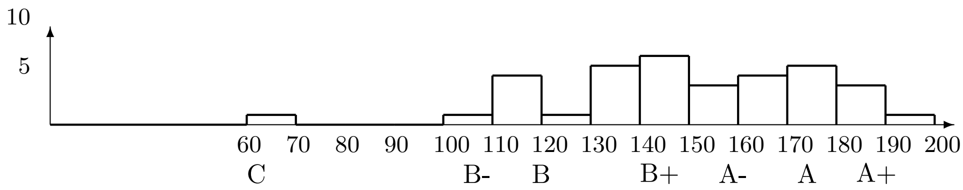
18.100B/C, SECTION 1, MELROSE
SOLUTIONS TO FINAL EXAM

Median seems to be around 142. Grades are modified by 100C component (if relevant) and HW/Tests.



Problem 1

Let $K \subset \mathbb{R}^2$ be a closed and bounded set and let $f : [0,1] \times K \longrightarrow \mathbb{C}$ be a continuous function on this subset of $\mathbb{R}^3$ (all with respect to the Euclidean metric). Show that the set of functions $\{g_s\}_{s\in[0,1]}$ where

$$g_s(x) = f(s,x)$$

is equicontinuous on K.

Solution: The subset $[0,1] \times K \subset \mathbb{R}^3$ is compact, as the product of two compact sets. To see this consider a sequence in $[0,1] \times K$ which consists of a pair of sequences, one in $[0,1]$ and the other in K. The first has a convergent subsequence with limit in $[0,1]$ by the Heine-Borel theorem and taking the corresponding subsequence of the second this has in turn a convergent subsequence by the compactness of K and a result to this effect from Rudin. Or recall that we proved that the product of two compact sets is compact, or use Heine-Borel more directly. Since f is continuous on this compact metric space, by a result in Rudin, it is uniformly continuous. Thus, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|s_1 - s_2|^2 + |y - y'|^2 < \delta^2 \Longrightarrow |f(s_1, y) - f(s_2, y')| < \epsilon.$$

Taking $s_1 - s_2$ this is precisely what is needed to show that the collection of functions g_s is equicontinuous.

Comments. Quite a few people got to uniform continuity and then did not see that it gave equicontinuity of the g_s. Some people tried to bluff their way through. Some people used the continuity of the map $[0,1] \ni s \longmapsto g_s \in \mathcal{C}(K)$, Heine-Borel and Ascoli-Arzelà. This works if you prove continuity of the map, which is pretty much precisely the equicontinuity we want but okay.

Problem 2

Show that in any metric space, the closure of a connected set is connected.

Solution: See Test 1.

Problem 3

If $f : [0, 2] \longrightarrow \mathbb{R}$ is a continuous function, state a theorem which shows that $F(x) = \int_0^x f(s)ds$ is differentiable on $[0, 2]$ (or prove it directly) and show that there exists $c \in (0, 2)$ such that $\int_0^2 f(x)dx = 2f(c)$.

Solution: Sorry about the typo. The Fundamental Theorem of Calculus states that the integral of a Riemann integrable function $F(x) = \int_0^x f(s)ds$ is differentiable at each point of continuity of f with derivative there $F'(x) = f(x)$. Since f is continuous everywhere on $[0, 2]$, F is differentiable on $[0, 2]$ and by the Mean Value Theorem there exists $c \in (0, 2)$ such that

$$F(2) - F(0) = \int_0^2 f(s)ds = f(c)(2 - 0) = 2f(c).$$

Problem 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and suppose that 0 is a local minimum of f, i.e. for some $\epsilon > 0$ $f(x) \geq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that if $f''(0) > 0$ then 0 is a strict local minimum in the sense that there exists $\epsilon > 0$ such that $f(x) > f(0)$ for $0 \neq x \in (-\epsilon, \epsilon)$.

