18.100B, Fall 2002, Solutions to Homework 3

Rudin:

(1) Chapter 2, Problem 6

Done in class on Thursday September 26. Here $E \subset X$ is a subset of a metric space and E' is the set of limit points, in X, of E.

(a) Prove that E' is closed.

If $p \in X$ is a limit point of E' then for each $r > 0$, $B(p,r) \cap E' \ni q$ is not empty. Since q is a limit point of E and $r - d(p,q) > 0$, $B(q, r - d(p,q)) \cap E$ is an infinite set. By the triangle inequality, $B(q, r - d(p,q)) \subset B(p,r)$ so $B(p,r) \cap E$ is also infinite and p is therefore a limit point of E, i.e. $p \in E'$. Thus E' contains each of its limit points and it is therefore closed.

(b) Prove that E and $\overline{E}$ have the same limit points.

If p is a limit point of E then it is a limit point of $\overline{E}$ since $E \subset \overline{E}$. If p is a limit point of $\overline{E}$ then $B(p, \frac{1}{n}) \cap (E \setminus \{0\})$ decreases with n; either it is infinite for all n or it is empty for large n. We show that the second case cannot occur. Indeed this woould imply that $B(p, \frac{1}{n}) \cap (E' \setminus \{p\})$ is infinite for all n and hence that p is a limit point of E'; by the preceding result it is then a limit point of E contradicting the assumption that it is not. Thus a limit point of $\overline{E}$ is a limit point of E.

(c) Do E and E' have the same limit points?

No, not in general. A limit point of E' must be a limit point of E but the converse need not be true. For example consider $E = \{1/n \in \mathbb{R}; n \in \mathbb{N}\}$. This has a single limit point, 0 so $E' = \{0\}$ has no limit points at all.

(2) Chapter 2, Problem 8

(a) Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E?

Yes. If $E \subset \mathbb{R}^2$ is open then $B(p,r) \subset E$ for some $r' > 0$ and all $0 < r < r'$. Since $B(p,r) \subset \mathbb{R}^2$ is infinite if $r > 0$ it follows that p is a limit point of E.

(b) Same question for E closed?

No, not in general. For instance the set containing a single point $\{0\}$ is closed but has no limit points.

(3) Chapter 2, Problem 9.

Let E° denote all the interior points of $E \subset X$, meaning that $p \in E^\circ$ if $B(p,r) \subset E$ for some $r > 0$.

(a) Prove that E° is always open

If $p \in E^\circ$ then $B(p,r) \subset E$ for some $r > 0$ and if $q \in B(p,r)$ then, by the triangle inequality, $B(q, r - d(p,q)) \subset E$ so $B(p,r) \subset E^\circ$ and hence E° is open.

(b) Prove that E is open if and only if $E = E^\circ$.

Certainly if E is open then $E = E^\circ$ since for each $p \in E$ there exists $r > 0$ such that $B(p,r) \subset E$. Conversely if $E^\circ = E$ then this holds for each $p \in E$ so E is open.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

If $G \subset E$ is open then for each $p \in G$ there exists $r > 0$ such that $B(p,r) \subset G$, hence $B(p,r) \subset E$ so $p \in E^\circ$ and it follows that $G \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.
The complement $(E^\circ)^{\complement}$ consists of the points $p \in E$ such that $B(p,r) \cap E^{\complement} \neq \emptyset$ for all $r > 0$. Since $p \notin E^{\complement}$ this implies that p is a limit point of $E^{\complement}$ so

$$(E^\circ)^{\complement} \subset \overline{E^{\complement}}.$$

Conversely if $p \in \overline{E^{\complement}}$ then, by Problem 6 above, either $p \in E^{\complement}$ or $p \in (E^{\complement})'$ (or both). In the first case certainly $p \in (E^\circ)^{\complement}$, since $E^\circ \subset E$. So we may assume $p \in E$, i.e. $p \notin E^{\complement}$, and $p \in (E^{\complement})'$. Then for each $r > 0$ $B(p,r) \cap (E^{\complement}) \neq \emptyset$ (since p is a limit point not in the set) and this means $B(p,r)$ is NOT a subset of E for any $r > 0$, hence $p \notin E^\circ$. Thus $\overline{E^{\complement}} \subset (E^\circ)^{\complement}$ and these two sets are therefore equal.

(e) Do E and $\overline{E}$ have the same interiors?
Not necessarily. For instance $(0,1) \cup (1,2) \subset \mathbb{R}$ is open, so equal to its interior, but its closure is $[0,2]$ with interior $(0,2)$ which contains the extra point 1. It is always the case that $E^\circ \subset (\overline{E})^\circ$.

(f) Do E and E° have the same closures?
Again in general no. For example if $E = \{0\} \subset \mathbb{R}$ its interior is empty but it is closed and non-empty. Clearly the closure of E contains the closure of E°.

(4) Chapter 2, Problem 11.

(a) d_1 is not a metric since for the three points 0, $\frac{1}{2}$ and $\frac{1}{4}$

$$\frac{1}{4} = (0 - \frac{1}{2})^2 > (0 - \frac{1}{4})^2 + (\frac{1}{4} - \frac{1}{2})^2 = \frac{1}{8}.$$

(b) d_2 is a metric. It satisfies the first two axioms trivially. To see the triangle inequality first note that

$$|x - y| \leq |x - z| + |z - y|$$

for any three real numbers. Taking square-roots of both sides (using the montonicity of $\sqrt{}$) we find

$$\begin{aligned} d_2(x,y) = \sqrt{|x-y|} &\leq \sqrt{|x-z| + |z-y|} \\ &= \sqrt{(d_2(x,z))^2 + (d_2(z,y)^2} \leq d_2(x,z) + d_2(z,y) \end{aligned}$$

by the usual triangle inequality.

(c) d_3 is not a metric since $d_3(x, -x) = 0$ for all x.

(d) d_4 is not a metric since $d_4(1,2) \neq d_4(2,1)$.

(e) d_5 is a metric. Certainly it is symmetric and $d_5(x,y) = 0$ implies $|x - y| = 0$ and hence $x = y$. To get the triangle inequality we need to find the sign of

$$\frac{|x-z|}{1+|x-z|} + \frac{|y-z|}{1+|y-z|} - \frac{|x-y|}{1+|x-y|}.$$

Multplying by the product of the demoninators (which are all strictly positive) this is the same as the sign of

$$\begin{aligned}(1+|x-y|)(1+|y-z|)|x-z| + (1+|x-y|)(1+|x-z|)|y-z| \\ -(1+|x-z|)(1+|y-z|)|x-y| \\ = |x-y||y-z||x-z| + 2|y-z||x-z| + (|x-z|+|y-z|-|x-y|)\end{aligned}$$

All three terms here are non-negative, the last being the triangle inequality. Thus d_5 does also satisfy the triangle inequality.
Remark: If $d(x,y)$ is a metric then so is

$$\frac{d(x,y)}{1+d(x,y}.$$

The proof in general essentially the same.