From Microlocal to Global Analysis

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Contents

Preface

This semester, Spring 2008, I am trying to get these lectures notes close to a finished form. They represent accumulated notes from various different 'Microlocal Analysis' courses and seminars at MIT. In particular in the seminar this semester, which is a continuation of a course (also run as a seminar) last semester, we hope to complete a proof of the families index theorem of Atiyah and Singer and some version of Weyl asymptotics for self-adjoint elliptic pseudodifferential operators; maybe we will also get to Fourier integral operators.

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Introduction

I shall assume some familiarity with distribution theory, with basic analysis and functional analysis and a passing knowledge of the theory of manifolds. Any one or two of these prerequisites can be easily picked up along the way, but the prospective student with none of them should perhaps do some preliminary reading:

Distributions: A good introduction is Friedlander's book [**6**]. For a more exhaustive treatment see Volume I of Hörmander's treatise [**10**].

Analysis on manifolds: Most of what we need can be picked up from Munkres' book [**11**] or Spivak's little book [**14**].

CHAPTER 1

Preliminaries: Distributions, the Fourier transform and operators

Microlocal analysis is a geometric theory of distributions, or a theory of geometric distributions. Rather than study general distributions – which are like general continuous functions but worse – we consider more specific types of distributions which actually arise in the study of differential and integral equations. Distributions are usually defined by duality, starting from very "good" test functions; correspondingly a general distribution is everywhere "bad". The *conormal distributions* we shall study implicitly for a long time, and eventually explicitly, are usually good, but like (other) people have a few interesting faults, i.e. singularities. These singularities are our principal target of study. Nevertheless we need the general framework of distribution theory to work in, so I will start with a brief introduction. This is designed either to remind you of what you already know or else to send you off to work it out. (As noted above, I suggest Friedlander's little book [**5**] - there is also a newer edition with Joshi as coauthor as a good introduction to distributions.) Volume 1 of Hörmander's treatise [**9**] has all that you would need; it is a good general reference. Proofs of some of the main theorems are outlined in the problems at the end of the chapter.

1.1. Schwartz test functions

To fix matters at the beginning we shall work in the space of tempered distributions. These are defined by duality from the space of Schwartz functions, also called the space of test functions of rapid decrease. We can think of analysis as starting off from algebra, which gives us the polynomials. Thus in $\mathbb{R}^n$ we have the coordinate functions, $x_1, \dots, x_n$ and the constant functions and then the polynomials are obtained by taking (finite) sums and products:

$$\phi(x) = \sum_{|\alpha|\le k} p_\alpha x^\alpha,\ p_\alpha \in \mathbb{C},\ \alpha \in \mathbb{N}_0^n,\ \alpha = (\alpha_1, \dots, \alpha_n), \tag{1.1}$$

$$\text{where } x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} = \prod_{j=1}^n x_j^{\alpha_j} \text{ and } \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

A general function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is differentiable at $\bar{x}$ if there is a linear function $\ell_{\bar{x}}(x) = c + \sum_{j=1}^n (x_j - \bar{x}_j) d_j$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(x) - \ell_{\bar{x}}(x)| \le \epsilon |x - \bar{x}| \quad \forall\ |x - \bar{x}| < \bar{\delta}. \tag{1.2}$$

The coefficients d_j are the partial derivative of ϕ at the point $\bar{x}$. Then, ϕ is said to be differentiable on $\mathbb{R}^n$ if it is differentiable at each point $\bar{x} \in \mathbb{R}^n$; the partial derivatives are then also functions on $\mathbb{R}^n$ and ϕ is *twice* differentiable if the partial derivatives are differentiable. In general it is k times differentiable if its partial derivatives are $k-1$ times differentiable.

If ϕ is k times differentiable then, for each $\bar{x} \in \mathbb{R}^n$, there is a polynomial of degree k,

$$p_k(x;\bar{x}) = \sum_{|\alpha|\le k} a_\alpha i^{|\alpha|}(x-\bar{x})^\alpha/\alpha!,\ |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

(the factors of i are inserted just because the have been put into $D_j = \frac{1}{1}\frac{\partial}{\partial z_j}$) such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(x) - p_k(x,\bar{x})| \le \epsilon|x-\bar{x}|^k \quad \text{if } |x-\bar{x}| < \delta. \tag{1.3}$$

Then we set

$$D^\alpha\phi(\bar{x}) = a_\alpha. \tag{1.4}$$

If ϕ is infinitely differentiable all the $D^\alpha\phi$ are infinitely differentiable (hence continuous!) functions.

Definition 1.1. *The space of Schwartz test functions of rapid decrease consists of those $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that for every $\alpha, \beta \in \mathbb{N}_0^n$*

$$\sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha\phi(x)| < \infty; \tag{1.5}$$

it is denoted $\mathcal{S}(\mathbb{R}^n)$.

From (1.5) we construct norms on $\mathcal{S}(\mathbb{R}^n)$:

$$\|\phi\|_k = \max_{|\alpha|+|\beta|\le k} \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \phi(x)|. \tag{1.6}$$

It is straightforward to check the conditions for a norm:

(1) $\|\phi\|_k \ge 0,\ \|\phi\|_k = 0 \Longleftrightarrow \phi \equiv 0$
(2) $\|t\phi\|_k = |t|\|\phi\|_k,\ t \in \mathbb{C}$
(3) $\|\phi+\psi\|_k \le \|\phi\|_k + \|\psi\|_k\ \forall\ \phi,\psi \in \mathcal{S}(\mathbb{R}^n)$.

The topology on $\mathcal{S}(\mathbb{R}^n)$ is given by the metric

$$d(\phi,\psi) = \sum_k 2^{-k}\frac{\|\phi-\psi\|_k}{1+\|\phi-\psi\|_k}. \tag{1.7}$$

See Problem 1.4.

Proposition 1.1. *With the distance function* (1.7)*, $\mathcal{S}(\mathbb{R}^n)$ becomes a complete metric space (in fact it is a Fréchet space).*

Of course one needs to check that $\mathcal{S}(\mathbb{R}^n)$ is non-trivial; however one can easily see that

$$\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n). \tag{1.8}$$

In fact there are lots of smooth functions of compact support and

$$\mathcal{C}_c^\infty(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n); u = 0 \text{ in } |x| > R = R(u)\} \subset \mathcal{S}(\mathbb{R}^n) \text{ is dense.} \tag{1.9}$$

The two elementary operations of differentiation and coordinate multiplication give continuous linear operators:

$$\begin{aligned} x_j &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ D_j &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \end{aligned} \tag{1.10}$$

Other important operations we shall encounter include the exterior product,

$$\begin{aligned}\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m) \ni (\phi, \psi) \mapsto \phi \boxtimes \psi \in \mathcal{S}(\mathbb{R}^{n+m})\\ \phi \boxtimes \psi(x, y) = \phi(x)\psi(y).\end{aligned} \tag{1.11}$$

and pull-back or restriction. If $\mathbb{R}^k \subset \mathbb{R}^n$ is identified as the subspace $x_j = 0$, $j > k$, then the restriction map

$$\pi_k^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^k),\ \pi_k^* f(y) = f(y_1, \dots, y_k, 0, \dots, 0) \tag{1.12}$$

is continuous (and surjective).