Solution: By the definition of (second) derivative, $f''(0)$ is the limit as $x \to 0$ of the difference quotient $(f'(x) - f'(0))/x$. Since 0 is given to be an interior local minimum of f, $f'(0) = 0$ by a Theorem stating precisely this in Rudin's book. Thus if $f''(0) => 0$ then $f'(x)/x > 0$ in $(-\delta, \delta) \setminus \{0\}$ for some $\delta > 0$. Now, applying the mean value theorem if $x \in (-\delta, 0)$,

$$f(0) - f(x) = xf'(c) < 0$$

and similarly, if $x \in (0, \delta)$ then $f(x) - f(0) = xf'(c) > 0$ for some $c \in (0, \delta)$. Finally then $f(x) > f(0)$ so f has a strict local minimum at 0.

There are probably other ways to see this.

Problem 5

Let $g_n : [0, 1] \longrightarrow \mathbb{R}$ be a sequence of differentiable functions such that the sequence $g'_n : [0, 1] \longrightarrow \mathbb{R}$ is uniformly bounded.

(1) Show that there is a sequence of constants c_n such that sequence of functions $h_n(x) = g_n(x) - c_n$ has a uniformly convergent subsequence on $[0, 1]$.
(2) Show that if $\int_0^1 g_n dx$ is a bounded sequence in $\mathbb{R}$ then g_n has a uniformly convergent subsequence.

Solution: Note that I made a mistake in the first version of the proof by not reading the question and forgetting that I had made it harder by only supposing that the sequence of derivatives is uniformly bounded, not uniformly convergent! So maybe I concede that this was a little too tricky. Oh well, here is an answer to the actual question.

(1) Set $c_n = g_n(0)$ so then $h_n(0) = 0$ for all n. Since $h'_n(x) = g'_n(x)$ this sequence is unformly bounded and hence by the mean value theorem $|h_n(x)| = |h'_n(t_n)||x|$ for some $t_n \in (0, x)$ is uniformly bounded. It follows from Ascoli-Arzelà that h_n has a uniformly convergent

(2) The g_n are continuous, since they are differentiable, so the integral exists. Moroever, with h_n as in the previous part,

$$\int_0^1 h_n dx = \int_0^1 g_n dx - c_n.$$

Thus, if the integrals of the g_n are bounded then the c_n must be bounded, since we know that on the subsequence h_{n_i} converges uniformly so the corresponding subsequence of integrals on the left is convergent, by a result to this effect in Rudin. Thus, $\{c_{n)i}\}$ is bounded so it has a convergent subsequence, by the Heine-Borel Theorem (maybe Weierstrass in fact). The corresonding subsquence of g_n therefore converges uniformly, since it is $h_{n_i} + c_{n'_i}$.

Problem 6

Show that the series of functions

$$\sum_{n=1}^{\infty} \frac{x^n \sin(nx)}{n^3 + n^2 + 1}$$

converges uniformly on $[-1, 1]$ and defines a continuously differentiable function on $[-1, 1]$.

Solution: Since $|\sin y| \le 1$ the terms of the sequence are bounded by $(n^3 + n^2 + 1)^{-1}$ for $|x| \le 1$. This gives a convergent series, so the given series is uniformly convergent on $[-1, 1]$. Differentiating term by term, gives a new series with nth term

$$\frac{nx^{n-1} \sin(nx) + nx^n \cos(nx)}{n^3 + n^2 + 1}$$

This is bounded in absolute value by $n(n^3+n^2+1)^{-1}$ which again gives a convergent series, so the series of derivatives is also uniformly convergent. It follows from a result in Rudin's book that the limit is differentiable on $[-1,]$ and that the derivative is the uniform limit of this series of continuous functions, hence is also continuous.

Problem 7

(1) Explain carefully why the Riemann-Stieltjes integral

$$\int_{-1}^{1} x^2 \exp(x^3) d\alpha$$

exists for any increasing function $\alpha : [-1, 1] \longrightarrow \mathbb{R}$.

