ASCOLI-ARZELÀ THEOREM

Theorem. If K is a compact metric space then a subset $F \subset \mathcal{C}(K)$ of the space of continuous complex-valued functions on K equipped with the uniform distance, is compact if and only if it is closed, bounded and equicontinuous.

You should recall that a continuous function on a compact metric space is bounded, so the function

$$d(f, g) = \sup_{x \in K} |f(x) - g(x)|$$

is well-defined. We have shown previously that this is a distance, i.e. $\mathcal{C}(K)$ is a metric space, that convergence with respect to this distance is equivalent to uniform convergence and that as a metric space $\mathcal{C}(K)$ is complete. We will use all these results freely. The definition of equicontinuity of a subset of $\mathcal{C}(K)$ is 'uniform (in $f \in F$) uniform (in the point in K) continuity'. That is, given $\epsilon > 0$ there must exist $\delta > 0$ such that

$$(1) \qquad |f(x) - f(y)| < \epsilon \ \forall \ x, y \in K \text{ with } d(x, y) < \delta \text{ and } \forall \ f \in F.$$

So, the *failure* of $F \subset \mathcal{C}(K)$ to be equicontinuous means that this condition fails. That is, for *some* $\epsilon > 0$ there exists *no* $\delta > 0$ for which the condition holds. Restated fully, this means that for some $\epsilon > 0$ and every $\delta > 0$ there exists $x, y \in K$ with $d(x, y) < \delta$ and some $f \in F$ such that $|f(x) - f(y)| \geq \epsilon$. Since each f is uniformly continuous, as δ gets smaller different choices of f must be involved.

Proof. Necessity:- We know that a compact set in any metric space is closed and bounded, so we only need to show that a compact set in $\mathcal{C}(K)$ is equicontinuous. Suppose it were not equicontinuous. From what we have just seen, this means there exists $\epsilon > 0$ such that for each $n \in \mathbb{N}$ there is a pair of points $x_n, y_n \in K$ and and a function $f_n \in F$ such that $d(x_n, y_n) < 1/n$ but $|f_n(x) - f_n(y)| \geq \epsilon$. This fixes a sequence $\{f_n\}$ in F and we claim that this can have no (uniformly) convergent subsequence. The point of course is that it can have no equicontinuous subsequence – because taking a subsequence n_j we can still violate the condition (1) for the same $\epsilon > 0$ and every $\delta > 0$ just by taking $n_j > 1/\delta$. Now, we showed before that a uniformly convergent sequence is equicontinuous so this implies that F is not compact – since as a compact set any sequence in it would have to have a (uniformly) convergent subsequence. [1]

[1] In case you forgot the proof of equicontinuity for a uniformly convergent sequence $f_n \in \mathcal{C}(K)$ here it is in a nutshell:- Let f be the limit, so by uniform convergence, given $\epsilon > 0$ there exists N such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon/3$ for all $x \in K$. Now, for each $j \leq N$ the function f_j is uniformly continuous so there exist $\delta_j > 0$ such that $d(x, y) < \delta_j$ implies $|f_j(x) - f_j(y)| < \epsilon$. The limit f is also uniformly continuous, so there exists $\delta' > 0$ such that $|f(x) - f(y)| < \epsilon/3$ whenever $d(x, y) < \delta'$. Set $\delta = \min(\delta', \min_{j \leq N} \delta_j) > 0$. If $d(x, y) < \delta \leq \delta'$ then for $n > N$,

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f_n(y) - f(y)| < \epsilon.$$

Thus this in fact holds for all n since $\delta \leq \delta_j$ for $j \leq N$ as well, so the set of functions forming the sequence is equicontinuous.