1.2. Linear transformations

A linear transformation acts on $\mathbb{R}^n$ as a matrix (this is the standard action, but it is potentially confusing since it means that for the basis elements $e_j \in \mathbb{R}^n$, $Le_j = \sum\limits_{k=1}^n L_{kj} e_k$)

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ (Lx)_j = \sum_{k=1}^n L_{jk} x_k. \tag{1.13}$$

The Lie group of invertible linear transformations, $\mathrm{GL}(n, \mathbb{R})$ is fixed by several equivalent conditions

$$\begin{aligned} L \in \mathrm{GL}(n, \mathbb{R}) &\Longleftrightarrow \det(L) \neq 0\\ &\Longleftrightarrow \exists\ L^{-1} \text{ s.t. } (L^{-1})Lx = x\ \forall\ x \in \mathbb{R}^n\\ &\Longleftrightarrow \exists\ c > 0 \text{ s.t. } c|x| \leq |Lx| \leq c^{-1}|x|\ \forall\ x \in \mathbb{R}^n.\end{aligned} \tag{1.14}$$

Pull-back of functions is defined by

$$L^*\phi(x) = \phi(Lx) = (\phi \circ L)(x).$$

The chain rule for differentiation shows that if ϕ is differentiable then

$$D_j L^*\phi(x) = D_j\phi(Lx) = \sum_{k=1}^n L_{kj} (D_k\phi)(Lx) = L^*((L_* D_j)\phi)(x), \tag{1.15}$$

$$L_* D_j = \sum_{k=1}^n L_{kj} D_k$$

(so D_j transforms as a basis of $\mathbb{R}^n$ as it should, despite the factors of i.) From this it follows that

$$L^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ is an isomorphism for } L \in \mathrm{GL}(n, \mathbb{R}). \tag{1.16}$$

1.3. Tempered distributions

As well as exterior multiplication (1.11) there is the even more obvious multiplication operation

$$\begin{aligned}\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)\\ (\phi, \psi) \mapsto \phi(x)\psi(x)\end{aligned} \tag{1.17}$$

which turns $\mathcal{S}(\mathbb{R}^n)$ into a commutative algebra without identity. There is also integration

$$\int : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}. \tag{1.18}$$

Combining these gives a *pairing*, a bilinear map

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \longmapsto \int_{\mathbb{R}^n} \phi(x)\psi(x)dx. \tag{1.19}$$

If we fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ this defines a continuous linear map:

$$T_\phi : \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \int \phi(x)\psi(x)dx. \tag{1.20}$$

Continuity becomes the condition:

$$\exists\ k, C_k \text{ s.t. } |T_\phi(\psi)| \le C_k \|\psi\|_k \ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{1.21}$$

We generalize this by denoting by $\mathcal{S}'(\mathbb{R}^n)$ the dual space, i.e. the space of all continuous linear functionals

$$u \in \mathcal{S}'(\mathbb{R}^n) \Longleftrightarrow u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$
$$\exists\ k, C_k \text{ such that } |u(\psi)| \le C_k \|\psi\|_k \ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 1.1. *The map*

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto T_\phi \in \mathcal{S}'(\mathbb{R}^n) \tag{1.22}$$

is an injection.

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $T_\phi(\phi) = \int |\phi(x)|^2 dx$, so $T_\phi = 0$ implies $\phi \equiv 0$. $\square$

If we wish to consider a topology on $\mathcal{S}'(\mathbb{R}^n)$ it will normally be the *weak* topology, that is the weakest topology with respect to which all the linear maps

$$\mathcal{S}'(\mathbb{R}^n) \ni u \longmapsto u(\phi) \in \mathbb{C}, \quad \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.23}$$

are continuous. This just means that it is given by the seminorms

$$\mathcal{S}(\mathbb{R}^n) \ni u \longmapsto |u(\phi)| \in \mathbb{R} \tag{1.24}$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is fixed but arbitrary. The sets

$$\{u \in \mathcal{S}'(\mathbb{R}^n); |u(\phi_j)| < \epsilon_j,\ \phi_j \in \Phi\} \tag{1.25}$$

form a basis of the neighbourhoods of 0 as $\Phi \subset \mathcal{S}(\mathbb{R}^n)$ runs over finite sets and the ϵ_j are positive numbers.

Proposition 1.2. *The continuous injection $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, given by* (1.22), *has dense range in the weak topology.*

See Problem 1.8 for the outline of a proof.

The maps x_i, D_j extend by continuity (and hence uniquely) to operators

$$x_j,\ D_j : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1.26}$$

This is easily seen by defining them by duality. Thus if $\phi \in \mathcal{S}(\mathbb{R}^n)$ set $D_j T_\phi = T_{D_j\phi}$, then

$$T_{D_j\phi}(\psi) = \int D_j\phi\psi = -\int \phi D_j \psi, \tag{1.27}$$

the integration by parts formula. The definitions

$$D_j u(\psi) = u(-D_j\psi),\ x_j u(\psi) = u(x_j\psi),\ u \in \mathcal{S}'(\mathbb{R}^n), \psi \in \mathcal{S}(\mathbb{R}^n) \tag{1.28}$$

satisfy all requirements, in that they give continuous maps (1.26) which extend the standard definitions on $\mathcal{S}(\mathbb{R}^n)$.

To characterize the action of $L \in \mathrm{GL}(n, \mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^n)$ consider the distribution associated to $L^*\phi$:

$$\begin{aligned} T_{L^*\phi}(\psi) &= \int_{\mathbb{R}^n} \phi(Lx)\psi(x)dx \\ &= \int_{\mathbb{R}^n} \phi(y)\psi(L^{-1}y)|\det L|^{-1}dy = T_\phi(|\det L|^{-1}(L^{-1})^*\psi). \end{aligned} \tag{1.29}$$

Since the operator $|\det L|^{-1}(L^{-1})^*$ is an ismorphism of $\mathcal{S}(\mathbb{R}^n)$ it follows that if we take the definition by duality

$$\begin{aligned} L^*u(\psi) = u(|\det L|^{-1}(L^{-1})^*\psi),\ u \in \mathcal{S}'(\mathbb{R}^n),\ \psi \in \mathcal{S}(\mathbb{R}^n),\ L \in \mathrm{GL}(n, \mathbb{R}) \\ \Longrightarrow L^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \end{aligned} \tag{1.30}$$

is an isomorphism which extends (1.16) and satisfies

$$D_j L^*u = L^*((L_*D_j)u), L^*(x_j u) = (L^*x_j)(L^*u),\ u \in \mathcal{S}'(\mathbb{R}^n),\ L \in \mathrm{GL}(n, \mathbb{R}), \tag{1.31}$$

as in (1.15).