(2) Evaluate this integral when

$$\alpha = \begin{cases} x & x < 0 \\ x + 1 & x \ge 0. \end{cases}$$

This one surely was a gift.

Solution: The function $x^2 \exp(x^3)$ is continuous since the polynomials x^2 and x^3 are continuous, as is exp so by composition and product theorem continuity follows. Since α is increasing, the Riemann-Stieltjes integrability follows from the theorem in Rudin that any continuous function on a compact interval is integrable.

By a decomposition result in Rudin if we write $\alpha = \alpha_1 + \alpha_2$, where here α_2 is the Heaviside function with jump at 0 and $\alpha_1(x) = x$ then

$$\int_{-1}^{1} f d\alpha = \int_{-1}^{1} f d\alpha_1 + \int_{-1}^{1} f d\alpha_2$$

for any continuous function. The first integral is the Riemann integral and since $x^2 \exp(x^3) = \frac{1}{3}\frac{d}{dx}\exp(x^3)$ this evaluates to $\frac{1}{3}(e - e^{-1})$. On the other hand the value of f at 0 is zero, so this in fact is the value of the integral.

Problem 8

Let$f : X \longrightarrow \mathbb{R}$ be a continuous function on a compact metric space, X. If $\{x_n\}$ is a sequence in X show that $\{f(x_n)\}$ has a convergent subsequence with limit in $f(X)$.

If you didn't get this one you were not paying attention.

Solution: if $\{x_n\}$ is a sequenc in X then $\{f(x_n)\}$ is a sequence in $f(X)$ which is compact, as the continuous image of a compact set. Thus it has a convergent subsequence with limit in $f(X)$ as this is a property of any sequence in a compact set.

Problem 9

If $a_n > 0$, $n \in \mathbb{N}$, is a sequence of real numbers such that the sequence $b_N = \sum\limits_{n=1}^{N} a_n$ is a bounded, show that $\sum\limits_n a_n^{\frac{1}{2}} a_{n+1}^{\frac{1}{2}}$ converges.

Solution: Since $2a_n a_{n+1} \le a_n^2 + a_{n+1}^2$ taking square-roots shows that the increasins sequence b_N is bounded above by a multiple of the limit if the series, so is itself convergent. Hence the series converges.

Problem 10

Show that a non-empty open subset of $(-1, 1)$ can be written as an at most countable union of open intervals (a_n, b_n) where $(a_n, b_n) \cap (a_k, b_k) = \emptyset$ if $k \ne n$.

Solution: Let O be the open subset. Consider the rational points in O. For each such q let $O_q = \bigcup (a, b)$, where $q \in (a, b) \subset O$. Each O_q is open, being the union of open sets, and is non-empty since O being open contains an interval around q. Set $a(q) = \inf O_q$ and $b(q) = \sup O_q$, which exist since $O_q \subset (-1, 1)$. Then in fact $O_q = (a(q), b(q))$ is an interval. Indeed, $a(q) \notin O_q$ since otherwise the complement could not be closed, and similarly $b(q) \notin O)q$. However, given $\epsilon > 0$ there exists $a < a(q) + \epsilon$ with $(a, q] \subset O_q$ since $a(q)$ is the limit of lower points of intervals containing q in O. Similarly $[q, b) \subset O_q$ for some $b > b(q) - \epsilon$. Thus in fact $O_q = (a(q), b(q))$ as claimed. Now, $O_q = O_{q'}$ for any other $q' \in O_q \cap \mathbb{Q}$, since O_q is an interval containing q' so $O_q \subset O_{q'}$ but then $O_{q'}$ is an interval containing q (in O of course) so $O_{q'} \subset O_q$ and they are equal. It follows that any two of the O_q's for rational points in O are either equal or disjoint. Now, enumerating the rational points in O and dropping repeated intervals gives the intervals (a_n, b_n) which form a decomposition of O as an at most countable union of open intervals, since any point of O is in an interval $(a, b) \subset O$ and hence in an (a_n, b_n).