(e), y) \le M$ for all $e \in E$. Let $\mathcal{B}(E; Y)$ be the set of these bounded maps and define for each f, $g \in \mathcal{B}(E; Y)$

$$d(f, g) = \sup_{e \in E} d_Y(f(e), g(e)).$$

Show that this is a metric on $\mathcal{B}(E; Y)$ and that if Y is complete so is $\mathcal{B}(E; Y)$.

Solution: If $f : E \longrightarrow Y$ is bounded then for any y' in Y the function $d_Y(f(e), y') \le D_Y(f(e), y) + d_Y(y, y')$ is bounded. It follows that $d(f, g)$ is well defined if E and Y are non-empty since

$$d(f(e), g(e)) \le d(f(e), y) + d(y, g(e))$$

is bounded on E. Certainly $d(f, g) \ge 0$ and $d(f, g) = 0$ implies that $d_Y(f(e), g(e)) = 0$ for all $e \in E$ so $f = g$. Moreover $d(f, g) = d(g, f)$ so only the triangle inequality needs to be checked. This follows from the triangle inequality in Y since for any three bounded maps

$$d_Y(f(e), g(e)) \le d_Y(f(e), h(e)) + d_Y(h(e), g(e)) \le d(f, h) + d(h, e)$$

so taking the supremum on the left gives $d(f, g) \le d(f, h) + d(h, g)$.

A sequence $\{f_n\}$ in $\mathcal{B}(E; Y)$ is Cauchy if to any $\epsilon > 0$ there corresponds $N \in \mathbb{N}$ such that

$$\sup_{e \in E}(d_Y(f_n(e), f_m(e))) < \epsilon \forall\ n, m > N.$$

This implies that each sequence $\{f_n(e)\}$ is Cauchy in Y and hence convergent, by the assumed completeness of Y. The limits define a function $f : E \longrightarrow Y$. Passing to the limit in the inequalities

$$d_Y(f_n(e), f_m(e)) < \epsilon \forall\ n, m > N.$$

it follows that $d(f_n(e), f(e)) \le \epsilon$ if $n > N$ and hence that f is bounded – since $d_Y(f(e), y) \le d_Y(f_n(e), y) + \epsilon \le M + \epsilon$. Thus in fact $f \in \mathcal{B}(E; Y)$ and $d(f_n, f) \le e\epsilon$ if $n > N$, so $f_n \to f$ in $\mathcal{B}(E; Y)$ which is therefore complete.

(2) Show that if X is a metric space and Y is a (complete if you want because of the previous question but it is not needed here) metric space then the set $\mathcal{C}(X; Y)$ of continuous and bounded maps $f : X \longrightarrow Y$ is a closed subset of $\mathcal{B}(X; Y)$ defined in the previous question.

Solution: If X is a metric space and $\mathcal{C}(X; Y) \subset \mathcal{B}(X; Y)$ is the subset of bounded and continuous maps then a limit point $f \in \mathcal{B}(X; Y)$ is the limit of a sequence $f_n \in \mathcal{C}(X;Y)$. It follows that f is continuous, since for x, $x' \in X$,

$$d_Y(f(x), f(x')) \le d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(x')) + d_Y(f_n(x'), f(x')).$$

Fixing x', and choosing $\epsilon > 0$ there exists n such that the both the first and third terms on the right are bounded by $\epsilon/3$, by the uniform convergence of f_n to f. For this n the continuity of f_n means that there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies that the second term on the right is less than $\epsilon/3$ and hence $d_Y(f(x), f(x')) < \epsilon$, so f is continuous. As note above, it is necessaril bounded so $f \in \mathcal{C}(X;Y)$ which is therefore closed.

(3) Rudin Chap 7 No 24. Let X be a metric space with metric d and suppose $a \in X$ is a fixed point. Assign to each $p \in X$ the function $f_p : X \longrightarrow \mathbb{R}$ where $f_p(x) = d(p, x) - d(a, x)$ for each $x \in X$. Show that $f_p \in \mathcal{C}(X)$, i.e. it is a continuous bounded function, and so this construction defines a map $\Phi : X \longrightarrow \mathcal{C}(X)$. Prove that

$$\sup_{x \in X} |f_p(x) - f_q(x)| = d(p, q) \ \forall \ p, q \in X.$$

Hence conclude that Φ is an *isometry* (a map between metric spaces preserving the distance) which maps X 1-1 onto a subset of $\mathcal{C}(X)$. Since the latter is complete deduce that the closure of the range of Φ is a metric completion of X – is a complete metric space which has X (represented as the range of Φ) as a dense subset and which has a distance which restricts to the distance on X.

Solution: For each $p \in X$, the function $f_p(x) = d(p, x) - d(a, x)$ is continuous since the distance is continuous in each variable. It is bounded by the 'reverse triangle inequality' $|d(p, x) - d(a, x)| \le d(p, a)$. Thus $f_p \in \mathcal{C}(X)$. Consider the map $\Phi : X \ni p \longmapsto f_p(\cdot) \in \mathcal{C}(X)$ that this defines. Again by the triangle inequality, for any three points p, q, $x \in X$,

$$|f_p(x) - f_q(x)| = |d(p, x) - d(q, x)| \le d(p, q)$$

with equality when $x = q$. Thus

$$d(f_p, f_q) = \sup_{x \in X} |f_p(x) - f_q(x)| = d(p, q)$$

shows that the map is an isometry. It is 1-1, since $f_p = f_q$ in $\mathcal{C}(X)$ implies that $p = q$. Let $\bar{X}$ be the closure of $\Phi(X) \subset \mathcal{C}(X)$. As a closed subset of a complete metric space this is itself a complete metric space. Moreover we may think of $X \subset \bar{X}$ by identifying p with f_p. Then, by construction the distance on X is the same as the distance induced by this 1-1 inclusion and X is a dense subset of $\bar{X}$. Thus, $\bar{X}$ is a metric completion of X.

Remark. It is straightforward to show that any two metric completions are 'naturally isometric' – there is a unique bijection between them which leaves the points of X fixed and is an isometry.

(4) Let $f_n : [0, 1] \longrightarrow [0, \infty)$ be a sequence of continuous functions which is pointwise decreasing with $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0, 1]$. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Solution: We are not given uniform convergence only pointwise convergence of the decreasing sequence. However, Rudin's Theorem 7.13 (also proved in class) shows that $f_n \to 0$ uniformly on the compact set $[0, 1]$, since the functions are continuous, monotonic decreasing and have a continuous limit. The the convergence of the integral follows from another result in Rudin's book.

(5) Suppose that $f : [0, 1] \longrightarrow \mathbb{R}$ is a Riemann integrable function. Show that, as a consequence of the Stone-Weierstrass theorem, there is a sequence of polynomials p_n such that

$$\lim_{n\to\infty} \int_0^1 |f(x) - p_n(x)|dx = 0.$$

Hint: Use a problem from earlier homework about Riemann integrable functions.

Solution: In an earlier homework you showed that if f is Riemann integrable on $[0, 1]$ then there is a sequence of continuous functions, f_n, such that $\int_0^1 |f(x) - f_n(x)|dx \to 0$. Using the Stone-Weierstrass theorem, we can choose a sequence of polynomials p_n such that

$$\sup |f_n(x) - p_n(x)| < 2^{-n} \Longrightarrow \int_0^1 |f_n(x) - p_n(x)| < 2^{-n}.$$

Then

$$\int_0^1 |f(x) - p_n(x)| \le \int_0^1 |f(x) - f_n(x)| + \int_0^1 |f_n(x) - p_n(x)| \to 0 \text{ as } n \to \infty.$$