18.100B, Fall 2002, Homework 10

Due in 2-251, by Noon, Thursday December 5
Rudin:

(1) Chapter 7, Problem 14

Solution. There is a function f as described, just set

$$f(t) = \begin{cases} 0 &) \le t \le \frac{1}{3} \\ 3(t - \frac{1}{3}) & \frac{1}{3} \le t \le \frac{2}{3} \\ 1 & \frac{2}{3} \le t \le 1 \end{cases}$$

and for instance $f(2-t) = f(t)$ for $1 \le t \le 2$ and then $f(2k+t) = f(t)$ for all $k \in \mathbb{N}$, $k \neq 0$, $t \in [0,2]$. This gives a continuous function. Consider

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t),\ y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t). \tag{1}$$

The nth term in the series for (1) is bounded

$$|2^{-n} f(3^{2n-1}t)| \le 2^{-n}.$$

By Theorem 7.10, the series converges uniformly. Thus $x(t)$ is continuous by Theorem 7.12. The same argument applies to $y(t)$ so $\Phi(t) = (x(t), y(t))$ is continuous by Theorem 4.10. Now, to see that Φ is surjective, follow the hint. Each point in $[0,1]$ has a dyadic decomposition

$$x = \sum_{n=1}^{\infty} 2^{-n} b_n,\ b_n = 0 \text{ or } 1.$$

Indeed one can compute the successive b_n, $n = 1, \dots, k$ to arrange that $0 \le x - \sum_{n=1}^{k} 2^{-n} b_n \le 2^{-k}$. Then choose $b_{k+1} = 0$ or 1 depending on whether $0 \le x - \sum_{n=1}^{k} 2^{-n} b_n < 2^{-k-1}$ or $2^{-k-1} \le x - \sum_{n=1}^{k} 2^{-n} b_n \le 2^{-k}$. Now we let the a_{2n-1} be these numbers for x_0 and a_{2n} those for y_0.

It follows that $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$ converges to a number in $[0,1]$ and that $3^k t_0 = 2N + \frac{2}{3}a_k + t_k$ where N is an integer, $a_k = 0$ or 1 and $t_k \le \frac{1}{3}$. The properties of f show that $f(3^k t_0) = f(\frac{2}{3}a_k + t_k) = a_k$ since as $a_k = 0$ or 1 then $\frac{2}{3}a_k + t_k \in [0, \frac{1}{3}]$ of $[\frac{2}{3}, 1]$. From the definition of $x(t)$ and $y(t)$ it follows that $\Phi(t_0) = (x_0, y_0)$ and hence Φ is surjective.

These points t_0 are the ones appearing in the Cantor set as defined in section 2.44. □

(2) Chapter 7, Problem 18

Solution. If f_n is a uniformly bounded sequence of functions on $[a,b]$ there exists a constant M such that $|f_n(x)| \le M$ for all $x \in [a,b]$ and all n. Since the f_n are Riemann integrable the functions

$$F_n(x) = \int_a^x f_n(t)dt.$$

are continuous. We show that the sequence F_n is equicontinuous. In fact if $x \le x' \in [a,b]$ then

$$|F_n(x) - F_n(x')| \le \int_x^{x'} |f_n(t)|dt \le M|x - x'|.$$

Thus, if $|x-x'| < \delta = \epsilon/M$ then $|F_n(x) - F_n(x')| < \epsilon$, showing the equicontinuity. We also have a uniform bound $|F_n(x)| \le (b-a)M$. As a sequence of uniformly bounded and equicontinuous functions on a compact metric space we see from Theorem 7.25 that $\{F_n\}$ has a uniformly convergent subsequence. $\square$

(3) Chapter 7, Problem 24

Solution. We define a map $X \longrightarrow \mathcal{C}(X)$ into the space of bounded continuous functions on X. Fixing a point $a \in X$ let

$$f : X \ni p \longmapsto f_p,\ \ f_p(x) = d(x,p) - d(x,a).$$

By the triangle inequality $d(x,p) \le d(x,a) + d(a,p)$ and $d(x,a) \le d(x,p) + d(a,p)$ shows that $|f_p(x)| \le d(a,p)$. Thus f_p is a bounded function. It is continuous, again by the triangle inequality

$$|f_p(x) - f_p(y)| \le |d(x,p) - d(y,p)| + |d(x,a) - d(y,a)| \le 2d(x,y).$$

Thus $f_p \in \mathcal{C}(X)$. Now, consider

$$\begin{aligned}\|f_p(x) - f_q(x)\| &= \sup_{x\in X} |d(x,p) - d(x,a) - d(x,q) + d(x,a)| \\ &= \sup_{x\in X} |d(x,p) - d(x,q)| = d(p,q).\end{aligned}$$

The last inequality follows from the triangle inequality and the fact that $f_p(q) - f_q(q) = d(p,q)$.

