(1) If $f : [0,1] \longrightarrow \mathbb{R}$ is continuous show that there exists $c \in [0,1]$ such that

$$\int_0^1 f(x)dx = f(c).$$

(2) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and bounded, show that there is a point $x \in \mathbb{R}$ such that $f(x) = \pi - x$.

(3) Show that the sum

$$\sum_{n=0}^{\infty} e^{-n} cos(nx), \; x \in \mathbb{R}$$

converges uniformly on the real line to a function f for which the kth derivative $f^{(k)}$ exists for all k.

(4) Given two continuous functions

$$f_0 : [0,1] \longrightarrow \mathbb{R}, \; g : [0,1] \longrightarrow \mathbb{R}$$

Define a sequence of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ for $n \in \mathbb{N}$ by

$$f_n(x) := \int_0^x \frac{f_{n-1}(t)}{2} + g(t)dt$$

Show that $\{f_n\}$ converges uniformly to a differentiable function f which satisfies

$$f'(x) = \frac{f(x)}{2} + g(x)$$

(Hint: show first that $\sup|f_{n+1}(x) - f_n(x)| \leq \frac{1}{2}\sup|f_n(x) - f_{n-1}(x)|$)

(5) Show that if $\alpha : [-1,1] \longrightarrow \mathbb{R}$ is monotone increasing, and is not continuous, then $\alpha \notin \mathcal{R}(\alpha)$. In other words, show that

$$\int_0^1 \alpha d\alpha$$

is not well defined as a Riemann-Stieltjes integral when α is not continuous.

(6) Given n points in the plane, $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, prove that there exists some point x which minimizes the sum of the distances to $\{x_1, \ldots, x_n\}$. In other words,

$$\sum_{i=1}^{n} d(x, x_i) = \inf_{y \in \mathbb{R}^2} \sum_{i=1}^{n} d(y, x_i)$$

where $d(\cdot,\cdot)$ indicates the usual Euclidean metric on $\mathbb{R}^2$.

(7) Suppose that X is a metric space and $K_i \subset X$, $i \in \mathbb{N}$, are compact subsets such that

$$X = \bigcup_{i=1}^{\infty} K_i.$$

Show that X has a countable dense subset.

(8) Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function which is differentiable at every point and which is periodic of period 1, that is

$$u(x+1) = u(x) \; \forall \; x \in \mathbb{R}.$$

Show that there are two points, x_1, $x_2 \in \mathbb{R}$ with $f'(x_1) = f'(x_2) = 0$ and $|x_1 - x_2| < 1$.

(9) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuously differentiable function, meaning it is differentiable and its derivative is continuous on $[0, 1]$. Show that there is a sequence of polynomials $\{p_n\}$ such that $p_n \longrightarrow f$ uniformly and $p'_n \longrightarrow f'$ uniformly on $[0, 1]$.

(10) Show that the series

$$\sum_{i=1}^{\infty} \sqrt{n} x^n \tag{1}$$

converges for each $x \in (-1, 1)$, diverges for $|x| \geq 1$ and that the limit is a continuous function of $x \in (0, 1)$.