1.4. Two big theorems

The association, by (1.22), of a distribution to a function can be extended considerably. For example if $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a bounded and continuous function then

$$T_u(\psi) = \int u(x)\psi(x)dx \tag{1.32}$$

still defines a distribution which vanishes if and only if u vanishes identically. Using the operations (1.26) we conclude that for any $\alpha, \beta \in \mathbb{N}_0^n$

$$x^\beta D_x^\alpha u \in \mathcal{S}'(\mathbb{R}^n) \text{ if } u : \mathbb{R}^n \longrightarrow \mathbb{C} \text{ is bounded and continuous.} \tag{1.33}$$

Conversely we have the *Schwartz representation Theorem:*

THEOREM 1.1. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$ there is a finite collection $u_{\alpha\beta} : \mathbb{R}^n \longrightarrow \mathbb{C}$ of bounded continuous functions, $|\alpha| + |\beta| \le k$, such that*

$$u = \sum_{|\alpha|+|\beta|\le k} x^\beta D_x^\alpha u_{\alpha\beta}. \tag{1.34}$$

Thus tempered distributions are just products of polynomials and derivatives of bounded continuous functions. This is important because it says that distributions are "not too bad".

The second important result (long considered very difficult to prove, but there is a relatively straightforward proof using the Fourier transform) is the *Schwartz kernel theorem.* To show this we need to use the exterior product (1.11). If $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ this allows us to define a linear map

$$O_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.35}$$

by

$$O_K(\psi)(\phi) = \int K \cdot \phi \boxtimes \psi \, dxdy. \tag{1.36}$$

THEOREM 1.2. *There is a* 1-1 *correspondence between continuous linear operators*

$$A : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.37}$$

and $\mathcal{S}'(\mathbb{R}^{n+m})$ *given by* $A = O_K$.

Brief outlines of the proofs of these two results can be found in Problems 1.15 and 1.16.

1.5. Examples

Amongst tempered distributions we think of $\mathcal{S}(\mathbb{R}^n)$ as being the 'trivial' examples, since they are the test functions. One can say that the study of the singularities of tempered distributions amounts to the study of the quotient

$$\mathcal{S}'(\mathbb{R}^n)/\mathcal{S}(\mathbb{R}^n) \tag{1.38}$$

which could, reasonably, be called the space of tempered microfunctions.

The sort of distributions we are interested in are those like the Dirac delta "function"

$$\delta(x) \in \mathcal{S}'(\mathbb{R}^n), \quad \delta(\phi) = \phi(0). \tag{1.39}$$

The definition here shows that δ is just the Schwartz kernel of the operator

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi(0) \in \mathbb{C} = \mathcal{S}(\mathbb{R}^0). \tag{1.40}$$

This is precisely one reason it is interesting. More generally we can consider the maps

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto D^\alpha \phi(0), \quad \alpha \in \mathbb{N}_0^n. \tag{1.41}$$

These have Schwartz kernels $(-D)^\alpha \delta$ since

$$(-D)^\alpha \delta(\phi) = \delta(D^\alpha \phi) = D^\alpha \phi(0). \tag{1.42}$$

If we write the relationship $A = O_K \longleftrightarrow K$ as

$$(A\psi)(\phi) = \int K(x,y)\phi(x)\psi(y)dxdy \tag{1.43}$$

then (1.42) becomes

$$D^\alpha \phi(0) = \int (-D)^\alpha \delta(x)\phi(x)dx. \tag{1.44}$$

More generally, if $K(x,y)$ is the kernel of an operator A then the kernel of $A \cdot D^\alpha$ is $(-D)_y^\alpha K(x,y)$ whereas the kernel of $D^\alpha \circ A$ is $D_x^\alpha K(x,y)$.

1.6. Two little lemmas

Above, some of the basic properties of tempered distributions have been outlined. The main "raison d'être" for $\mathcal{S}'(\mathbb{R}^n)$ is the *Fourier transform* which we proceed to discuss. We shall use the Fourier transform as an almost indispensable tool in the treatment of pseudodifferential operators. The description of differential operators, via their Schwartz kernels, using the Fourier transform is an essential motivation for the extension to pseudodifferential operators.

Partly as simple exercises in the theory of distributions, and more significantly as preparation for the proof of the inversion formula for the Fourier transform we consider two lemmas.

First recall that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then we have defined $D_j u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$D_j u(\phi) = u(-D_j \phi) \ \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.45}$$

In this sense it is a "weak derivative". Let us consider the simple question of the form of the solutions to

$$D_j u = 0, \ u \in \mathcal{S}'(\mathbb{R}^n). \tag{1.46}$$

Let I_j be the integration operator:

$$\begin{gathered} I_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1}) \\ I_j(\phi)(y_1, \dots, y_{n-1}) = \int \phi(y_1, \dots y_{j-1}, x, y_j, \dots y_{n-1}) dx. \end{gathered} \tag{1.47}$$

Then if $\pi_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ is the map $\pi_j(x) = (x_1, \dots, x_{j-1}, x_{j+1} \dots, x_n)$, we *define*, for $v \in \mathcal{S}'(\mathbb{R}^{n-1})$,

$$\pi_j^* v(\phi) = v(I_j \phi) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.48}$$

It is clear from (1.47) that $I_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1})$ is continuous and hence $\pi_j^* v \in \mathcal{S}'(\mathbb{R}^n)$ is well-defined for each $v \in \mathcal{S}'(\mathbb{R}^{n-1})$.

LEMMA 1.2. *The equation* (1.46) *holds if and only if* $u = \pi_j^* v$ *for some* $v \in \mathcal{S}'(\mathbb{R}^{n-1})$.

PROOF. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi = D_j \psi$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ then $I_j \phi = I_j(D_j \psi) = 0$. Thus if $u = \pi_j^* v$ then

$$u(-D_j \phi) = \pi_j^* v(-D_j \phi) = v(I_j(-D_j \phi)) = 0. \tag{1.49}$$

Thus $u = \pi_j^* v$ does always satisfy (1.46).

Conversely suppose (1.46) holds. Choose $\rho \in \mathcal{S}(\mathbb{R})$ with the property

$$\int \rho(x) dx = 1. \tag{1.50}$$

Then each $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be decomposed as

$$\phi(x) = \rho(x_j) I_j \phi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n) + D_j \psi, \quad \psi \in \mathcal{S}(\mathbb{R}^n). \tag{1.51}$$

Indeed this is just the statement

$$\zeta \in \mathcal{S}(\mathbb{R}^n),\ I_j\zeta = 0 \Longrightarrow \psi(x) \in \mathcal{S}(\mathbb{R}^n) \text{ where}$$
$$\psi(x) = \int_{-\infty}^{x_j} \zeta(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)dt$$
$$= \int_{\infty}^{x_j} \zeta(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)dt.$$

Using (1.51) and (1.46) we have

$$u(\phi) = u\left(\rho(x_j) I_j \phi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)\right). \tag{1.52}$$

Thus if

$$v(\psi) = u\left(\rho(x_j)\psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)\right)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^{n-1}) \tag{1.53}$$

then $v \in \mathcal{S}'(\mathbb{R}^{n-1})$ and $u = \pi_j^* v$. This proves the lemma. □

Of course the notation $u = \pi_j^* v$ is much too heavy-handed. We just write $u(x) = v(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and regard 'v as a distribution in one additional variable'.

