Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly!

**Problem 1**

Consider the set $S$ defined as follows. The elements of $S$ are sequences, $(s_n)_{n=1}^\infty$ with all entries either 1 or 2 and with the additional property that every 2 is followed by a 1. Said more precisely, for every $n$, $s_n = 1$ or $s_n = 2$ and if $s_n = 2$ then $s_{n+1} = 1$.

Say why precisely one of the following is true:

(a) $S$ is finite
(b) $S$ is countably infinite
(c) $S$ is uncountably infinite

and then decide which one is true and prove it.

**Solution and remarks:** By definition a set $S$ is finite if it is either empty or else is in 1-1 correspondence with the set $\{1, \ldots, n\}$ for some $n$. It is countably infinite if it is in 1-1 correspondence with $\mathbb{N}$ and it is uncountably infinite if it is neither finite nor countably infinite. Only one of these can hold (since we know that $\mathbb{N}$ cannot be in 1-1 correspondence with $\{1, \ldots, n\}$ for any finite $n$).

The set $S$ is uncountably infinite. Here is a proof that reduces it to the case we looked at in class. Namely, we know that $S'$ which consists of the set of sequences with values 0 or 1 is uncountably infinite. We show that $S$ and $S'$ are in 1-1 correspondence (and hence have the same cardinality by definition). Take a sequence in $S'$ and replace every occurrence of 0 by two terms, 2 followed by 1. This gives a sequence in $S$. Moreover no two sequences in $S'$ are mapped to the same sequence in $S$. Thus the map is injective. We can construct an inverse the same way, replace every pair 2,1 by one element 0. Thus $S$ is indeed uncountably infinite.

It is also fairly straightforward to use the diagonalization procedure, but not completely trivial since you have to make sure that the new sequence is in $S$ and different from the others.

**Problem 2**

Consider the metric space $M = [0, 1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\}$ with the usual metric, $d(x, y) = |x - y|$. Is the set $A = [0, \frac{1}{2}) = \{x \in \mathbb{R}; 0 \leq x < \frac{1}{2}\}$ open as a subset of $M$? What is the closure of $A$ as a subset of $M$? Is $A$ compact? Is the closure of $A$ compact? In each case justify your answer.

**Solution and remarks:** Everything is relative to the metric space $M = [0, 1]$.

1. As a subset of $M$, $A$ is indeed open. If $x \in A$ then $B(x, \epsilon) \subset A$ if $\epsilon = \frac{1}{2} - x$, since $|y - x| < \epsilon$, $y \in [0, 1]$ implies $y < \frac{1}{2}$ and hence $y \in A$.

2. Clearly $\frac{1}{2}$ is a limit point of $A$ so the closure $\bar{A} \supset [0, \frac{1}{2})$. By the same argument as above $[\frac{1}{2}, 1]$ is open in $[0, 1]$ so this set is closed and hence $\bar{A} = [0, \frac{1}{2}]$. 

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(3) Since $A$ is not closed it cannot be compact.
(4) By the Heine-Borel theorem $\bar{A} = [0, \frac{1}{2}]$ is compact since it is closed and bounded.

Usual errors with this sort of question are to say that $A$ is not open, thinking of it as a subset of $\mathbb{R}$ this is certainly true but it is open as a subset of $M$. Similarly in the third part it does not follow directly from the fact that $A$ is open that it is not compact! It does follow from the fact that it is not closed, but in a finite metric space (which this is not) there are open compact sets so something else has to be said.

**Problem 3**

Let $M$ be a compact metric space. Suppose $A \subset M$ is not compact. Show, directly from the definition or using a theorem proved in class, that $A$ is not closed.

**Solution:** By a theorem in class every closed subset of a compact metric space is compact, hence if $A$ is not compact it is not closed.

**Problem 4**

Recall that a set $S$ in a metric space $M$ is connected if any separated decomposition of it, $S = A \cup B$ where $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, is ‘trivial’ in the sense that either $A$ or $B$ is empty. Show that the whole metric space $M$ is connected if and only if the only subsets $A \subset M$ of it which are both open and closed are the ‘trivial’ cases $A = \emptyset$ and $A = M$.

**Solution:** Suppose first that $M$ is connected. Let $A$ be a subset of $M$ which is both open and closed. Show that the whole metric space $M$ is connected if and only if the only subsets $A \subset M$ of it which are both open and closed are the ‘trivial’ cases $A = \emptyset$ and $A = M$.

Conversely, suppose that the only subsets of $M$ which are both open and closed are $M$ and $\emptyset$. Then let $A$ and $B$ be separated sets in $M$ such that $M = A \cup B$. This means that $B = M \setminus A$ is the complement of $A$ in $M$. The conditions that $A$ and $B$ be separated imply that $\overline{A} \cap B = \emptyset$, so $A \subset M \setminus B = A$ hence $A$ must be closed. Similarly $B$ must be closed and hence $A$ must be both open and closed. Thus one of $A$ or $B$ must be empty and hence, by definition, $M$ is connected.