Since I did not have time to write out complete solutions to this exam (from 2007) I am just giving you very, very strong hints. Sorry about any typos. If you need more help, ask!

(1) If $f : [0,1] \longrightarrow \mathbb{R}$ is continuous show that there exists $c \in [0,1]$ such that

$$\int_0^1 f(x)dx = f(c).$$

Hint: By a theorem in Rudin (namely the Fundamental Theorem of Calculus), the integral of a continuous function is differentiable in the upper limit, so $F(t) = \int_0^t f(x)dx$ is differentiable on $[0,1]$. By the Mean Value Theorem, $F(1) - F(0) = F'(c)(1-0)$ for some $c \in (0,1)$.

Or, as Michael suggests: Bounds on the integral give $\inf f \leq \int_0^1 f(x)dx \leq \sup f$. Since it is a continuous function on a compact set, $\inf f$ and $\sup f$ are in the range of f so by the intermediate value theorem so is $\int_0^1 f(x)dx = f(c)$ for some $c \in [0,1]$.

(2) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and bounded, show that there is a point $x \in \mathbb{R}$ such that $f(x) = \pi - x$.

Hint: The function $g(x) = f(x) + x$ is continuous and by a theorem in Rudin, the continuous image of a connected set is connected, so $I = g(\mathbb{R}) \subset \mathbb{R}$ is connected. Since f is bounded, say $|f(x)| \leq M$ Taking $x_1 = M + \pi$ ensures that there is a point $g(x_1) = f(x_1) + x_1 \geq \pi$ in I and taking $x_2 = -M + \pi$ that there is a point $g(x_2) \leq \pi$ in I. A theorem in Rudin states that if $p \leq y \leq q$ $p, q \in I$ where $I \subset \mathbb{R}$ is connected then $y \in I$, so ....

(3) Show that the sum

$$\sum_{n=0}^{\infty} e^{-n} cos(nx), \ x \in \mathbb{R}$$

converges uniformly on the real line to a function f for which the kth derivative $f^{(k)}$ exists for all k.

Hint: Compute the kth derivative of series term by term. Since sin and cos are both bounded it follows that each of these series is uniformly convergetn on the whole real line. Apply a theorem from Ruding that if a series of differentiable functions converges uniformly (just at one point is enoug) and the differentiated series also converges then .... Now, apply this repeatedly the successive derivatives.

(4) Given two continuous functions

$$f_0 : [0,1] \longrightarrow \mathbb{R}, \ g : [0,1] \longrightarrow \mathbb{R}$$

Define a sequence of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ for $n \in \mathbb{N}$ by

$$f_n(x) := \int_0^x \frac{f_{n-1}(t)}{2} + g(t)dt$$

Show that $\{f_n\}$ converges uniformly to a differentiable function f which satisfies

$$f'(x) = \frac{f(x)}{2} + g(x)$$

(Hint: show first that $\sup|f_{n+1}(x) - f_n(x)| \leq \frac{1}{2}\sup|f_n(x) - f_{n-1}(x)|$)

(5) Show that if $\alpha : [-1,1] \longrightarrow \mathbb{R}$ is monotone increasing, and is not continuous, then $\alpha \notin \mathcal{R}(\alpha)$. In other words, show that

$$\int_0^1 \alpha d\alpha$$

is not well defined as a Riemann-Stieltjes integral when α is not continuous.

Hint: If a monotone increasing function is not continuous at a point p then the limit from the right R at p (which exists ...) must be strictly greater than the limit from the left L (which also exists ...). Let $\delta > 0$ be the larger of the differences $R - \alpha(p)$ and $\alpha(p) - L$. It follows that for any partition of the interval, the difference

$$U(\mathcal{P}, \alpha, \alpha) - L(\mathcal{P}, \alpha, \alpha) \geq \delta^2$$

since this point must be in one intervals – either as an interior or an end-point. It follows from a criterion for integrability in the book that α is not Riemann-Stieltjes integrable with respect to α.

(6) Given n points in the plane, $\{x_1, \dots, x_n\} \subset \mathbb{R}^2$, prove that there exists some point x which minimizes the sum of the distances to $\{x_1, \dots, x_n\}$. In other words,

$$\sum_{i=1}^n d(x, x_i) = \inf_{y \in \mathbb{R}^2} \sum_{i=1}^n d(y, x_i)$$

where $d(\cdot, \cdot)$ indicates the usual Euclidean metric on $\mathbb{R}^2$.

Hint: Let f be the sum of the distances. This is a continuous non-negative function on $\mathbb{R}^2$; let L be its infimum. Then L this is also the infimum of f over the set $|x| \leq L + \sum_{i=1}^n |x_i|$ because outside that set $|f(x)| > L$ (say why exactly). Thus, as the infimum of a continuous function on a compact set L is attained.

(7) Suppose that X is a metric space and $K_i \subset X$, $i \in \mathbb{N}$, are compact subsets such that

$$X = \bigcup_{i=1}^\infty K_i.$$

Show that X has a countable dense subset.

Hint: A compact metric space has a countable dense subset. The union over i of these subsets is countable and dense in X.

(8) Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function which is differentiable at every point and which is periodic of period 1, that is

$$u(x+1) = u(x) \ \forall \ x \in \mathbb{R}.$$

Show that there are two points, x_1, $x_2 \in \mathbb{R}$ with $f'(x_1) = f'(x_2) = 0$ and $|x_1 - x_2| < 1$.

Hint: By periodicity, $u(0) = u(1)$ so the MVT shows the existence of $c \in (0,1)$ with $u'(c)(1-0) = u(1) - u(0) = 0$. Apply the same argument again to the points c and $c+1$, so there exists $d \in (c, c+1)$ such that $u'(d)(c+1-c) = u(c+1) - u(c) = 0$. Since $|c - d| < 1$ this is what we want.

(9) Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuously differentiable function, meaning it is differentiable and its derivative is continuous on $[0,1]$. Show that there is a

sequence of polynomials $\{p_n\}$ such that $p_n \longrightarrow f$ uniformly and $p_n' \longrightarrow f'$ uniformly on $[0,1]$.

Hint: Since $f' : [0,1] \longmapsto \mathbb{R}$ is given to be continuous, there is a sequence of polynomials q_n converging uniformly to f' by the Stone-Weierstrass Theorem. Integrating from 0 set

$$p_n(x) = f(0) + \int_0^x q_n(s)ds.$$

This is a sequence of polynomials and certainly $p_n(0) = f(0)$ so converges to $f(0)$. A theorem in Rudin's book asserts that uniform convergence of the sequence of derivatives and convergence at one point implies uniform convergence of the sequence. Another theorem asserts that uniform convergence of a sequence of Riemann(-Stietjes) integable functions implies convergence of the integrals, so $p_n \to f$ uniformly (since the sequence converges uniformly and the limit is f).

(10) Show that the series

$$\sum_{i=1}^{\infty} \sqrt{n} x^n \tag{1}$$

converges for each $x \in (-1,1)$, diverges for $|x| \geq 1$ and that the limit is a continuous function of $x \in (0,1)$.

Hint: Convergence in $(-1,1)$ follows from the ratio test, divergence if $|x| \geq 1$ follows from the fact that the terms of a converent series must converge to zero. Moreover, the convergence on any compact subset $[-T,T]$, $T < 1$, is uniform – since the comparison is with a fixed absolutely convergent sequence. Thus the limit is in fact continuous on $(-1,1)$ since it is continuous on $[-T,T]$ for each $T < 1$ by the continuity of a uniformly convergent sequence of continuous functions. Hence, by restriction, it is continuous on $(0,1)$ (maybe put in to deliberately confuse?)