It follows that f is a continuous map. It is an isometry, namely the distance in $\mathcal{C}(X)$ is $\|f - g\|$. In particular this shows that f is 1-1. Let Y be the closure of $f(X) \subset \mathcal{C}(X)$. Then we may regard $X \subset Y$ using f. As a closed subset of $\mathcal{C}(X)$, Y is complete as a metric space. Thus X is isometric to a subset of the complete space Y in which it is dense. $\square$

(4) Chapter 5, Problem 26

Solution. We assume that f is differentiable on $[a,b]$, $f(a) = 0$ and there is a real number A such that $|f'(x)| \le A|f(x)|$ for all $x \in [a,b]$. Following the hint, for a chosen $x_0 \in (a,b]$, set $M_0 = \sup|f(x)|$ and $M_1 = \sup|f'(x)|$ with the suprema over $[a,x_0]$; where the second exists because of the assumption. It follows that $M_1 \le AM_0$. By the mean value theorem $f(x) - f(a) = f'(y)(x-a)$ for some $y \in (a,x_0)$ so

$$|f(x) - f(a)| = |f(x)| = |f'(y)||x-a| \le M_1(x_a) \le AM_0(x_0 - a).$$

Taking the sup over $x \in [a,x_0]$ we see that $M_0 \le AM_0(x_0 - a)$. Taking $x_0 - a$ so small that $A(x_0 - a) < 1$ we see $M_0 \le 0$ and hence $M_0 = 0$. Thus $f(x) = 0$ on $[a,x_0]$. Set $z = \sup\{x_0; f(x) = 0 \text{ on } [a,x_0]\}$. If $z = b$ we are finished, since f is continuous so $f(z) = 0$. If $z < b$ then we may apply the argument again to find a contradiction. $\square$

(5) Chapter 5, Problem 27

Solution. I did this in class in a slightly different way, so I am just asking you to write down the proof. Namely if $f_i(x)$ are two solutions, for $i = 1, 2$, then we may integrate the equation to see that

$$f_i(x) = c + \int_a^x \phi(t, f_i(t))dt.$$

It follows that the difference $f(x) = f_2(x) - f_1(x)$ satifies

$$f(a) = 0, \;\; f'(x) = \phi(x, f_1(x)) - \phi(x, f_2(x).$$

We see that $|f'(x)| \leq A|f(x)|$ where A is the constant in the Lipschitz condition. Applying the previous problem, we conclude that $f = 0$.

Differentiating $y = \frac{1}{4}x^2$, $y' = \frac{1}{2}x = y^{\frac{1}{2}}$ shows that it is a solution in $[0, 1]$. Thus both $y \equiv 0$ and $y = \frac{1}{4}x^2$ are both solutions.

These are not the only solutions, since if $x_0 > 0$ and we define

$$y = \begin{cases} 0 & x < x_0 \\ \frac{1}{4}(x - x_0)^2 & x \geq x_0 \end{cases} \tag{2}$$

we get a continuously differentiable solution by the same argument. Conversely every solution, $y \geq 0$, of $y' = y^{\frac{1}{2}}$ is of this form. Indeed, if $y(t) > 0$ for some $t \in (0, 1)$ then, being continuous, it is positive nearby. Thus we can divide by $y^{\frac{1}{2}}$ and conclude that $\frac{d}{dx}\left(2y^{-\frac{1}{2}}\right) = 1$ on any interval where $y > 0$. This implies that $2y^{-\frac{1}{2}} = x - x_0$ for some constant x_0 and hence that (2) holds in any interval where $y > 0$. In principle there could be different values of x_0 on different intervals. However, y in (2) is increasing, so the set on which it is strictly positive must be of the form $(x_0, 1]$ for some $x_0 \in [0, 1]$, since $y(0) = 0$. Thus the general solution is (2) for some $x_0 \in [0, 1]$.

PS. I don't have the book with me, perhaps the upper limit of the interval here is ∞, but the argument is the same. $\square$