TOPIC 1

RICHARD MELROSE

Abstract.

Overview, one and two-dimensional manifolds. Basics of Riemann surfaces

0.1. **Overview.** In this course I want to use the Riemann Moduli spaces as a central theme to introduce many of the methods of use in differential geometry, differential analysis, differential topology and so on; I do expect you to be able to differentiate!

So, what is the Riemann moduli space of a Riemann surface S with n 'marked points' forming a set $P \subset S$:

$$\begin{aligned} \mathcal{M}(S) = \mathcal{M}_g &= \mathcal{J}(S)/\operatorname{Dff}^+(S) \text{ (unmarked)}, \\ \mathcal{M}(S,P) = \mathcal{M}_{g,n} &= \mathcal{J}(S)/\operatorname{Dff}^+(S;P),\ \#(P) = n, \\ \operatorname{Dff}^+(S;P) &= \{F \in \operatorname{Dff}^+(S); F(p) = p\ \forall\ p \in P\}. \end{aligned} \tag{1}$$

To understand this definition we first have to comes to grips with S. First S is a compact two dimensional *oriented manifold*. The notation 'g' is for the *genus* of S, $g \in \mathbb{N}_0$ which actually determines S up to diffeomorphism. The diffeomorphisms of a compact manifold form an infinte-dimensional group $\operatorname{Dff}(S)$). The labelling of oriented surfaces by the genus is a form of the uniformization theorem. Then $\mathcal{J}(S)$ is the (infinite-dimensional manifold) of complex structures on S, $\operatorname{Dff}^+(S) \subset \operatorname{Dff}(S)$ is the subgroup of index 2 of orientation-preserving diffeomorphisms. Further restricting, $\operatorname{Dff}^+(S;P) \subset \operatorname{Dff}^+(S)$ is the subgroup of diffeomorphisms which fix each point of P. Not totally surprisingly, this has real codimension $2n$ in $\operatorname{Dff}^+(S)$ where $n = \#(P)$ is the number of points in P, although even the precise meaning of this is for the moment unclear.

So, what is $\mathcal{M}_{g,n}$ like? Well, it only makes good sense in the 'stable' range $2g + n > 2$ which is actually a condition on the *Euler characteristic* of S – in real money this means $n \geq 3$ if $g = 0$ (the sphere) and $n \geq 1$ if $g = 1$ (the torus) for the surfaces with $g \geq 2$ there is no restriction. Then for instance $\mathcal{M}_{0,3}$ is a point and $\mathcal{M}_{0,4}$ is the 2-sphere with three points removed. The space $\mathcal{M}_{1,1}$ is what you would normally think of as the 'moduli space'. As we shall see $\mathcal{M}_{g,n}$ is 'almost' a complex manifold of (complex) dimension $3g - 3 + n$ where the almost is because it is an *orbifold* – no big deal but we need to sort it out – and a *stack*, into which I will not go so deeply but you will see what this means too. Basically it is smooth; however it is non-compact, except for $\mathcal{M}_{0,3}$, and that is something we really do need to get into – namely we need to discuss its compactification $\overline{\mathcal{M}}_{g,n}$. (This is standard notation for the Delign-Mumford compactification, although the 'bar' was not a very good choice since it can get confused with complex conjugation to which it is not related.)

The space $M_{g,n}$ is the (coarse) moduli space of complex structures on an n-pointed Riemann surface of genus g.

We will have a lot more to say about these spaces and the concepts here that I have left undefined. First we need to get to the point that everyone can honestly say they understand everything in the definition. We are probably not there yet.

0.2. **Brief version of first lecture.** Let S be a compact oriented 2-manifold S. A conformal structure on S is given by a metric-up-to-conformal-equivalence. That is, a conformal structure is just a class of metrics of the form $e^{2\phi}g$ where g is some fixed metric and $\phi \in \mathcal{C}^\infty(S;\mathbb{R})$ is an arbitrary; thus $e^{2\phi}$ is actually an arbitrary strictly positive smooth function on S. The reason such a conformal structure is interesting in this case is it is the same as an almost complex structure and hence the same thing as a complex structure. Initially we just see that it is the same as an *almost complex structure*. I will continue to distinguish between complex and almost complex structures until we see that they are the same.

Definition 1. An almost complex structure is a smooth real bundle map

$$j : TS \longrightarrow TS \tag{2}$$

(so mapping T_sS to T_sS linearly for each $s \in S$ with smooth dependence on s) such that $j^2 = -\operatorname{Id}$.

