18.100B/C, FALL 2009 FINAL EXAM
3 HOURS

This exam is closed book: no books, notes or electronic devices permitted. You may use theorems from class, or the book, provided you can recall them correctly. This includes standard properties of the exponential and trigonometric functions. Remember, the thing we most want to see is clarity! The problems are of equal value but partial answers will receive limited credit. To achieve full marks you should answer all seven questions.

Problem 1

Show that the set $\{z \in \mathbb{C}; z = \exp(it^{24} + 23t^7) \text{ for some } t \in \mathbb{R}\}$ is connected.

Problem 2

Explain why there is no continuous map from the disk $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ onto the interval $(0, 1) \in \mathbb{R}$.

Problem 3

Suppose that a number s is the upper limit (limit supremum) of a subsequence of a sequence $\{x_n\}$ in the reals. Show that s is the limit of some subsequence of $\{x_n\}$.

Problem 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and suppose that 0 is a local maximum of f, i.e. for some $\epsilon > 0$, $f(x) \leq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that $f''(0) \leq 0$.

Problem 5

Let $\{\phi_n\}$ be a uniformly bounded sequence of continuous functions on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 x^k \phi_n(x) dx = 0$$

for every $k = 0, 1, 2, \ldots$. Show that for any continuous function $f : [0, 1] \to \mathbb{R}$, the limit

$$\lim_{n\to\infty} \int_0^1 f(x)\phi_n(x) dx$$

exists.

Problem 6

Using standard properties of the cosine function show that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \cos(nx)$$

defines a continuously differentiable function on the real line.

Problem 7

(1) Explain why the Riemann-Stieltjes integral

$$\int_{-1}^{1} \exp(x^2/3) d\alpha$$

exists for any increasing function $\alpha : [-1, 1] \longrightarrow \mathbb{R}$.

(2) Evaluate this integral when

$$\alpha = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$