Sufficiency:- So, now we get to assume that F is closed, bounded and equicontinuous and have to prove that it is compact in $\mathcal{C}(K)$. The first thing to recall is that any compact metric space has a countable dense subset. This follows directly from the definition of compactness. Namely, given any $k \in \mathbb{N}$, cover K by all the balls of radius $1/k$ (centred at all the points of K.) By compactness this has a finite subcover, let $Q_k \subset K$ be the set of centers of such a finite subcover. Then every point of K is in one of the balls, so it is distant at most $1/k$ from (at least) one of the points in Q_k. The union, Q, of these finite sets is (at most) countable and is clearly dense in K – any point in K is the limit of a sequence in Q.

So, let $\{f_n\}$ be a sequence in F. Since F is equicontinuous, so is the sequence, and we need to show that it has a (uniformly) convergent subsequence; since F is closed the limit will be in F. Take a point $q \in Q$, then $\{f_n(q)\}$ is a bounded sequence in $\mathbb{C}$. So, by Heine-Borel, we may extract a subsequence of f_n so that $\{f_n(q)\}$ converges in $\mathbb{C}$. Since Q is countable we can construct successive subsequences, $f_{n_{k,j}}$ of the preceding subsequence $f_{n_{k-1,j}}$, so that the kth subsequence converges at the kth point of Q. Now, the diagonal sequence $f_{n_i} = f_{n_{i,i}}$ is 'eventually' a subsequence of $f_{n_{k,j}}$ for each k, i.e. after a finite number of terms it is a subsequence. So along this subsequence $f_{n_i}(q)$ converges for each point in Q – convergence being a property of the 'tail' of the sequence. So, let this sequence be $\{g_n\}$, it is a subsequence of the original sequence $\{f_n\}$ and we want to show that it converges uniformly; it suffices to show that it is uniformly Cauchy.

Suppose $\epsilon > 0$ is given. By the equicontinuity of the sequence we can choose $\delta > 0$ so that $|g_n(x) - g_n(y)| < \epsilon/3$ whenever $d(x, y) < \delta$. Next choose $k > 1/\delta$. Since there are only finitely many points in Q_k we may choose N so large that $|g_n(q) - g_m(q)| < \epsilon/3$ if $q \in Q_k$ and $n,\ m > N$. Then for a general point $x \in K$ there exists $q \in Q_k$ with $d(x, q) < 1/k < \delta$ so

$$|g_n(x) - g_m(x)| \le |g_n(x) - g_n(q)| + |g_n(q) - g_m(q)| + |g_m(q) - g_m(x)| < \epsilon \tag{2}$$

whenever $n,\ m > N$. Thus the sequence is uniformly Cauchy, hence uniformly convergent and we have proved the compactness of F. □

Next consider Peano's (or the Cauchy-Peano) existence theorem for ordinary differential equations. Here we are looking for a function $u : [0, \epsilon] \longrightarrow \mathbb{R}$ on a (possibly small) interval which is differentiable and satisfies

$$u'(x) = f(x, u(x)),\ u(0) = 0 \tag{3}$$

where $f : [0, 1] \times [-a, a] \longrightarrow \mathbb{R}$ is a given continuous function of two arguments with $a > 0$. You might hope that the solution exists for $x \in [0, 1]$ but this is generally not the case. Let $M = \sup |f|$ then we can conclude that the solution exists for at least as long as $\epsilon = a/M$. This is reasonable from (3) since we must have (of course assuming the solution exists) $|u(\epsilon)| = |u(\epsilon) - u(0)| \le \epsilon M$ by the Mean Value Theorem, so there is no opportunity for the solution to 'escape' from the domain before $\epsilon = a/M$.