So I am assuming you know about the tangent bundle and bundle maps – see the expanded discussion below.

Such a map is a bundle isomorphism since $-j = j^{-1}$ and it turns each T_sS into a complex vector space – we already have linearity over $\mathbb{R}$ and if you define 'fibre multiplication by $i = \sqrt{-1}$' to be j_s then this gives the axioms of a 1-D complex vector space, a complex line if you prefer. In 2 dimensions there is no integrability condition (as there is in higher dimensions) and giving an almost complex structure is equivalent to giving a complex structure, in the sense of a covering by complex coordinate systems. It is still necessary to show this, to see that there are automatically local complex coordinates (and that these are compatible on overlaps). Namely, j induces a dual isomorphism of T^*S so this is complex too and then we need to find real coordinates x and y such that

$$j(dx) = dy,\ j(dy) = -dx \text{ everywhere in the coordinate patch.} \tag{3}$$

This is not completely obvious! It is precisely what we need to prove to identify almost and 'actual' complex structures.

The relationship between almost complex structures and conformal stuctures stated above is obtained by noting that the choice of a metric induces an action of SO(2) on T_sS for each s – these are the orientation-preserving linear maps which fix the metric. If we take linear coordinates in T_sS in terms of which the metric is the standard Euclidean metric (which we always can) the elements of SO(2) act as (counterclockwise) rotations, $R(\theta)$, $\theta \in [0, 2\pi)$. Then set

$$J_s = R(\pi/2) \text{ on } T_sS \tag{4}$$

gives an almost complex structure.

Conversely if we have an almost complex structure and we take any local non-vanishing real vector field v – a then v, jv determine a metric if we require them to be orthonormal. You should check that a different choice of v leads to a conformal metric. It follows that the conformal class of the metric can be recovered from the (almost) complex structure. If you prefer, given j we can define a conformal class of metrics by locally taking any function y_1 which has non-vanishing differential and defining a metric by specifying dy_1, $j(dy_1)$ as an orthonormal basis of T^*_sS, locally.

So conformal classes of metrics and almost complex structures are one in the same – and $\mathcal{M}_g$ is supposed to be the 'moduli space' of these.

It is therefore important to think about what the collection, $\mathcal{J}(S)$, of *all* the almost complex structures on S is like. To approach this we do a little linear algebra and think of the almost complex structure in a slightly different way. The linear space T_sS is 2-dimensional over the reals and we can pass to it complexification, $\mathbb{C}T_sS = \mathbb{C} \otimes_{\mathbb{R}} T_sS$ just by allowing complex coefficients. This is a complex 2-dimensional vector space. The reason for doing this is that j_s at each point is real but its eigenvalues are necessarily complex, since of course they must be $\pm i$ – so it is necessary to extend to the complexification to 'see' these. Each of the eigenvalues must correspond to a 1-dimensional eigenspace – of necessarily complex vectors, so the complexification splits at each point, and in fact smoothly, into the $\pm i$ eigenspaces:

$$\mathbb{C}TS = V \oplus \overline{V},\ V = T^{1,0}S,\ \mathbb{C}T^*S = V^* \oplus \overline{V^*} = \Lambda^{1,0} \oplus \Lambda^{0,1}. \tag{5}$$

In fact TS as a complex (line) bundle is naturally isomorphic to $T^{1,0}S$.

Once we choose an almost complex structure and perform the splitting (5) we can see what the others are like – pointwise and then globally over S. Namely, each j tensor gives a splitting as in (5) into two complex lines which are conjugate to each other (note that the complex conjugate is also a complex vector space of dimension one the 'bar' refers to the fact that it is naturally isomorphic to V over the real numbers but by an antilinear isomorphism)

$$\mathbb{C}T_sS = W_s \oplus \overline{W_s},\ W_s \cap \overline{W_s} = \{0\}. \tag{6}$$

This also gives an orientation of T_sS which should be the correct one – which means we cannot exchange W_s and $\overline{W_s}$ since that would reverse the orientation. So, using the fixed almost complex struture (5), any other almost complex structure gives a linear map by applying the decomposition

$$\gamma : \overline{V_s} \longrightarrow V_s,\ \overline{v} + \gamma(\overline{v}) \in \overline{W_s},\ \forall\ \overline{v} \in \overline{V}_s. \tag{7}$$

Thus the original structure corresponds to $\gamma = 0$. Observe that for a 1-dimensional complex vector space the linear maps as in (7) have a norm. Taking a basis element $\overline{e}$ of $\overline{V}_s$ it follows that e is a basis element of V_s and the map is given by a complex number, $\gamma(\overline{e}) = \lambda e$; change of basis just corresponds to scaling and it follows that $|\lambda|$ (but not λ itself) is independent of the choice of $\overline{e}$.