The second, related, lemma is just a special case of a general result of Schwartz concerning the support of a distribution. The particular result is:

Lemma 1.3. *Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ and $x_j u = 0, j = 1, \dots n$ then $u = c\delta(x)$ for some constant c.*

Proof. Again we use the definition of multiplication and a dual result for test functions. Namely, choose $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\rho(x) = 1$ in $|x| < \frac{1}{2}$, $\rho(x) = 0$ in $|x| \geq 3/4$. Then any $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be written

$$\phi = \phi(0) \cdot \rho(x) + \sum_{j=1}^{n} x_j \psi_j(x), \qquad \psi_j \in \mathcal{S}(\mathbb{R}^n). \tag{1.54}$$

This in turn can be proved using Taylor's formula as I proceed to show. Thus

$$\phi(x) = \phi(0) + \sum_{j=1}^{n} x_j \zeta_j(x) \text{ in } |x| \leq 1, \text{ with } \zeta_j \in \mathcal{C}^\infty. \tag{1.55}$$

Then,

$$\rho(x)\phi(x) = \phi(0)\rho(x) + \sum_{j=1}^{n} x_j \rho\zeta_j(x) \tag{1.56}$$

and $\rho\zeta_j \in \mathcal{S}(\mathbb{R}^n)$. Thus it suffices to check (1.54) for $(1-\rho)\phi$, which vanishes identically near 0. Then $\zeta = |x|^{-2}(1-\rho)\phi \in \mathcal{S}(\mathbb{R}^n)$ and so

$$(1-\rho)\phi = |x|^2\zeta = \sum_{j=1}^{n} x_j(x_j\zeta) \tag{1.57}$$

finally gives (1.54) with $\psi_j(x) = \rho(x)\zeta_j(x) + x_j\zeta(x)$. Having proved the existence of such a decomposition we see that if $x_j u = 0$ for all j then

$$u(\phi) = u(\phi(0)\rho(x)) + \sum_{j=1}^{n} u(x_j\psi_j) = c\phi(0), \quad c = u(\rho(x)), \tag{1.58}$$

i.e. $u = c\delta(x)$. □

1.7. Fourier transform

Our normalization of the Fourier transform will be

$$\mathcal{F}\phi(\xi) = \int e^{-i\xi\cdot x}\phi(x)dx. \tag{1.59}$$

As you all know the inverse Fourier transform is given by

$$\mathcal{G}\psi(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\psi(\xi)d\xi. \tag{1.60}$$

Since it is so important here I will give a proof of this invertibility. First however, let us note some of the basic properties.

Both $\mathcal{F}$ and $\mathcal{G}$ give continuous linear maps

$$\mathcal{F}, \mathcal{G} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{1.61}$$

To see this observe first that the integrals in (1.59) and (1.60) are absolutely convergent:

$$|\mathcal{F}\phi(\xi)| \le \int |\phi(x)|dx \le \int (1+|x|^2)^{-n}dx \times \sup_{x\in\mathbb{R}^n}(1+|x|^2)^n|\phi(x)|, \tag{1.62}$$

where we use the definition of $\mathcal{S}(\mathbb{R}^n)$. In fact this shows that $\sup|\mathcal{F}\phi| < \infty$ if $\phi \in \mathcal{S}(\mathbb{R}^n)$. Formal differentiation under the integral sign gives an absolutely convergent integral:

$$D_j\mathcal{F}\phi(\xi) = \int D_{\xi_j}e^{-ix\xi}\phi(x)dx = \int e^{-ix\cdot\xi}(-x_j\phi)dx$$

since $\sup\limits_x (1+|x|^2)^n|x_j\phi| < \infty$. Then it follows that $D_j\mathcal{F}\phi$ is also bounded, i.e. $\mathcal{F}\phi$ is differentiable, and (1.7) holds. This argument can be extended to show that $\mathcal{F}\phi$ is $\mathcal{C}^\infty$,

$$D^\alpha\mathcal{F}\phi(\xi) = \mathcal{F}\big((-x)^\alpha\phi\big). \tag{1.63}$$

Similarly, starting from (1.59), we can use integration by parts to show that

$$\xi_j\mathcal{F}\phi(\xi) = \int e^{-ix\xi}\xi_j\phi(x)dx = \int e^{-ix\cdot\xi}(D_j\phi)(x)dx$$

i.e. $\xi_j\mathcal{F}\phi = \mathcal{F}(D_j\phi)$. Combining this with (1.63) gives

$$\xi^\alpha D_\xi^\beta\mathcal{F}\phi = \mathcal{F}\big(D^\alpha\cdot[(-x)^\beta\phi]\big). \tag{1.64}$$

Since $D_x^\alpha((-x)^\beta\phi) \in \mathcal{S}(\mathbb{R}^n)$ we conclude

$$\sup|\xi^\alpha D_\zeta^\beta\mathcal{F}\phi| < \infty \Longrightarrow \mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.65}$$

This map is continuous since

$$\sup|\xi^\alpha D_\xi^\beta\mathcal{F}\phi| \le C\cdot\sup_x|(1+|x|^2)^n D_x^\alpha[(-x)^\beta\phi]$$
$$\Longrightarrow \|\mathcal{F}\phi\|_k \le C_k\|\phi\|_{k+2n}, \ \forall\ k.$$

The identity (1.64), written in the form

$$\begin{aligned}\mathcal{F}(D_j\phi) &= \xi_j\mathcal{F}\phi\\ \mathcal{F}(x_j\phi) &= -D_{\xi_j}\mathcal{F}\phi\end{aligned} \tag{1.66}$$

is already the key to the proof of invertibility:

THEOREM 1.3. *The Fourier transform gives an isomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longleftrightarrow \mathcal{S}(\mathbb{R}^n)$ with inverse $\mathcal{G}$.*

PROOF. We shall use the *idea* of the Schwartz kernel theorem. It is important *not* to use this theorem itself, since the Fourier transform is a key tool in the (simplest) proof of the kernel theorem. Thus we consider the composite map

$$\mathcal{G}\circ\mathcal{F} :\ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \tag{1.67}$$

and write down its kernel. Namely

$$\begin{gathered}K(\phi) = (2\pi)^{-n}\iiint e^{iy\cdot\xi - ix\cdot\xi}\phi(y,x)dxd\xi dy\\ \forall\ \phi\in\mathcal{S}(\mathbb{R}^n_y\times\mathbb{R}^n_x)\Longrightarrow K\in\mathcal{S}'(\mathbb{R}^{2n}).\end{gathered} \tag{1.68}$$

The integrals in (1.68) are iterated, i.e. should be performed in the order indicated. Notice that if $\psi, \zeta \in \mathcal{S}(\mathbb{R}^n)$ then indeed

$$\begin{aligned}(\mathcal{G}\circ\mathcal{F}(\psi))(\zeta) = \int\zeta(y)(2\pi)^{-n}\left(\int e^{iy\cdot\xi}\int e^{-ix\cdot\xi}\psi(x)dxd\xi\right)dy\\ = K(\zeta\boxtimes\psi)\end{aligned} \tag{1.69}$$

so K is the Schwartz kernel of $\mathcal{G}\circ\mathcal{F}$.

