HOMEWORK 1 FOR 18.100B/C, FALL 2010
SECTION 1 (MELROSE)
DUE 11AM THURSDAY, 16 SEPTEMBER IN 2-108.

Blurb: Paper homework is due on Thursdays in 2-108 by 11AM. There will be a tray to put it in and I will remove it after lecture. No paper homework will be accepted late, period. HOWEVER, you may do your homework with TeX (I encourage you to do this, preferably using LaTeX) or scan your homeowork to a pdf file and email it to me (rbm at math dot mit dot edu works best). The deadline for scanned homework is 5PM on the Thursday on which it is due. Late homework will be accepted, and will be graded, but will in general incur a substantial penalty.

Suggestion. First think through how you will do a problem and then try to write down your argument as clearly and briefly as you can. The objective is to convince me that you really do understand why the statement is true, not to right a thesis just yet. It is always a good idea to think of your (possible) audience when writing. In this case it is ME, maybe it will be a grader next week but I will still look at your work. So, ask yourself when you write down an argument – Is this really convincing? Are the gaps to large? Are there missing steps? Is all this obvious stuff really necessary? Try to find the right level for you target audience – in this case it is clear who it is.

The rational numbers are $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$.

HW1.1 Rudin Chap 1, Prob 1. If r is rational and non-zero and x is irrational show that $r+x$ and rx are irrational.

Hint: An irrational number is an element of $\mathbb{R}$ which is not in $\mathbb{Q}$. There are only two possibilities, that $r+x$ is rational or irrational. By assumption r is rational. If $r+x$ were rational then x would also be rational (why?) .... The second part is similar.

Solution: We know that the rational numbers form a field. It follows that if $r \neq 0$ is rational then so are $-r$ and $1/r$. Thus if $x+r$ were rational then $x = (x+r) - r$ would also be rational. Similarly if xr were rational then $x = (xr)/r$ would be rational. Thus if x is irrational, meaning just that it is not rational, then so are $x+r$ and xr.

HW1.2 Show that there is no rational number $r = m/n$ such that $r^3 = 3$.

Hint: Suppose there is such a rational number r, choose a 'presentation' $r = m/n$ where the integers m and n do not both have a factor of 3 in their prime decompositions and deduce a contradiction.

Solution: Supposing there is such a rational number $r = m/n$ we will arrive at a contradiction. We may assume that not *both* m and n have a factor of 3 in their prime factorizations, which is to say they are not *both* divisible by 3. Since $r^3 = 3$, $m^3 = 3n^3$. Thus m^3 is a multiple of 3 and hence the same is true for m. You can see this using the prime decomoposition, since m^3 must have a factor of 3 in its prime factorization and 3 is prime, it must divide one of the factors, i.e. m. If you are not very comfortable with that then show that if m is *not* a multiple of 3 then neither is m^3. To do so, observe that if m is not a multiple of 3 then it is either of the form $3k+1$ or $3k+2$ where k is an integer. Compute the cube and you will see it is not divisibile by 3. So $m = 3k$ but then $n^3 = 9k^3$ and the same argument applies to n which contradicts the fact that the have been chosen not to both be multiples of 3.

HW1.3 Rudin Chap 1, Prob 4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound for E and β is an upper bound. Prove that $\alpha \le \beta$.

Hint: Look at the order relations between α, β and some element $\gamma \in E$.

Solution: By definition to say that α is a lower bound for E means that $\alpha \le \gamma$ for all $\gamma \in E$. To say that β is an upper bound for E is to say $\gamma \le \beta$ for all $\gamma \in E$. Since E is non-empty there is an element such that $\alpha \le \gamma \le \beta$ so by the transitivity property of an order $\alpha \le \beta$.

HW1.4 Show that the real numbers of the form $q_1 + q_2\sqrt{3}$, where q_1 and q_2 are rational, form a field. Hint: First check that the negative and inverse of a non-zero number of this form is of the same form, then check that the sum and product of two of these numbers is also of the same form. Then go through the list of axioms of a field checking that they are true because they are true for the reals.

Solution: You can use the fact that the real numbers form a field. Let F be the set of real numbers of the form $q_1 + q_2\sqrt{3}$. First check that F is 'closed' under addition and multiplication – the result of these operations is a real number, we just need to show that it is in F. Computing directly

$$(a+b\sqrt{3})+(c+d\sqrt{3}) = (a+b)+(c+d)\sqrt{3},\ (a+b\sqrt{3})(c+d\sqrt{3}) = (ac+3bd)+(ad+bc)\sqrt{3} \tag{1}$$

where of course we have used the fact that $(\sqrt{3})^2 = 3$. Now, the rationals form a field so these are both elements of f. It follows that the associativity, commutativity and distribution axioms hold for F since the hold for $\mathbb{R}$. Also, 0 and 1 are in F and have the required properties since this is true in $\mathbb{R}$. So, what remains is to show the existence of additive and multiplicative inverses. Clearly

$$(a+b\sqrt{3}) + (-a-b\sqrt{3}) = 0 \tag{2}$$

and observe that

$$(a+b\sqrt{3})(a-b\sqrt{3}) = a^2 - 3b^2. \tag{3}$$

Now, we know – following the argument in class for $\sqrt{2}$ – that $\sqrt{3}$ is not rational, which means that the only rational solution of $a^2 - 3b^2$ is $a = b = 0$. Thus, if $a + b\sqrt{3} \neq 0$, which means that a and b are not both zero, then $I = a^2 - 3b^2 \neq 0$ is rational and

$$(a+b\sqrt{3})^{-1} = \frac{a}{I} - \frac{b}{I}\sqrt{3} \tag{4}$$

is in F. Thus, F is indeed a field.

HW1.5 Rudin Chap 2, Prob 2: A complex number z is said to be *algebraic* if there are integers a_i, $i = 0, \dots, n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Assuming the easy part of the fundamental theorem of algebra, that such an equation can have at most n distinct solutions, prove that the set of all algebraic numbers is countable. Rudin's hint: For every positive integer N there are only finitely many such equations with

$$n + |a_0 + |a_1| + \cdots + |a_n| = N.$$

Using this write the algebraic numbers as a countable union of finite sets.

Solution: We can assume that the set of solutions of any one algebraic equation

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \tag{5}$$

is finite, provided at least one of the coefficients is non-zero. In fact there are at most n solutions – at the end of the course we will prove that there is always a solution (provided some $a_j \neq 0$ with $j > 0$. Now, consider the set of all $(n+2)$-tuples $A = \{(n, a_0, \dots, a_n)\}$ where $n \in \mathbb{N}$ and the $a_j \in \mathbb{Z}$ with at least one non-zero. Then A is a countable set – it is clearly not finite and it can be written as the union

$$A = \bigcup_N A_N, \ A_N = \{(n, a_0, \dots, a_n) \in A; n + |a_0| + \cdots + |a_n| \leq N\}.$$

Since each of the entries in an element of A_N is bounded by in absolute value by N, A_N is finite. Thus A is countable and if we let $F(n, a_0, \dots, a_n) \subset \mathbb{C}$ be the set of solutions of (5) then the set of algebraic numbers is

$$\bigcup_{(n,a_0,\dots,a_n) \in A_N} F(n, a_0, \dots, a_n)$$

is a countable union of finite sets hence is at most countable. All rationals are algebriac, since the satisfy linear equations $mz - n = 0$ hence the set of algebraic numbers is infinite and therefore countable.