Lemma 1. *Given one almost complex structure on S corresponding to the decomposition* (5)*, the space of almost complex structures on a S is identified with the smooth bundle maps*

$$\mathcal{J}(S) = \{\gamma : T^{0,1}S \longrightarrow T^{1,0}S; |\gamma|_s < 1\ \forall\ s \in S\}. \tag{8}$$

So this tells us what $\mathcal{J}(S)$ is – it is an open set in the Fréchet space of bundle maps as in (8) and hence in particular a Fréchet manifold, but of a very simple type.

Finally we come to our first theorem. Namely

Theorem 1 (Isothermal coordinates)**.** [1] *For any almost complex structure on S there exist local coordinate near each point, x and y, in terms of which $\Lambda^{1,0}$ is spanned by $dx + idy = dz$, $z = x + iy$.*

Why 'isothermal'? Better ask Gauss, but it is a bit late.

Proof. Here is a proof using a black-box (maybe for real if I work out how to do it in TeX). By this I mean we use a major result which you might or might not know. As here, I will typically state this in a more general form than needed. At some point this needs to be discussed properly, but of course I am claiming that the black box does not depend on the result we are using it to prove. I could try to proceed strictly linearly but it is unwise to do so.

As a more general aside, the theory of Riemann surfaces was developed before and, later, to some extent separately from the general theory of manifolds and differential analysis – using special methods (in this case, Beltrami differentials) that do not extend easily to higher dimensions. The existence of isothermal coordinates is an example of this and there are direct proofs which apply to complex vector fields in two, but not higher, dimensions. I eschew such proofs here! One point of the course after all is to introduce general results and methods!

If we take a given almost complex structure on S (this does not work in higher dimensions but see 'Newlander-Nirenberg' maybe later) then locally near a given point $\bar{s} \in S$, there is a smooth, non-vanishing section of $T^{0,1}S$, call it V. This is a complex vector field on S and what we look for is a solution to the differential equation

$$Vz = 0,\ dz(s) \neq 0\ z \in \mathcal{C}^\infty(O),\ \bar{s} \in O \subset S \text{ open.} \tag{9}$$

The black box, in a form that I do plan to discuss later, is:-

Theorem 2. *Any (elliptic) Dirac operator (so of order* 1*),* $\eth$*, (between sections of vector bundles) on a manifold is locally solvable in the sense that each point has a neighbourhood O such that if f is smooth and has support in some compact subset $K \Subset O$ then there is a smooth solution to*

$$\eth u = f \text{ in } O \text{ satisfying } \|u\|_{\mathcal{C}^{k+1,\alpha}(O)} \le C\|f\|_{\mathcal{C}^{k,\alpha}(O)}. \tag{10}$$

I will not say much about this at the moment – these are 'Schauder estimates' and it can be proved by using Fredholm propeties of global elliptic operators. I do plan to include this in these notes, but just how much I will do in the lectures is the sort of thing that I am seeking guidance for. In any case, not just yet!

In particular I have suppressed the notation for vector bundles etc in (10). The integers k need to be chosen as does the Hölder exponent $0 < \alpha < 1$, (because even when O and K are fixed the constants depend on them) then there is an open set O and then once $K \Subset O$ is fixed there is a solution and a constant such that (10) holds. I am not even going to bother explaining what these Hölder norms are at the moment, but the $\mathcal{C}^{k,\alpha}$ norm controls (is bigger than) the usual $\mathcal{C}^k$ norm (supremum of first k derivatives) and conversely is controlled by the $\mathcal{C}^{k+1}$ norm.

[1] Understood by Gauss for surfaces in three space, proved by Korn and Lichenstein for almost complex structure in two real dimensions (reference needed). The proof I outline here is due to Deturck and Kazdan (reference needed).