The two identities (1.66) translate (with essentially the same proofs) to the conditions on K :

$$\begin{cases}(D_{x_j} + D_{y_j})K(x,y) = 0\\ (x_j - y_j)K(x,y) = 0\end{cases}\qquad j = 1,\dots,n. \tag{1.70}$$

Next we use the freedom to make linear changes of variables, setting

$$\begin{gathered}K_L(x,z) = K(x,x-z),\ K_L\in\mathcal{S}'(\mathbb{R}^{2n})\\ \text{i.e. } K_L(\phi) = K(\psi),\ \psi(x,y) = \phi(x,x-y)\end{gathered} \tag{1.71}$$

where the notation will be explained later. Then (1.70) becomes

$$D_{x_j}K_L(x,z) = 0 \text{ and } z_jK_L(x,z) = 0 \text{ for } j = 1,\dots n. \tag{1.72}$$

This puts us in a position to apply the two little lemmas. The first says $K_L(x,z) = f(z)$ for some $f \in \mathcal{S}'(\mathbb{R}^n)$ and then the second says $f(z) = c\delta(z)$. Thus

$$K(x,y) = c\delta(x-y)\Longrightarrow\mathcal{G}\circ\mathcal{F} = c\,\mathrm{Id}\,. \tag{1.73}$$

It remains only to show that $c = 1$. That $c \neq 0$ is obvious (since $\mathcal{F}(\delta) = 1$). The easiest way to compute the constant is to use the integral identity

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \pi^{\frac12} \tag{1.74}$$

to show that[1]

[1]See Problem 1.9.

$$
\begin{aligned}
&\mathcal{F}(e^{-|x|^2}) = \pi^{\frac{n}{2}} e^{-|\xi|^2/4} \\
(1.75) \qquad &\Longrightarrow \mathcal{G}(e^{-|\xi|^2/4}) = \pi^{-\frac{n}{2}} e^{-|x|^2} \\
&\Longrightarrow \mathcal{G} \cdot \mathcal{F} = \mathrm{Id}\,.
\end{aligned}
$$

□

Now $(2\pi)^n\mathcal{G}$ is actually the *adjoint* of $\mathcal{F}$:

$$
(1.76) \qquad \int \phi(\zeta)\overline{\mathcal{F}\psi}(\zeta)d\zeta = (2\pi)^n \int (\mathcal{G}\phi)\cdot\overline{\psi}dx \ \forall\ \phi,\psi \in \mathcal{S}(\mathbb{R}^n).
$$

It follows that we can extend $\mathcal{F}$ to a map on tempered distributions

$$
(1.77) \qquad
\begin{aligned}
&\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \\
&\mathcal{F}u(\overline{\phi}) = u((2\pi)^n\overline{\mathcal{G}\phi}) \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n)
\end{aligned}
$$

Then we conclude

Corollary 1.1. *The Fourier transform extends by continuity to an isomorphism*

$$
(1.78) \qquad \mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longleftrightarrow \mathcal{S}'(\mathbb{R}^n)
$$

with inverse $\mathcal{G}$, *satisfying the identities* (1.66).

Although I have not discussed Lebesgue integrability I assume familiarity with the basic Hilbert space

$$
\begin{aligned}
&L^2(\mathbb{R}^n) = \\
&\qquad \left\{u : \mathbb{R}^n \longrightarrow \mathbb{C}; f \text{ is measurable and } \int_{\mathbb{R}^n} |f(x)|^2dx < \infty\right\} / \sim, \\
&\qquad\qquad f \sim g \Longleftrightarrow f = g \text{ almost everywhere.}
\end{aligned}
$$

This also injects by the same integration map (1.104) with $\mathcal{S}(\mathbb{R}^n)$ as a dense subset

$$
\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^p) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).
$$

Proposition 1.3. *The Fourier transform extends by continuity from the dense subspace* $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, *to an isomorphism*

$$
\mathcal{F} : L^2(\mathbb{R}^n) \longleftrightarrow L^2(\mathbb{R}^n)
$$

satisfying $\|\mathcal{F}u\|_{L^2} = (2\pi)^{\frac{1}{2}n}\|u\|_{L^2}$.

Proof. Given the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, this is also a consequence of (1.76), since setting $\phi = \mathcal{F}u$, for $u \in \mathcal{S}(\mathbb{R}^n)$, gives Parseval's formula

$$
\int \mathcal{F}u(\zeta)\overline{\mathcal{F}v(\zeta)} = (2\pi)^n \int u(x)\overline{v(x)}dx.
$$

Setting $v = u$ gives norm equality (which is Plancherel's formula).

An outline of the proof of the density statement is given in the problems below.

□

1.8. Differential operators

The simplest examples of the Fourier transform of distributions are immediate consequences of the definition and (1.66). Thus

$$\mathcal{F}(\delta) = 1 \tag{1.79}$$

as already noted and hence, from (1.66),

$$\mathcal{F}(D^\alpha \delta(x)) = \xi^\alpha \quad \forall\ \alpha \in \mathbb{N}_0^n. \tag{1.80}$$

Now, recall that the space of distributions with support the point 0 is just:

$$\big\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{sup}(u) \subset \{0\}\big\} = \big\{u = \sum_{\text{finite}} c_\alpha D^\alpha \delta\big\}. \tag{1.81}$$

Thus we conclude that the Fourier transform gives an isomorphism

$$\mathcal{F} : \big\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \subset \{0\}\big\} \longleftrightarrow \mathbb{C}[\xi] = \{\text{polynomials in } \xi\}. \tag{1.82}$$

Another way of looking at this same isomorphism is to consider partial differential operators with constant coefficients:

$$\begin{gathered} P(D) :\ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ P(D) = \sum c_\alpha D^\alpha. \end{gathered} \tag{1.83}$$

The identity becomes

$$\mathcal{F}(P(D)\phi)(\xi) = P(\xi)\mathcal{F}(\phi)(\xi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.84}$$

and indeed the same formula holds for all $\phi \in \mathcal{S}'(\mathbb{R}^n)$. Using the simpler notation $\hat{u}(\xi) = \mathcal{F}u(\xi)$ this can be written

$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi),\ P(\xi) = \sum c_\alpha \xi^\alpha. \tag{1.85}$$

The polynomial P is called the (full) characteristic polynomial of $P(D)$; of course it determines $P(D)$ uniquely.

It is important for us to extend this formula to differential operators with variable coefficients. Using (1.59) and the inverse Fourier transform we get

$$P(D)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} P(\xi)u(y)dyd\xi \tag{1.86}$$

where again this is an *iterated* integral. In particular the inversion formula is just the case $P(\xi) = 1$. Consider the space

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \big\{u : \mathbb{R}^n \longrightarrow \mathbb{C};\ \sup_x |D^\alpha u(x)| < \infty\ \forall\ \alpha\big\} \tag{1.87}$$

the space of $\mathcal{C}^\infty$ function with all derivatives bounded on $\mathbb{R}^n$. Of course

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n) \tag{1.88}$$

but $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ is much bigger, in particular $1 \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Now by Leibniz' formula

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u \cdot D^{\alpha-\beta} v \tag{1.89}$$

it follows that $\mathcal{S}(\mathbb{R}^n)$ is a *module* over $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. That is,

$$u \in \mathcal{C}^\infty_\infty(\mathbb{R}^n), \phi \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow u\phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.90}$$

From this it follows that if

$$P(x,D)=\sum_{|\alpha|\le m}p_\alpha(x)D^\alpha,\ p_\alpha\in\mathcal{C}^\infty_\infty(\mathbb{R}^n) \tag{1.91}$$

then $P(x,D):\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)$. The formula (1.86) extends to

$$P(x,D)\phi=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)\phi(y)dyd\xi \tag{1.92}$$

where again this is an iterated integral. Here

$$P(x,\xi)=\sum_{|\alpha|\le m}p_\alpha(x)\xi^\alpha \tag{1.93}$$

is the (full) characteristic polynomial of P.

