

18.100B, FALL 2002
PRACTICE FINAL EXAM: DECEMBER 16

Some of these are similar to the questions on the final and some are rejects.

Try each of the questions; the first seven are worth 12 points each, the last one is harder and is worth 16. You may use theorems from class, or the book, provided you can recall them correctly! This includes standard properties of the exponential and trigonometric functions. No books or papers are permitted.

PROBLEM 1

Show that the set $\{z \in \mathbb{C}; 1 < |z| < 2\}$ is connected as a subset of \mathbb{C} with the usual metric.

PROBLEM 2

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable and satisfy $|g'(x)| \leq \frac{1}{2}$. Show that the function $f(x) = x - g(x)$ is 1-1.

PROBLEM 3

- (1) Why is the function $f(x) = |x|^{\frac{5}{2}} \cos(|x|^{\frac{3}{2}})$ continuously differentiable on $[0, 1]$?
- (2) Why does f have a minimum value on this interval?

PROBLEM 4

Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. Show that there exists $c \in (0, 1)$ such that $\int_0^1 f(x) dx = f(c)$.

PROBLEM 5

- (1) For what values of $x \in \mathbb{R}$ does the series

$$\sum_{n=0}^{\infty} n \exp(-nx)$$

converge?

- (2) For what intervals $[a, b]$ does it converge uniformly?
- (3) On what intervals $[a, b]$ is the sum of the series differentiable?

PROBLEM 6

Consider the (power) series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

Show that this series converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$; let $f(x)$ denote the sum. Show that the series obtained by term-by-term differentiation converges uniformly in the same set and explain why the limit is $f'(x)$. Is $f'(x)$ a rational function?

PROBLEM 7

- (1) Explain carefully why the Riemann-Stieltjes integral

$$\int_0^2 \exp(3(|x|^{\frac{3}{2}} - 1)) d\alpha$$

exists for any increasing function $\alpha : [0, 1] \rightarrow \mathbb{R}$.

- (2) Evaluate this integral when

$$\alpha(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 3 & 1 \leq x \leq 2. \end{cases}$$

PROBLEM 8

Suppose that $f_n : [0, 1] \rightarrow \mathbb{R}$ is a sequence of continuous functions which is uniformly bounded and satisfies

$$f_n(x) = \frac{1}{n} + \int_0^x f_n^2(t) dt, \quad x \in [0, 1].$$

Show that $\{f_n\}$ is uniformly convergent on $[0, 1]$ and prove that the limit is identically zero.