Since we are thinking of an operator – a vector field – on functions here everything is much simpler than the general theorem. All we really need for the moment is the existence and the (somewhat misleading) estimate for a solution

$$\|u\|_{\mathcal{C}^1(O)} \le C\|f\|_{\mathcal{C}^1(O)}. \tag{11}$$

Of course applying the theorem requires us, apart from having some blind faith, to check that it is 'elliptic' and 'Dirac' – neight of which is not yet defined. It is! A vector field cannot be elliptic in dimensions greater than two, and ellipticity in dimension two amounts to the real and imaginary parts being linearly independent.

To use this black box we go back to the equation (9) that we want to solve and work near some point. Take local coordinates y_1, y_2 based at the point and such that the conformal structure corresponds to a metric $dy_1^2 + dy_2^2 + y_1 g_1 + y_2 g_2$ as we can arrange by a linear change of coordinates. Then a smooth section V of $T^{0,1}$ can be written

$$V = a(y_1, y_2)(\partial_{y_1} + i\partial_{y_2}) + y_1 V_1 + y_2 V_2,\ a \ne 0,\ V_i \text{ smooth}. \tag{12}$$

Since we only want to solve $Vz = 0$ we can divide by a – changing the error terms. Now, we proceed in 'formal power series'. Try to choose

$$Z_k = y_1 + iy_2 + \sum_{0<|\alpha|\le k+1,} b_\alpha y^\alpha \text{ s.t. } VZ_k = O(|y|^{k+2}). \tag{13}$$

Here I am using multiindex notation for polynomials, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $|\alpha| = \alpha_1 + \alpha_2$, $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2}$,and we want the error to vanish to order $k+1$ at the origin – our chosen point. For $k = 0$ we have this already. To proceed inductively, suppose we have found a Z_j such that

$$VZ_j = \sum_{|\beta|=j+1} c_\alpha y^\beta + O(|y|^{j+2}) \tag{14}$$

where the leading term in the Taylor series has been made explicit. In fact we can introduce the complex variable $\zeta = y_1 + iy_2$ and write the leading part of the error as

$$\sum_{0\le l\le j+1} d_l \zeta^l \overline{\zeta}^{j+1-l}$$

for some other constants. Then we want to add a homogeneous polynomial of degree $j+2$ to Z_j to give Z_{j+1}. Since

$$V(\zeta^j \overline{\zeta}^{j+2-l}) = 2(j+2-l)\zeta^j \overline{\zeta}^{j+2-l} + O(|y|^{j+3}) \tag{15}$$

it suffices to take

$$Z_{j+1} = Z_j - \sum_{0\le l\le j+1} d_l \frac{1}{2(j+2-l)} \zeta^l \overline{\zeta}^{j+2-l}. \tag{16}$$

We really only need (13) with $k = 1$. If O' is the coordinate neighbourhood, choose $\phi \in \mathcal{C}_c^\infty(O')$ (undefined notation should always appear in the expanded version of the lecture – this is the space of smooth functions each of which vanishes outside a compact subset of O') so

$$f = \phi V Z_2 = \sum_{|\alpha|=2} y^\alpha \phi \psi_\alpha(y_1, y_2),\ \psi_\alpha \in \mathcal{C}^\infty(O'). \tag{17}$$

We will choose a sequence ϕ_n of 'cut-off' functions. Take $\chi \in \mathcal{C}^\infty_{\text{c}}(O')$, $0 \le \chi \le 1$, $\chi = 1$ in $|y| < \epsilon$ for some $\epsilon > 0$, $\chi = 0$ in $|y| > 2\epsilon$ and set

$$\phi_n(y_1, y_2) = \chi(ny_1, ny_2),\ f_n = \phi_n V Z_2 = \sum_{|\alpha|=2} y^\alpha \phi_n \psi_\alpha \tag{18}$$

The $\mathcal{C}^1$ norm of f_n is the sum of the three terms

$$\sup|f_n| \le Cn^{-2},\ \sup|\partial_{y_1} f_n| \le Cn^{-1},\ \sup|\partial_{y_y} f_n| \le Cn^{-1} \tag{19}$$

where the vanishing as n increases comes from the fact that the support is in $|y| < \epsilon/n$ and there is at least one factor of $|y|$ 'left over'. Of course you could get faster vanishing by increasing j in the first place.