1.9. Radial compactification

For later purposes, and general propaganda, consider the three standard compactifications of of $\mathbb{R}^n$. They are the one-point, the quadratic and the radial compactifications.

1.9.1. One-point compactification. This is most familiar in the case of $\mathbb{R}^2$ as $\mathbb{C}$ compactified to the Riemann sphere. However, it works in general by the stereographic map

$$\mathbb{R}^n\ni z\longmapsto\left(\frac{4-|z|^2}{4+|z|^2},\frac{4z}{4+|z|^2}\right).\in\mathbb{S}^n\subset\mathbb{R}^{n+1} \tag{1.94}$$

We will mainly consider this in the case of $n=1$ when it gives an smooth map from $\mathbb{R}$ into the unit circle. Rotating the axes so that the origin is mapped to the point $(1,0)$ (rather than $i=(0,1)$) in complex notation this is

$$\mathbb{R}\ni t\longmapsto e^{i\theta(t)}\in\mathbb{S}\subset\mathbb{C},\ \theta(t)=\arctan(\frac{4t}{4+t^2}). \tag{1.95}$$

1.9.2. Quadratic compactification. The smooth map

$$\mathrm{QRC}:\mathbb{R}^n\ni x\longmapsto\frac{x}{(1+|x|^2)^{\frac12}}\in\mathbb{R}^n \tag{1.96}$$

is 1-1 and maps onto the interior of the unit ball, $\mathbb{B}^n=\{|x|\le1\}$. Consider the subspace

$$\dot{\mathcal{C}}^\infty(\mathbb{B}^n)=\{u\in\mathcal{S}(\mathbb{R}^n);\operatorname{supp}(u)\subset\mathbb{B}^n\}. \tag{1.97}$$

This is just the set of smooth functions on $\mathbb{R}^n$ which vanish outside the unit ball. Then the composite ('pull-back') map

$$\mathrm{QRC}^*:\dot{\mathcal{C}}^\infty(\mathbb{B}^n)\ni u\longmapsto u\circ\mathrm{QRC}\in\mathcal{S}(\mathbb{R}^n) \tag{1.98}$$

is a topological isomorphism. A proof is indicated in the problems below.

The dual space of $\dot{\mathcal{C}}^\infty(\mathbb{B}^n)$ is generally called the space of 'extendible distributions' on $\mathbb{B}^n$ – because they are all given by restricting elements of $\mathcal{S}'(\mathbb{R}^n)$ to $\dot{\mathcal{C}}^\infty(\mathbb{B}^n)$. Thus QRC also identifies the tempered distributions on $\mathbb{R}^n$ with the extendible distributions on $\mathbb{B}^n$. We shall see below that various spaces of functions on $\mathbb{R}^n$ take interesting forms when pulled back to $\mathbb{B}^n$. I often find it useful to 'bring infinity in' in this way.

Why is this the 'quadratic' radial compactification, and not just the radial compactification? There is a good reason which is discussed in the problems below.

1.9.3. Radial compactification. The actual radial compactification is a closely related map which identifies Euclidean space, $\mathbb{R}^n$, with the interior of the upper half of the n-sphere in $\mathbb{R}^{n+1}$:

$$\begin{aligned}\text{RC} : \mathbb{R}^n \ni x \longmapsto &\left(\frac{1}{(1+|x|^2)^{\frac{1}{2}}}, \frac{x}{(1+|x|^2)^{\frac{1}{2}}} \right) \\ &\in \mathbb{S}^{n,1} = \{X = (X_0, X') \in \mathbb{R}^{n+1}; X_0 \geq 0, X_0^2 + |X'|^2 = 1\}\end{aligned} \tag{1.99}$$

Since the half-sphere is diffeomorphic to the ball (as compact manifolds with boundary) these two maps can be compared – they are not the same. However it is true that RC also identifies $\mathcal{S}(\mathbb{R}^n)$ with $\dot{\mathcal{C}}^\infty(\mathbb{S}^{n,1})$.

1.10. Problems

PROBLEM 1.1. Suppose $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a function such that for *each* point $\bar{x} \in \mathbb{R}^n$ and each $k \in \mathbb{N}_0$ there exists a constant $\epsilon_k > 0$ and a polynomial $p_k(x; \bar{x})$ (in x) for which

$$|\phi(x) - p_k(x; \bar{x})| \leq \frac{1}{\epsilon_k} |x - \bar{x}|^{k+1} \quad \forall \ |x - \bar{x}| \leq \epsilon_k. \tag{1.100}$$

Does it follow that ϕ is infinitely differentiable – either prove this or give a counterexample.

PROBLEM 1.2. Show that the function $u(x) = \exp(x) \cos[e^x]$ 'is' a tempered distribution. Part of the question is making a precise statement as to what this means!

PROBLEM 1.3. Write out a careful (but not necessarily long) proof of the 'easy' direction of the Schwartz kernel theorem, that any $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ defines a continuous linear operator

$$O_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.101}$$

[with respect to the weak topology on $\mathcal{S}'(\mathbb{R}^n)$ and the metric topology on $\mathcal{S}(\mathbb{R}^m)$] by

$$O_K \phi(\psi) = K(\psi \boxtimes \phi). \tag{1.102}$$

[Hint: Work out what the continuity estimate on the kernel, K, means when it is paired with an exterior product $\psi \boxtimes \phi$.]

PROBLEM 1.4. Show that d in (1.6) is a metric on $\mathcal{S}(\mathbb{R}^n)$. [Hint: If $\| \cdot \|$ is a norm on a vector space show that

$$\frac{\|u+v\|}{1+\|u+v\|} \leq \frac{\|u\|}{1+\|u\|} + \frac{\|v\|}{1+\|v\|}.]$$

PROBLEM 1.5. Show that a sequence ϕ_n in $\mathcal{S}(\mathbb{R}^n)$ is Cauchy, resp. converges to ϕ, with respect to the metric d in Problem 1.4 if and only if ϕ_n is Cauchy, resp. converges to ϕ, with respect to each of the norms $\| \cdot \|_k$.

Problem 1.6. Show that a linear map $F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^p)$ is continuous with respect to the metric topology given in Problem 1.4 if and only if for each k there exists $N(k) \in \mathbb{N}$ a constant C_k such that

$$\|F\phi\|_k \le C_k \|\phi\|_{N(k)} \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Give similar equivalent conditions for continuity of a linear map $f : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ and for a bilinear map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^p) \longrightarrow \mathbb{C}$.

Problem 1.7. Check the continuity of (1.12).

