

**18.100B, FALL 2002**  
**PRACTICE TEST 1 WITH SOLUTIONS**

Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly!

PROBLEM 1

Consider the set  $S$  defined as follows. The elements of  $S$  are sequences,  $\{s_n\}_{n=1}^{\infty}$  with all entries either 1 or 2 and with the additional property that every 2 is followed by a 1. Said more precisely, for every  $n$ ,  $s_n = 1$  or  $s_n = 2$  and if  $s_n = 2$  then  $s_{n+1} = 1$ . Say why precisely one of the following is true

- (a)  $S$  is finite
- (b)  $S$  is countably infinite
- (c)  $S$  is uncountably infinite

and then decide which one is true and *prove* it.

Solution and remarks: By definition a set  $S$  is finite if it is either empty or else is in 1-1 correspondence with the set  $\{1, \dots, n\}$  for some  $n$ . It is countably infinite if it is in 1-1 correspondence with  $\mathbb{N}$  and it is uncountably infinite if it is neither finite nor countably infinite. Only one of these can hold (since we know that  $\mathbb{N}$  cannot be in 1-1 correspondence with  $\{1, \dots, n\}$  for any finite  $n$ .)

The set  $S$  is uncountably infinite. Here is a proof that reduces it to the case we looked at in class. Namely, we know that  $S'$  which consists of the set of sequences with values 0 or 1 is uncountably infinite. We show that  $S$  and  $S'$  are in 1-1 correspondence (and hence have the same cardinality by definition). Take a sequence in  $S'$  and replace every occurrence of 0 by two terms, 2 followed by 1. This gives a sequence in  $S$ . Moreover no two sequences in  $S'$  are mapped to the same sequence in  $S$ . Thus the map is injective. We can construct an inverse the same way, replace every pair 2, 1 by one element 0. Thus  $S$  is indeed uncountably infinite.

It is also fairly straightforward to use the diagonalization procedure, but not completely trivial since you have to make sure that the new sequence is in  $S$  and different from the others.

PROBLEM 2

Consider the metric space  $M = [0, 1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\}$  with the usual metric,  $d(x, y) = |x - y|$ . Is the set  $A = [0, \frac{1}{2}) = \{x \in \mathbb{R}; 0 \leq x < \frac{1}{2}\}$  open as a subset of  $M$ ? What is the closure of  $A$  as a subset of  $M$ ? Is  $A$  compact? Is the closure of  $A$  compact? In each case justify your answer.

Solution and remarks: Everything is relative to the metric space  $M = [0, 1]$ .

- (1) As a subset of  $M$ ,  $A$  is indeed open. If  $x \in A$  then  $B(x, \epsilon) \subset A$  if  $\epsilon = \frac{1}{2} - x$ , since  $|y - x| < \epsilon$ ,  $y \in [0, 1]$  implies  $y < \frac{1}{2}$  and hence  $y \in A$ .
- (2) Clearly  $\frac{1}{2}$  is a limit point of  $A$  so the closure  $\bar{A} \supset [0, \frac{1}{2}]$ . By the same argument as above  $(\frac{1}{2}, 1]$  is open in  $[0, 1]$  so this set is closed and hence  $\bar{A} = [0, \frac{1}{2}]$ .

- (3) Since  $A$  is not closed it cannot be compact.  
 (4) By the Heine-Borel theorem  $\bar{A} = [0, \frac{1}{2}]$  is compact since it is closed and bounded.

Usual errors with this sort of question are to say that  $A$  is not open, thinking of it as a subset of  $\mathbb{R}$  this is certainly true but it is open as a subset of  $M$ . Similarly in the third part it does not follow directly from the fact that  $A$  is open that it is not compact! It does follow from the fact that it is not closed, but in a finite metric space (which this is not) there are open compact sets so something else has to be said.

### PROBLEM 3

Let  $M$  be a *compact* metric space. Suppose  $A \subset M$  is *not* compact. Show, directly from the definition or using a theorem proved in class, that  $A$  is *not* closed.

Solution: By a theorem in class every closed subset of a compact metric space is compact, hence if  $A$  is not compact it is not closed.

### PROBLEM 4

Recall that a set  $S$  in a metric space  $M$  is connected if any separated decomposition of it,  $S = A \cup B$  where  $\bar{A} \cap B = \emptyset = A \cap \bar{B}$ , is ‘trivial’ in the sense that either  $A$  or  $B$  is empty. Show that the whole metric space  $M$  is connected if and only if the only subsets  $A \subset M$  of it which are *both open and closed* are the ‘trivial’ cases  $A = \emptyset$  and  $A = M$ .

Solution: Suppose first that  $M$  is connected. Let  $A$  be a subset of  $M$  which is both open and closed. Then  $B = M \setminus A$  is also both open and closed and  $M = A \cup B$ . Since  $A$  and  $B$  are separated ( $A \cap B = \emptyset$  and  $\bar{A} = A$ ,  $\bar{B} = B$ ) it follows, from the assumption that  $M$  is connected, that one of them is empty, so  $A = \emptyset$  or  $A = M$  are the only sets which are both open and closed.

Conversely, suppose that the only subsets of  $M$  which are both open and closed are  $M$  and  $\emptyset$ . Then let  $A$  and  $B$  be separated sets in  $M$  such that  $M = A \cup B$ . This means that  $B = M \setminus A$  is the complement of  $A$  in  $M$ . The conditions that  $A$  and  $B$  be separated imply that  $\bar{A} \cap B = \emptyset$ , so  $\bar{A} \subset M \setminus B = A$  hence  $A$  must be closed. Similarly  $B$  must be closed and hence  $A$  must be both open and closed. Thus one of  $A$  or  $B$  must be empty and hence, by definition,  $M$  is connected.