Now we apply Theorem 2 and the estimate in the form (11) on the solution to find $u_n \in \mathcal{C}^\infty(O)$ (where $O \subset O'$ may be a smaller neighbourhood) such that

$$Vu_n = f_n,\ \|u_n\|_{\mathcal{C}^1(O)} \le C\|f_n\|_{\mathcal{C}^1(O)} \le Cn^{-1}. \tag{20}$$

Then set $z = Z_2 - u_n$ where n is choses so large that $\|u_n\|_{\mathcal{C}^1} \le \frac{1}{2}$. The derivatives of Z_2 at the origin of the coordinates are of size 1 so it follows that

$$Vz = 0 \text{ in } |y| < \delta < \epsilon/n,\ |\partial_{y_i} z(0)| \ne 0,\ i = 1, 2. \tag{21}$$

Thus we win. □

This allows us to introduce complex coordinates $x = \text{Re}(z)$, $y = \text{Im}(z)$ – these are coordinates by the inverse function theorm – and $z = x + iy$. It follows in turn that

$$V = a(\partial_x + i\partial_y),\ a \ne 0 \text{ near } 0 \tag{22}$$

where we can again drop the factor of a. Now the only functions in the null space of V near 0 are the holomorphic functions of z. Maybe this requires a proof.

0.3. **Riemann surfaces and manifolds.** In the second part of each lecture (really 'topic' since I do not anticipate these corresponding to actual lectures) I/we will add some more detail and context to the briefer version, as above.

So, to the basic setup for Riemann surfaces. The 'collection' of two-dimensional compact manifolds is still quite manageable. Namely for each non-negative integer $g \in \mathbb{N}_0$ there is just one oriented compact surface (=2D manifold) up to (orientation-preserving) diffeomorphism. There are lots of ways to prove this – but I will not try to give an elementary 'cut and paste' proof but later use some more general machinery to prove it. Lets explore some of the (remarkable) things that are special to the two-dimensional case and proceed a bit later to examples of each 'genus' once we have defined it and then approach the uniqueness problem.

Perhaps we should have the explicit definition of a manifold here.

From the definition of a manifold (which of course arose from trying to treat Riemann surfaces systematically but that is a long time ago) we know we can cover S by coordinate charts. I will typically use variables y_1 and y_2, but x and y would be more conventional. In fact there is a finite cover by such coordinate patches

$$F : S \supset U \longrightarrow U' \subset \mathbb{R}^2_{y_1,y_2}. \tag{23}$$

The definition means that S comes equipped with a topology, in which U is an open set, $U' \subset \mathbb{R}^2$ is required to be open and F to be a homeomorphism. The smooth structure on S – part of the definition of a ($\mathcal{C}^\infty$) manifold is that for two such coordinate patches F_i, $i = 1, 2$, such that $U_1 \cap U_2 \neq \emptyset$ the map

$$F_2 F_1^{-1} : F_1(U_1 \cap U_2) \longrightarrow F_2(U_1 \cap U_2) \text{ is a diffeomorphism} \tag{24}$$

which just means that the corresponing pull-back map $(F_2 F_1^{-1})^*$ from functions on $F_2(U_1 \cap U_2)$ to functions on $F_1(U_1 \cap U_2)$ is a bijection from $\mathcal{C}^\infty(F_2(U_1 \cap U_2))$ to $\mathcal{C}^\infty(F_1(U_1 \cap U_2))$.

This allows us, as on any manifold, to define

$$\begin{gathered}\mathcal{C}^\infty(S) = \\ \{u : S \longrightarrow \mathbb{C}; (F^{-1})^*(u\big|_U) \in \mathcal{C}^\infty(U'); \text{ for all coordinate charts } F : U \longrightarrow U'\}.\end{gathered} \tag{25}$$

To show that a function $f : S \longrightarrow \mathbb{C}$ is in $\mathcal{C}^\infty(S)$ it suffices to check the condition in (25) for a cover of S by coordinate charts.

The space $\mathcal{C}^\infty(S)$ is a ring and is actually a '$\mathcal{C}^\infty$ algebra' – meaning if you take k real-valued elements and insert them into a smooth function on $\mathbb{R}^k$ you get another element). The real and imaginary parts of a function are smooth so $\mathcal{C}^\infty(S; \mathbb{R})$ is a vector space over the reals such that

$$\mathcal{C}^\infty(S) = \mathcal{C}^\infty(S; \mathbb{R}) \otimes \mathbb{C}. \tag{26}$$

A map between manifolds $F : S_1 \longrightarrow S_2$ is smooth if and only if

$$F^* : \mathcal{C}^\infty(S_2) \longrightarrow \mathcal{C}^\infty(S_1). \tag{27}$$