Problem 1.8. Prove Proposition 1.2. [Hint: It is only necessary to show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ is fixed then for any of the open sets in (1.1), B, (with all the $\epsilon_j > 0$) there is an element $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $u - T_\phi \in B$. First show that if $\phi'_1, \dots \phi'_p$ is a basis for Φ then the set

$$B' = \{v \in \mathcal{S}'(\mathbb{R}^n); |\langle v, \phi'_j\rangle| < \delta_j \tag{1.103}$$

is contained in B if the $\delta_j > 0$ are chosen small enough. Taking the basis to be orthonormal, show that $u - \psi \in B'$ can be arranged for some $\psi \in \Phi$.]

Problem 1.9. Compute the Fourier transform of $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$. [Hint: The Fourier integral is a product of 1-dimensional integrals so it suffices to assume $x \in \mathbb{R}$. Then

$$\int e^{-i\xi x} e^{-x^2} dx = e^{-\xi^2/4} \int e^{-(x+\frac{i}{2}\xi)^2} dx.$$

Interpret the integral as a contour integral and shift to the new contour where $x + \frac{i}{2}\xi$ is real.]

Problem 1.10. Show that (1.20) makes sense for $\phi \in L^2(\mathbb{R}^n)$ (the space of (equivalence classes of) Lebesgue square-integrable functions and that the resulting map $L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an injection.

Problem 1.11. Suppose $u \in L^2(\mathbb{R}^n)$ and that

$$D_1 D_2 \cdots D_n u \in (1+|x|)^{-n-1} L^2(\mathbb{R}^n),$$

where the derivatives are defined using Problem 1.10. Using repeated integration, show that u is necessarily a bounded continuous function. Conclude further that for $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\begin{gathered} D^\alpha u \in (1+|x|)^{-n-1} L^2(\mathbb{R}^n) \ \forall\ |\alpha| \le k+n \\ \Longrightarrow D^\alpha u \text{ is bounded and continuous for } |\alpha| \le k. \end{gathered} \tag{1.104}$$

[This is a weak form of the Sobolev embedding theorem.]

Problem 1.12. The *support* of a (tempered) distribution can be defined in terms of the support of a test function. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ the support, $\operatorname{supp}(\phi)$, is the closure of the set of points at which it takes a non-zero value. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\operatorname{supp}(u) = O^{\complement},\ O = \bigcup \{O' \subset \mathbb{R}^n \text{ open}; \operatorname{supp}(\phi) \subset O' \Longrightarrow u(\phi) = 0\}. \tag{1.105}$$

Show that the definitions for $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are consistent with the inclusion $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Prove that $\operatorname{supp}(\delta) = \{0\}$.

Problem 1.13. For simplicity in $\mathbb{R}$, i.e. with $n = 1$, prove Schwartz theorem concerning distributions with support the origin. Show that with respect to the norm $\|\cdot\|_k$ the space

$$\{\phi \in \mathcal{S}(\mathbb{R}); \phi(x) = 0 \text{ in } |x| < \epsilon,\ \epsilon = \epsilon(\phi) > 0\} \tag{1.106}$$

is dense in

$$\left\{\phi \in \mathcal{S}(\mathbb{R}); \phi(x) = x^{k+1}\psi(x),\ \psi \in \mathcal{S}(\mathbb{R})\right\}. \tag{1.107}$$

Use this to show that

$$u \in \mathcal{S}'(\mathbb{R}),\ \operatorname{supp}(u) \subset \{0\} \Longrightarrow u = \sum_{\ell,\text{ finite}} c_\ell D_x^\ell \delta(x). \tag{1.108}$$

Problem 1.14. Show that if P is a differential operator with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ then P is local in the sense that

$$\operatorname{supp}(Pu) \subset \operatorname{supp}(u) \quad \forall\ u \in \mathcal{S}'(\mathbb{R}^n). \tag{1.109}$$

The converse of this, for an operator $P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ where (for simplicity) we assume

$$\operatorname{supp}(Pu) \subset K \subset \mathbb{R}^n \tag{1.110}$$

for a fixed compact set K, is *Peetre's theorem*. How would you try to prove this? (No full proof required.)

Problem 1.15. (Schwartz representation theorem) Show that, for any $p \in \mathbb{R}$ the map

$$R_p \colon \mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto (1 + |x|^2)^{-p/2}\mathcal{F}^{-1}[(1 + |\xi|^2)^{-p/2}\mathcal{F}\phi] \in \mathcal{S}(\mathbb{R}^n) \tag{1.111}$$

is an isomorphism and, using Problem 1.11 or otherwise,

$$p \geq n + 1 + k \Longrightarrow \|R_p\phi\|_k \leq C_k\|\phi\|_{L^2},\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.112}$$

Let $R_p^t \colon \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ be the dual map (defined by $R_p^t u(\phi) = u(R_p\phi)$). Show that R_p^t is an isomorphism and that if $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|u(\phi)| \leq C\|\phi\|_k,\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.113}$$

then $R_p^t u \in L^2(\mathbb{R}^n)$, if $p \geq n+1+k$, in the sense that it is in the image of the map in Problem 1.10. Using Problem 1.11 show that $R_{n+1}(R^t_{n+1+k}u)$ is bounded and continuous and hence that

$$u = \sum_{|\alpha|+|\beta|\leq 2n+2+k} x^\beta D^\alpha u_{\alpha,\beta} \tag{1.114}$$

for some bounded continuous functions $u_{\alpha,\beta}$.

Problem 1.16. (Schwartz kernel theorem.) Show that any continuous linear operator

$$T : \mathcal{S}(\mathbb{R}^m_y) \longrightarrow \mathcal{S}'(\mathbb{R}^n_x)$$

extends to a continuous linear operator

$$T : (1 + |y|^2)^{-k/2}H^k(\mathbb{R}^m_y) \longrightarrow (1 + |x|^2)^{-q/2}H^q(\mathbb{R}^n_x)$$

for some k and q. Deduce that the operator

$$\tilde{T} = (1 + |D_x|^2)^{(-n-1-q)/2}(1 + |x|^2)^{q/2} \circ T \circ (1 + |y|^2)^{k/2}(1 + |D|^2)^{-k/2} : L^2(\mathbb{R}^m) \longrightarrow \mathcal{C}_\infty(\mathbb{R}^n)$$

is continuous with values in the bounded continuous functions on $\mathbb{R}^n$. Deduce that $\tilde{T}$ has Schwartz kernel in $\mathcal{C}_\infty(\mathbb{R}^n; L^2(\mathbb{R}^m)) \subset \mathcal{S}'(\mathbb{R}^{n+m})$ and hence that T itself has a tempered Schwartz kernel.

PROBLEM 1.17. Radial compactification and symbols.

PROBLEM 1.18. Series of problems discussing double polyhomogeneous symbols.

CHAPTER 2

Pseudodifferential operators on Euclidean space

Formula (1.92) for the action of a differential operator (with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$) on $\mathcal{S}(\mathbb{R}^n)$ can be written

$$
\begin{aligned}
P(x,D)u &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)u(y)dyd\xi \\
&= (2\pi)^{-n}\int e^{ix\cdot\xi}P(x,\xi)\hat{u}(\xi)d\xi
\end{aligned}
\tag{2.1}
$$

where $\hat{u}(\xi) = \mathcal{F}u(\xi)$ is the Fourier transform of u. We shall generalize this formula by generalizing $P(x,\xi)$ from a polynomial in ξ to a *symbol*, which is to say a smooth function satisfying certain uniformity conditions at infinity. In fact we shall also allow the symbol, or rather the *amplitude*, in the integral (2.1) to depend in addition on the 'incoming' variables, y :

$$
A(x,D)u = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi, u\in\mathcal{S}(\mathbb{R}^n). \tag{2.2}
$$

Of course it is not immediately clear that this integral is well-defined.

