18.100B, FALL 2002
SOLUTIONS TO RUDIN, CHAPTER 4, PROBLEMS 2,3,4,6

Problem 2

If $f : X \longrightarrow Y$ is a continuous map then $f^{-1}(C) \subset X$ is closed for each closed subset $C \subset Y$. For any map and any subset $G \subset Y$, $f(f^{-1}(G)) = G$. Now, if $E \subset X$ then $C = \overline{f(E)}$ is closed and $E \subset f^{-1}(C)$ (since $x \in E$ implies $f(e) \in f(E) \subset C$ implies $e \in f^{-1}(C)$). By the continuity condition $f^{-1}(C)$ is closed so $\overline{E} \subset f^{-1}(C)$ which implies $f(\overline{E}) \subset \overline{f(E)}$.

Consider $X = [0, 1) \cup [1, 2]$ as a subset of $\mathbb{R}$ with the usual metric. Then $f : X \longrightarrow [0, 1]$ given by

$$f(x) = \begin{cases} x & x \in [0,1) \\ 1 & x \in [1,2] \end{cases}$$

is continuous since a convergent sequence $\{x_n\}$ in X is eventually in $[0, 1)$ or $[1, 2]$ and so $\{f(x_n)\}$ converges by the continuity of x and constant maps. On the other hand, $E = [0, 1)$ is closed and $f(E)$ is not, with $\overline{f(E)} = [0, 1]$ so $f(\overline{E})$ is strictly contained in $\overline{f(E)}$.

Problem 3

By definition $Z(f) = f^{-1}(\{0\})$. The set $\{0\}$ is closed and f is continuous, so $Z(f)$ is closed.

Problem 4

If $y \in f(X)$ then there exists $x \in X$ such that $f(x) = y$. By the density of E in X there is a sequence $\{x_n\}$ in E with $x_n \to x$ in X. By the continuity of f, $f(x_n) \to f(x) = y$ so $f(E)$ is dense in $f(X)$.

Suppose $g(p) = f(p)$ for all $p \in E$. Given $x \in X$, by the result above, there exists $\{x_n\}$ in E such that $x_n \to x$ and $f(x_n) \to f(x)$. The continuity of g means that $g(x_n) = f(x_n) \to g(x)$ so $f(x) = g(x)$ for all $x \in X$.

Problem 6

The distance on $X \times Y$ is the sum of the distances on X and Y. I will do it with sequences.

Suppose E is compact and $f : E \longrightarrow Y$ is continuous. Now suppose $\{p_n\}$ is a sequence in graph(f). Thus, $p_n = (x_n, f(x_n))$ for some sequence $\{x_n\}$ in E. By the compactness of E, there is a convergent subsequence $\{x_{n(k)}\}$. By the continuity of f, $f(x_{n(k)})$ is convergent, and hence $p_n = (x_{n(k)}, f(x_{n(k)}))$ is convergent, so each sequence in graph(f) has a convergent subsequence. It follows that it is compact.

Conversely, suppose that E and graph(f) are both compact. Let $\{x_n\}$ be a convergent sequence in E, $x_n \to x$. Then $\{(x_n, f(x_n))\}$ is a sequence in graph(f) so by its compactness has a convergent subsequence, $(x_{n(k)}, f(x_{n(k)})) \to (x, q)$. Since the graph is closed, this must be a point in it, so $q = f(x)$. This argument applies to any subsequence of $\{x_n\}$, so we see that any subsequence of $\{f(x_n)\}$ has a convergent subsequence with limit $f(x)$. This however implies that $f(x_n) \to f(x)$, since if not there would exist a sequence $f(x_{n(k)}$ with $d(f(x), f(x_{n(k)}) \geq c > 0$ and this cannot have such a convergent subsequence. Thus in fact $x_n \to x$ implies that $f(x_n) \to f(x)$, so f is continuous.