18.100B/C, FALL 2009
FINAL EXAM, SOLUTIONS

These solutions are about as short as I think you should expect to get away with and still get full marks! I hope this wasn't all too painful, have a pleasant Winter break. RBM.

Problem 1

Show that the set $\{z \in \mathbb{C}; z = \exp(it^{24} + 23t^7) \text{ for some } t \in \mathbb{R}\}$ is connected.

Solution. The polynomial $p(t) = it^{24} + 23t^7$ is continuous on the real line. The function exp is continuous from $\mathbb{C}$ to $\mathbb{C}$ and hence the composite $\exp(p(t))$ is continuous on $\mathbb{R}$. The real line is connected and the continuous image of a connected set is connected, hence the given set is connected.

Problem 2

Explain why there is no continuous map from the disk $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ onto the interval $(0, 1) \in \mathbb{R}$.

Solution: By the Heine-Borel theorem the disc $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ is compact, being closed and bounded. Hence its image under a continuous map must be compact but $(0, 1) \subset \mathbb{R}$ is not compact, since it is not closed, so there cannot be a continuous map from the closed disc onto the open interval.

Problem 3

Suppose that a number s is the upper limit (limit supremum) of a subsequence of a sequence $\{x_n\}$ in the reals. Show that s is the limit of some subsequence of $\{x_n\}$.

Solution: By assumption there exists a subsequence, which we can denote y_k, of x_n so that $s = \lim_{n\to\infty} t_n$, $t_n = \sup_{n\geq k} y_k$. By the definition of sup there is a subsequence $y_{p(k)}$ such that $y_{p(k)}) > t_k - 1/k$. Since $t_k \geq y_{p_k} > t_k - 1/k$ it follows that $y_{n(k)} \to s$ as $k \to \infty$. Since $y_{n(k)}$ is a subsequence of y_k it is a subsequence of the original sequence x_n.

Problem 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and suppose that 0 is a local maximum of f, i.e. for some $\epsilon > 0$, $f(x) \leq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that $f''(0) \leq 0$.

Solution. Since 0 is a local maximum of f which is differentiable, it follows that $f'(0) = 0$ by a theorem in Rudin. For $0 < x < \epsilon$ the mean value theorem shows the existence of y with $0 < y < x$ such that

$$f(x) - f(0) = f'(y)(x - 0) \leq 0, \tag{1}$$

so $f'(y) \leq 0$. Thus the difference quotient for f'', $\frac{f'(y)-f'(0)}{y-0} = f'(y)/y$ is non-positive for some small positive y, arbitrary small. As the limit as $y \downarrow 0$ it follows that $f''(0) \leq 0$.

Problem 5

Let $\{\phi_n\}$ be a uniformly bounded sequence of continuous functions on $[0,1]$ such that

$$\lim_{n\to\infty}\int_0^1 x^k\phi_n(x)dx = 0 \tag{2}$$

for every $k = 0, 1, 2, \ldots$. Show that for any continuous function $f : [0,1] \to \mathbb{R}$, the limit

$$\lim_{n\to\infty}\int_0^1 f(x)\phi_n(x)dx$$

exists.

Solution: Since any polynomial is a finite sum of products of monomials x^k and constants it follows from (2) that

$$\lim_{n\to\infty}\int_0^1 p(x)\phi_n(x)dx = \sum_{k=1}^{N} c_k \lim_{n\to\infty}\int_0^1 x^k\phi_n(x)dx = 0 \tag{3}$$

for any polynomial p.

The uniform boundedness means that there exists a constant M such that $|\phi_n(x)| \le M$ for all n and for all $x \in [0,1]$. Given a continuous function $f : [0,1] \longrightarrow \mathbb{C}$, the Stone-Weierstrass Theorem shows that there is a sequence of polynomials $p_l(x)$ converging uniformly to f on $[0,1]$. Thus, given $\epsilon > 0$, there is a polynomial p such that

$$|p(x) - f(x)| < \epsilon/2(M+1) \Longrightarrow |\int_0^1 f(x)\phi_n(x)dx - \int_0^1 p(x)\phi_n(x)dx| < \epsilon/2. \tag{4}$$

Now, since p is a polynomial, (3) shows the existence of N such that

$$n > N \Longrightarrow |\int_0^1 p(x)\phi_n(x)dx| < \epsilon/2 \Longrightarrow |\int_0^1 f(x)\phi_n(x)dx| < \epsilon. \tag{5}$$

Thus $\lim_{n\to\infty}\int_0^1 f(x)\phi_n(x)dx = 0$ for any continuous function f on $[0,1]$.

Problem 6

Using standard properties of the cosine function show that the series

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\cos(nx)$$

defines a continuously differentiable function on the real line.

Solution: The terms in the series are all differentiable with derivatives

$$-\frac{1}{n^{3/2}}\sin(nx). \tag{6}$$

Since $|\sin(nx)| \le 1$ this series is absolutely and uniformly convergent on $\mathbb{R}$, by the comparison test with $\sum_n n^{-3/2}$. The same applies to the original series so, by a theorem in Rudin, the sum f is differentiable on $\mathbb{R}$ and the derivative is the sum of a uniformly convergent series of continuous functions, hence continuous. Thus f is continuously differentiable as a function on $\mathbb{R}$.

Problem 7

(1) Explain why the Riemann-Stieltjes integral

$$\int_{-1}^{1} \exp(x^2/3)d\alpha \tag{7}$$

exists for any increasing function $\alpha : [-1, 1] \longrightarrow \mathbb{R}$.

(2) Evaluate this integral when

$$\alpha = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

Solution: By a theorem in Rudin any continuous function is Riemann-Stieltjes integrable with respect to any increasing function α. Since $\exp(x^2/3)$ (or was it supposed to be $\exp(x^{\frac{2}{3}})$?) is continuous, as the composite of continuous functions, the integral (7) exists.

To compute it, consider the partition with two interior division points at $-\delta$ and δ, where $\delta > 0$ is small. The upper and lower Riemann-Stieltjes sums are then

$$\inf_{[-\delta,\delta]} \exp(x^2/3)(\alpha(\delta) - \alpha(-\delta)) \text{ and } \sup_{[-\delta,\delta]} \exp(x^2/3)(\alpha(\delta) - \alpha(-\delta)). \tag{8}$$

As $\delta \to 0$ these both approach 1 which must therefore be the value of the integral

$$\int_{-1}^{1} \exp(x^2/3)d\alpha = 1. \tag{9}$$

[Or by quoting some other result from Rudin.]