The space $\mathcal{C}^\infty(S)$ is the space of global sections of a *sheaf.* Namely, any open set $O \subset S$ is itself a smooth 2-dimensional manifold and so $\mathcal{C}^\infty(O)$ is well-defined. There is a restriction map

$$\big|_{OO'} : \mathcal{C}^\infty(O') \longrightarrow \mathcal{C}^\infty(O),\ O \subset O' \subset S \text{ open} \tag{28}$$

which is 'functorial' under composition (so the identity when $O = O'$ and composes properly for triples $O'' \subset O' \subset O$). That makes 'it' (meaning the functor $O \longmapsto \mathcal{C}^\infty(O)$ of open sets into abelian rings) a *presheaf.* It is a sheaf because one can assemble elements. If $u \in \mathcal{C}^\infty(O)$ and $u' \in \mathcal{C}^\infty(O')$ are such that

$$\begin{gathered} u\big|_{O\cap O'} = u'\big|_{O\cap O'} \\ \text{then there exists a unique } v \in \mathcal{C}^\infty(O \cup O') \text{ with} \\ v\big|_O = u,\ v\big|_{O'} = u'. \end{gathered} \tag{29}$$

The support, $\operatorname{supp}(u)$, of $u \in \mathcal{C}^\infty(S)$ is associated to this structure – it is the largest closed set (which exists as a consequence of the conditions above) on the complement of which the function vanishes. In this case the support is also the closure of the set on which the function is non-zero, but better use the first definition in general). All this is fine on any manifold.

Basic real analysis allows us to construct *partitions of unity* on any manifold (not necessarily compact) S. If O_a, $a \in A$, is any covering of S by open sets then there is a partition of unity subordinate to it. Namely there is a countable collection of functions $\chi_i \in \mathcal{C}^\infty(S)$, $i \in N$, a map $\alpha : N \longrightarrow A$ such that $\operatorname{supp}(u_i) \Subset O_{\alpha(i)}$ (is a compact subset of) and additionally

$$K \Subset S \Longrightarrow \{i \in I; \operatorname{supp}(u)i) \cap K \neq \emptyset\} \text{ is finite}$$
$$\sum_i \chi_i(x) = 1 \ \forall \ x \in S. \tag{30}$$

The first condition implies that the sum in the second is finite over any compact set.

Now, $\mathcal{C}^\infty(S)$ is actually a topological vector space, in fact it is a Fréchet space (on any manifold) and a Montel space (on any compact manifold). The topology is given by a collection of seminorms, which in the compact case can be combined into norms, given by the coordinate charts

$$u \longmapsto \|(F^{-1})^* \chi u\|_{\mathcal{C}^k}, \ \chi \in \mathcal{C}^\infty(S). \operatorname{supp}(u) \subset U \tag{31}$$

Although there are a lot of these '$\mathcal{C}^k$ norms' the countable collection given by a partition of unity subordinate to a coordinate cover suffices to give the topology.

So, once we have $\mathcal{C}^\infty(S)$ – for any manifold – we can define the 'usual objects'. Let's be quasi-algebraic-geometers and think of points in S as determining ideals

$$\mathcal{I}_x = \{u \in \mathcal{C}^\infty(S); u(x) = 0\}, \ \mathcal{C}^\infty(S)/\mathcal{I}_x = \mathbb{C}, \ u \longrightarrow u(x) \tag{32}$$

so the ideals also determine the points. Now let $\mathcal{I}_x^2$ be the ideal

$$\mathcal{I}_x = \left\{ u \in \mathcal{C}^\infty(S); u = \sum_{\text{finite}} u_i v_i, \ u_i, \ v_i \in \mathcal{I}_x \right\} \tag{33}$$

the finite span of products of elements. These are the functions vanishing to second order (or is it first order??) at x. The quotient is by definition

$$T_x^* S = \mathcal{I}_x / \mathcal{I}_x^2 \text{ the cotangent fibre at } x. \tag{34}$$

Taylor's theorem and a little manipulation tells us it is a vector space of the same dimension as S – in this case 2. Moreover, there is a (surjective) linear map

$$d : \mathcal{C}^\infty(O) \ni u \longmapsto [u - u(0)] \in T_x^* S, \ x \in O \text{ open} \tag{35}$$

which is the first manifestation of the deRham differential. Coordinates give a basis for $T_x^* S$ for any $x \in U$, namely dy_1 and dy_2 form a basis.

