18.100B, FALL 2002
PRACTICE TEST 2

Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly!

Problem 1

Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous real-valued function. Show that there exists $c \in (0, 1)$ such that

$$\int_0^1 f(x)dx = f(c).$$

Solution. If m and M are respectively the infimum and supremum of f on $[0, 1]$ then $f([0, 1]) = [m, M]$ since these values are attained and $f([0, 1])$ must be connected. Since

$$m \leq I = \int_0^1 f(x)dx \leq M$$

it follows that $I \in [m, M]$ so there exists $c \in [0, 1]$ with $f(c) = I = \int_0^1 f(x)dx$. □

Problem 2

(This is basically Rudin Problem 4.14)
Let $f : [0, 1] \longrightarrow [0, 1]$ be continuous.

(1) State why the the map $g(x) = f(x) - x$, from $[0, 1]$ to $\mathbb{R}$ is continuous.
(2) Using this, or otherwise, show that $L = \{x \in [0, 1]; f(x) \leq x\}$ is closed and $\{x \in [0, 1]; f(x) < x\}$ is open.
(3) Show that L is not empty.
(4) Suppose that $f(x) \neq x$ for all $x \in [0, 1]$ and conclude that L is open in $[0, 1]$ and that $L \neq [0, 1]$.
(5) Conclude from this, or otherwise, that there must in fact be a point $x \in [0, 1]$ such that $f(x) = x$.

I found the wording of this question a bit confusing.

Solution. (1) If f and g are continuous then so is $c_1 f + c_2 g$ for any constants and x is continuous directly from the definition, so $g(x) = f(x) - x$ is continuous.
(2) By definition, $L = \{x; g(x) \leq 0\} = g^{-1}([-\infty, 0])$ is the inverse image of a closed set, hence is closed. Similarly the second set is $g^{-1}((-\infty, 0))$ so is the inverse image of an open set under a continuous map, so is open in $[0, 1]$.
(3) Since $g(1) = f(1) - 1 \leq 0$, $1 \in L$.
(4) If $f(x) \neq x$ for all $x \in [0, 1]$ then $g(x) \neq 0$ for all $x \in [0, 1]$ and hence $L = g^{-1}((-\infty, 0))$ is open in $[0, 1]$. Thus L is both open and closed and is non-empty so $L = [0, 1]$. However, $g(0) = f(1) - 0 \geq 0$ so this is not possible and $L \neq [0, 1]$.

(5) Thus $f(x) \neq x$ for all $x \in [0,1]$ is not possible, so there must exist a point $x \in [0,1]$ with $f(x) = x$.

□

Problem 3

Consider the function
$$f(x) = \frac{-x(x+1)(x-100)}{x^{44} + x^{34} + 1}$$
for $x \in [0, 100]$.

(1) Explain why f has derivatives of all orders.
(2) Compute $f'(0)$.
(3) Show that there exists $\epsilon > 0$ such that $f(x) > 0$ for $0 < x < \epsilon$.
(4) Show that there must exist a point x with $f'(x) = 0$ and $0 < x < 100$.

Solution. (1) Polynomials are infinitely differentiable and the quotient p/q of two infinitely differentiable functions is infinitely differentiable on any interval on which $q \neq 0$. Since $x^{44} + x^{34} + 1 > 0$ for $x \in \mathbb{R}$ it follows that $f = p/q$ is infinitely differentiable on $\mathbb{R}$.

(2) Since $f'(0) = \frac{p'(0)q(0) - f(0)q'(0)}{q^2(0)}$ and $p(0) = 0$, $p'(0) = 100$, $q(0) = 1$, $q'(0) = 0$ it follows that $f'(0) = 100$.

(3) Since $f'(x)$ is continuous, there exists $\epsilon > 0$ such that $f'(x) > 0$ if $x \in [0, \epsilon)$. By the mean valued theorem for $x \in (0, \epsilon)$,
$$f(x) = xf'(y),\ y \in (0, \epsilon) \Longrightarrow f(x) > 0.$$

(4) From the form of f, $f(100) = 0$ so, again by the mean value theorem
$$f(100) - f(0) = 0 = 100f'(x)$$
for some $x \in (0, 100)$.

□

Problem 4

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are two functions which are continuous at 0, show that the function
$$h(x) = \max\{f(x), g(x)\},\ x \in \mathbb{R}$$
is also continuous at 0.

Solution. Either $h(0) = f(0)$ or $h(0) = g(0)$ (or both). Since h is unchanged if we exchange f and g we may assume that $h(0) = f(0)$.

If $g(0) \neq f(0)$ then $g(0) < f(0)$. By the continuity of f and g, given $\epsilon > 0$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that
$$|x| < \delta_1 \Longrightarrow |f(x) - f(0)| < \epsilon,\ |x| < \delta_2 \Longrightarrow |g(x) - g(0)| < \epsilon.$$
Taking $\delta = \min(\delta_1, \delta_2)$ and $\epsilon < \frac{1}{2}(f(0) - g(0))$ gives both $g(x) \leq g(0) + \frac{1}{2}\epsilon$ and $f(x) \geq f(0) - \frac{1}{2}\epsilon \geq g(x)$ on (δ, δ) so $h(x) = f(x)$ is continuous at 0.

On the other hand if $g(0) = f(0)$ then taking $\delta = \min(\delta_1, \delta_2)$ means that $f(x), g(x) \in (h(0) - \epsilon, h(0) + \epsilon)$ so $|h(x) - h(0)| < \epsilon$ and again the continuity of h follows. □