18.100B, Fall 2002, Homework 6

Due in 2-251, by Noon, Tuesday October 29. Rudin:

(1) Chapter 4, Problem 20

If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x,z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

(b) Prove that ρ_E is uniformly continuous on X by showing that

$$|\rho_E(x) - \rho_E(y)| \le d(x,y)$$

for all $x, y \in X$.

Solution. (a) If $\rho_E(x) = 0$ then there exists a sequence $z_n \in E$ such that $d(x, z_n) \to 0$. This implies $z_n \to x$ and hence $x \in \bar{E}$. Conversely if $x \in \bar{E}$ then either $x \in E$, in which case $\rho_E(x) = 0$, or else $x \in E'$, so there exists a sequence $z_n \in E$ with $z_n \to x$. This implies $d(x, z_n) \to 0$ so $\rho_E(x) = 0$.

(b) If $x, y \in X$ then for any $z \in E$, using the triangle inequality

$$\rho_E(x) \le d(x,z) \le d(x,y) + d(y,z).$$

Taking the infimum over $z \in E$ on the right-hand side shows that $\rho_E(x) - \rho_E(y) \le d(x,y)$. Interchanging the roles of x and y gives the desired estimate

$$|\rho_E(x) - \rho_E(y)| \le d(x,y).$$

This proves the uniform continuity of ρ_E, since given $\epsilon > 0$, $d(x,y) < \epsilon$ implies $|\rho_E(x) - \rho_E(y)| < \epsilon$.

□

(2) Chapter 4, Problem 23

A real valued function defined on (a,b) is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

whenever $x, y \in (a,b)$ and $\lambda \in (0,1)$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. If f is convex on (a,b) and if $a < s < t < u < b$ show that

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}.$$

Solution. (c) We do the third part first. Since $a < s < t < u < b$, $t = \lambda s + (1-\lambda)u$ with $\lambda = \frac{u-t}{u-s} \in (0,1)$. Thus

$$f(t) \le \frac{u-t}{u-s} f(s) + \frac{t-s}{u-s} f(u) \Longrightarrow (t-s)f(u) + (u-t)f(s) - (u-s)f(t) \ge 0.$$

This can be rewritten as $(t-s)(f(u)-f(s)) - (u-s)(f(t)-f(s)) \ge 0$ and $(u-s)(f(u)-f(t)) - (u-t)(f(u)-f(s)) \ge 0$ proving the two desired inequalities:

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}. \tag{1}$$

(a) Given $x \in (a,b)$ choose $\delta > 0$ so that $[x-\delta, x+\delta] \subset (a,b)$. Now, consider a point $z \in (x-\delta, x)$ applying the second inequality in (1) gives the first inequality in

$$\frac{f(x)-f(x-\delta)}{\delta} \le \frac{f(z)-f(x)}{z-x} \le \frac{f(x+\delta)-f(x)}{\delta}. \tag{2}$$

Applying the outer inequality in (1) to the three points $z < x < x+\delta$ gives the second inequality. Now consider the case $x < z < x+\delta$. Then the first inequality in (2) follows from the outer inequality in (1) applied to the three points $x-\delta, x, z$ and the second inequality in (2) follows from the first inequality in (1) applied to $x, z, x+\delta$. Now (2) implies that

$$|f(x)-f(z)| \le C|x-z| \ \forall \ z \in (x-\delta, x+\delta)$$

and hence proves the continuity of f (in fact the Lipschitz continuity).

(c) Let g be convex and increasing on (c,d) and f be convex on (a,b) with $f(a,b) \subset (c,d)$. Then set $h(x) = g(f(x))$. Since f is convex, $A = f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) = B$ and since g is increasing, $g(A) \le g(B)$ so

$$h(\lambda x + (1-\lambda)y) = g(f(\lambda x + (1-\lambda)y) \le g(\lambda f(x) + (1-\lambda)f(y) \le \lambda h(x) + (1-\lambda)h(y)$$

proving the convexity of h.

□

(3) Chapter 4, Problem 26

Suppose X, Y and Z are metric spaces and Y is compact. Let $f : X \longrightarrow Y$ and let $g : Y \longrightarrow Z$ be continuous and 1-1 and put $h(x) = g(f(x))$. Prove that f is uniformly continuous if h is uniformly continuous. Show that compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Solution. Consider the subset $Z' = g(Y)$ as a metric space with the metric induced from Z. Then $g : Y \longrightarrow Z'$ is 1-1 and onto. Since Y is compact, so is Z' and by a result from class, the inverse of g is continuous. Thus, again by a result from class, both g and $g^{-1} : Z' \longrightarrow Y$ are uniformly continuous. Note that the composite of two uniformly continuous maps is uniformly continuous[1]. Applying this to $f = g^{-1} \circ h$, $h = g \circ f$ shows that the uniform continuity of h implies that of f.

As a counterexample to the result when the compactness of Y is dropped, take $X = Z = [0,1]$ and $Y = [0, \frac{1}{2}) \cup [1, \frac{3}{2}]$. Let f be the discontinuous map $f(x) = x$ for $0 \le x < \frac{1}{2}$, $f(x) = x + \frac{1}{2}$ for $\frac{1}{2} \le x \le 1$. Then let g be the continuous map $g(y) = y$ for $0 \le y < \frac{1}{2}$ and $g(y) = y - \frac{1}{2}$ for $1 \le y \le \frac{3}{2}$. Observe that g is uniformly continuous, since $|g(y) - g(y')| \le |y - y'|$. The composite map is the identity on $[0,1]$, so uniformly continuous, but f is not even continuous (of course if it was continuous it would be uniformly continuous since $[0,1]$ is compact).

[1] If the maps are $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, both uniformly continuous then given $\epsilon > 0$ there exists $\eta > 0$ sucht that $d_Y(y,y') < \eta$ implies $d_Z(g(y), g(y')) < \epsilon$. Then from the uniform continuity of f there exists $\delta > 0$ such that $d_X(x,x') < \delta$ impies $d_Y(f(x), f(x')) < \gamma$ and hence $d(g(f(x)), g(f(x')) < \epsilon$. But this is the uniform continuity of $h = g \circ f$.

□

(4) Chapter 5, Problem 1

Let f be defined for all real x and suppose that

$$|f(x) - f(y)| \leq (x-y)^2 \ \forall\ x, u \in \mathcal{R}.$$

Prove that f is constant.

Solution. Certainly f is differentiable at each point with derivative zero, since

$$\lim_{0\neq h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{0\neq h\to 0} h = 0.$$

By the mean value theorem it follows that f is constant. □

(5) Chapter 5, Problem 2

Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b) and let g be its inverse function. Prove that g is differentiable and

$$g'(f(x)) = \frac{1}{f'(x)} \ \forall\ x \in (a,b).$$

Proof. By the mean value theorem, if $y > x$ are two points in (a,b) then there exists $z \in (x,y)$ such that $f(y) - f(x) = (y-x)f'(z) > 0$. Thus f is stricly increasing. It follows that it is $1-1$ as a map onto the (possibly infinite) interval $(c,d) = (\inf f, \sup f)$. Thus it has an inverse, g determined by the fact that $g(y) = x$ if $f(x) = y$. □