So, how to we show that (3) has a solution? We cannot do it directly, except in very special cases. For instance if f does not depend on the second variable at all, then we can just integrate and use the Fundamental Theorem of Calculus to see that

$$u(x) = \int_0^x f(s)ds.$$

Peano's idea was to try something similar in the general case and think a little about the definition of the Riemann integral. So, for each n divide up the interval $[0, \epsilon]$, $\epsilon = a/M$, into n equal pieces $[x_{i-1}, x_i]$ using the notation for partitions. Now, we define a function u_n 'as though $f(x, y)$ was constant in y on this interval' and assuming we know already what has happened in the previous interval. We can think of this as defining two functions u_n, $v_n : [0, \epsilon] \longrightarrow [-a, a]$ where v_n is constant on $[x_{i-1}, x_i]$ and is equal to $u_n(x_{i-1})$ and $u_n(x)$ is linear on $[x_{i-1}, x_i]$ and is given by

$$u_n(x) = u_n(x_{i-1}) + \int_{x_{i-1}}^{x} f(s, v_n) ds \text{ on } [x_{i-1}, x_i]. \tag{4}$$

It follows that $u_n : [0, \epsilon] \longrightarrow [-a, a]$ is continuous and $v_n; [0, \epsilon] \longrightarrow [-a, a]$ is piecewise constant but generally jumps across the ends of the intervals. From (4)

$$|u_n(x) - u_n(x_{i-1})| \le M\epsilon/n \text{ on } [x_{i-1}, x_i] \tag{5}$$

and hence that

$$|u_n(x) - v_n(x)| \le M\epsilon/n \text{ on } [0, \epsilon]. \tag{6}$$

Note that $|u_n| \le a$ and both functions do exist on the whole interval.

If x and x' are in the same interval $[x_{i-1}, x_i]$ then estimating the integral (or using the Mean Value Theorem) shows that

$$|u_n(x) - u_n(x')| \le M|x - x'| \tag{7}$$

So it follows that the same is true for any x, $y \in [0, \epsilon]$. This shows that the sequence $u_n \in \mathcal{C}([0, \epsilon])$ is equicontinuous. So, by the Ascoli-Arzelà Theorem, it has a uniformly convergent subsequence $u_n \to u$ uniformly on $[0, \epsilon]$. On the other hand, from (6) it follows that the sequence v_n must also be uniformly convergent. Thus in fact, $v_n \to u$ uniformly as well. Now, $f(x, v_n(x))$ is Riemann integrable and it is also a uniformly convergent sequence since f is uniformly continuous (in the second variable). Thus by our theorem on convergence of Riemann integrals,

$$\int_0^x f(s, v_n(s)) ds \to \int_0^x f(s, u(s)) ds$$

for each $x \in [0, \epsilon]$. But this means that

$$u(x) = \int_0^x f(s, u(s)) \ \forall \ x \in [0, \epsilon].$$

It now follows that u is differentiable – since $f(s, u(s))$ is continuous – and by the Fundamental Theorem of Calculus – satisfies (3).

You might think that there can be only one solution to such a differential equation, with given initial value as here. This is certainly the case if $f(x, y)$ is independent of the second variable. However, in general it is not true. For instance the function

$$u(x) = x^2/4 \text{ satisfies } u' = x/2 = u^{\frac{1}{2}} \text{ on } [0, 1] \tag{8}$$

which is (3) for $f(x, y) = |y|^{\frac{1}{2}}$ which is certainly continuous. On the other hand $u(x) \equiv 0$ is also a solution.

Theorem. If f is continuous on $[0, 1] \times [-a, a]$ and in addition is Lipschitz in the second variable,

$$|f(x, y) - f(x, z)| \le A|y - z| \tag{9}$$

then there is a unique solution to (3) on $[0, a/M]$.

Proof. If u_1 and u_2 are two solutions then by the Mean Value Theorem

$$|u_1(x) - u_2(x) - (u_1(0) - u_2(0))| \leq x \sup |u_1' - u_2'| \leq xA \sup_{0,s} |u_1(s) - u_2(s)|.$$

Since $u_1(0) = u_2(0)$, taking $x < \frac{1}{2}$ shows that $u_1(x) = u_2(x)$ on $[0, \delta]$ where δ is the smaller of ϵ and $\frac{1}{2}$. The argument can be iterated to show that they remain equal as long as both exist. $\square$