Using this

$$T^* S = \bigcup_{x \in S} T_x^* S \text{ becomes a manifold of dimension } 2 \dim S = 4. \tag{36}$$

Namely the coordinate charts are $U \times \mathbb{R}^n$ where $F : U \longmapsto U'$ and the coordinates are the maps

$$(x, d(\xi \cdot y)) \longmapsto (F(x), (F^{-1})^* (\xi \cdot dy) \in U' \times \mathbb{R}^n, \ \xi \cdot y = \sum_i \xi_i y_i \tag{37}$$

and the 'Jacobian matrix' $(F^{-1})^*$ is just the induced map on ideals coming from the pull-back map on functions.

I have defined the cotangent bundle first, but the tangent bundle is just as significant. It is most directly defined pointwise as the space of derivations on $\mathcal{C}^\infty(S)$:

$$T_x S = \{v : \mathcal{C}^\infty(S; \mathbb{R}) \longrightarrow \mathbb{R} \text{ linear and such that}$$
$$v(\phi\psi) = \phi(x) v(\psi) + (v(\phi))\psi(x) \ \forall \ \phi, \ \psi \in \mathcal{C}^\infty(S; \mathbb{R})\}. \tag{38}$$

Then it follows that $v\big|_{\mathcal{I}^2} = 0$ so there is a well-defined pairing

$$T_x^*S \times T_xS \longrightarrow \mathbb{R} \tag{39}$$

which is 'perfect', i.e. identifies each as the dual of the other. Similar arguments as above show that

$$TS = \bigcup_{x\in S} T_xS \tag{40}$$

is also a smooth manifold of dimension $2\dim S$.

Both these manifolds are *vector bundles*. A real (complex) vector bundle over a manifold S is another manifold V with a surjective smooth map

$$\pi : V \longrightarrow S \tag{41}$$

such that the preimage $V_x = \pi^{-1}(x) \subset V$ has the structure of a real (complex) vector space of fixed dimension (maybe only constant over components of S, usage varies) and such that each $x \in S$ has an neighbourhood O over which V is 'trivial' in the sense that there is a fibre-linear diffeomorphism τ giving a commutative diagramme with the projection

$$\begin{array}{ccc} \pi^{-1}(O) & \xrightarrow{\ \tau\ } & O \times \mathbb{R}^m(\mathbb{C}^m) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_O} & \\ & O & \end{array} \tag{42}$$

0.4. **Background.** Note that this can always be brought into the foreground!

- Manifolds recalled. Regularity sheaves on a model space. Our models are $\mathbb{R}^n$, or more generally $\mathbb{R}^n_+ = [0,\infty)^n$, and $\mathbb{C}^n$ – or correspondingly $\mathbb{C}^n_D$, which is $\mathbb{C}^n$ 'marked' by the normally intersecting divisors

$$D = \bigcup_{i=1}^{n} D_i,\ D_i = \{z_i = 0\}. \tag{43}$$

On $\mathbb{R}^n$ we consider a subsheaf $\mathcal{F}(U) \subset \mathcal{C}(U)$ of rings of the sheaf of continuous functions. The cases we might consider are things like

  - $\mathcal{C}^{k,\alpha}(U)$ (Hölder)
  - $H^s_{\mathrm{loc}}(U)$ (Sobolev), $s > n/2$
  - $\mathcal{C}^\infty(U)$ (smooth)
  - $\mathcal{A}(U)$ (real analytic)

on $U \subset \mathbb{R}^n$ open or $U \subset \mathbb{R}^n_+$ (relatively) open, or

  - $\mathcal{O}(V)$ (holomorphic) on $V \subset \mathbb{C}^n$ open.

In fact the main cases of interest here are the two cases of $\mathcal{C}^\infty$ and holomorphic (just called complex) structures.

All these spaces have the following 'formality' property.

Proposition 1. *Any linear bijection $L : \mathcal{F}(U_2) \longrightarrow \mathcal{F}(U_1)$, U_1, U_2 open, which preserves products (no continuity assumption required) is induced by pull-back under a uniquely determined map $F : U_1 \longrightarrow U_2$ with components in $\mathcal{F}(U_1)$ which has a two-sided inverse $G : U_2 \longrightarrow U_1$ with components in $\mathcal{F}(U_2)$.*

Department of Mathematics, Massachusetts Institute of Technology
Email address: rbm@math.mit.edu