To interpret (2.2) we first look into the definition and properties of symbols. Then we show how this integral can be interpreted as an oscillatory integral and that it thereby defines an operator on $\mathcal{S}(\mathbb{R}^n)$. We then investigate the properties of these *pseudodifferential operators* at some length.

2.1. Symbols

A polynomial, p, in ξ, of degree at most m, satisfies a bound

$$
|p(\xi)| \le C(1+|\xi|)^m \quad \forall\ \xi\in\mathbb{R}^n. \tag{2.3}
$$

Since successive derivatives, $D^\alpha_\xi p(\xi)$, are polynomials of degree $m-|\alpha|$, for any multiindex α, we get the family of estimates

$$
|D^\alpha_\xi p(\xi)| \le C_\alpha(1+|\xi|)^{m-|\alpha|} \quad \forall\ \xi\in\mathbb{R}^n, \alpha\in\mathbb{N}^n_0. \tag{2.4}
$$

Of course if $|\alpha| > m$ then $D^\alpha_\xi p\equiv 0$, so we can even take the constants C_α to be independent of α. If we consider the characteristic polynomial $P(x,\xi)$ of a differential operator of order m with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ (i.e. all derivatives of the coefficients are bounded) (2.4) is replaced by

$$
\left|D^\alpha_x D^\beta_\xi P(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall\ (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,\ \alpha,\beta\in\mathbb{N}^n_0. \tag{2.5}
$$

There is no particular reason to have the same number of x variables as of ξ variables, so in general we define:

DEFINITION 2.1. *The space* $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ *of symbols of order* m *(with coefficients in* $\mathcal{C}^\infty_\infty(\mathbb{R}^p)$*) consists of those functions* $a \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ *satisfying all the estimates*

$$\left|D^\alpha_z D^\beta_\xi a(z,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathbb{R}^p \times \mathbb{R}^n \quad \forall\ \alpha \in \mathbb{N}^p_0, \beta \in \mathbb{N}^n_0. \tag{2.6}$$

For later reference we even define $S^m_\infty(\Omega;\mathbb{R}^n)$ *when* $\Omega \subset \mathbb{R}^p$ *and* $\Omega \subset \operatorname{clos}(\operatorname{int}(\Omega))$ *as consisting of those* $a \in \mathcal{C}^\infty(\operatorname{int}(\Omega)\times\mathbb{R}^n)$ *satisfying* (2.6) *for* $(z,\xi) \in \operatorname{int}(\Omega)\times\mathbb{R}^n$.

The estimates (2.6) can be rewritten

$$\left\|a\right\|_{N,m} = \sup_{\substack{z\in\operatorname{int}(\Omega)\\ \xi\in\mathbb{R}^n}} \max_{|\alpha|+|\beta|\le N} (1+|\xi|)^{-m+|\beta|}\left|D^\alpha_z D^\beta_\xi a(z,\xi)\right| < \infty. \tag{2.7}$$

With these *norms* $S^m_\infty(\Omega;\mathbb{R}^n)$ is a Fréchet space, rather similar in structure to $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Thus the topology is given by the metric

$$d(a,b) = \sum_{N\ge 0} 2^{-N}\frac{\|a-b\|_{N,m}}{1+\|a-b\|_{N,m}},\ a,b \in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.8}$$

The subscript '∞' here is *not* standard notation. It refers to the assumption of uniform boundedness of the derivatives of the 'coefficients'. More standard notation would be just $S^m(\Omega\times\mathbb{R}^n)$, especially for $\Omega = \mathbb{R}^p$, but I think this is too confusing.

A more significant issue is: Why this class precisely? As we shall see below, there are other choices which are not only possible but even profitable to make. However, the present one has several virtues. It is large enough to cover most of the straightforward things we want to do (at least initially) and small enough to 'work' easily. It leads to what I shall refer to as the 'traditional' algebra of pseudodifferential operators.

Now to some basic properties. First notice that

$$(1+|\xi|)^m \le C(1+|\xi|)^{m'}\ \forall\ \xi \in \mathbb{R}^n \Longleftrightarrow m \le m'. \tag{2.9}$$

Thus we have an inclusion

$$S^m_\infty(\Omega;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\Omega;\mathbb{R}^n)\ \forall\ m' \ge m. \tag{2.10}$$

Moreover this inclusion is continuous, since from (2.7), $\|a\|_{N,m'} \le \|a\|_{N,m}$ if $a \in S^m(\Omega;\mathbb{R}^n)$ and $m' \ge m$. Since these spaces increase with m we think of them as a *filtration* of the big space

$$S^\infty_\infty(\Omega;\mathbb{R}^n) = \bigcup_m S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.11}$$

Notice that the two '∞s' here are quite different. The subscript refers to the fact that the 'coefficients' are bounded and stands for L^∞ whereas the superscript '∞' stands really for $\mathbb{R}$. The *residual* space of this filtration is

$$S^{-\infty}_\infty(\Omega;\mathbb{R}^n) = \bigcap_m S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.12}$$

In fact the inclusion (2.10) is *never* dense if $m' > m$. Instead we have the following rather technical, but nevertheless very useful, result.

LEMMA 2.1. *For any* $m \in \mathbb{R}$ *and any* $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ *there is a sequence in* $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ *which is bounded in* $S^m_\infty(\Omega;\mathbb{R}^n)$ *and converges to* a *in the topology of* $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ *for any* $m' > m$; *in particular* $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ *is dense in the space* $S^m_\infty(\Omega;\mathbb{R}^n)$ *in the topology of* $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ *for* $m' > m$.

The reason one cannot take $m' = m$ here is essentially the same reason that underlies the fact that $\mathcal{S}(\mathbb{R}^n)$ is not dense in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Namely any uniform limit obtained from a converging Schwartz sequence must vanish at infinity. In particular the constant function $1 \in S^0_\infty(\mathbb{R}^p;\mathbb{R}^n)$ cannot be in the closure in this space of $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ if $n > 0$.

Proof. Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $0 \le \phi(\xi) \le 1$, $\phi(\xi) = 1$ if $|\xi| < 1, \phi(\xi) = 0$ if $|\xi| > 2$ and consider the sequence

$$a_k(z,\xi) = \phi(\xi/k)a(z,\xi), \quad a \in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.13}$$

We shall show that $a_k \in S^{-\infty}_\infty(\Omega,\mathbb{R}^n)$ is a bounded sequence in $S^m_\infty(\Omega;\mathbb{R}^n)$ and that $a_k \longrightarrow a$ in $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m' > m$. Certainly for each N

$$|a_k(z,\xi)| \le C_{N,k}(1+|\xi|)^{-N} \tag{2.14}$$

since ϕ has compact support. Leibniz' formula gives

$$D^\alpha_z D^\beta_\xi a_k(z,\xi) = \sum_{\beta' \le \beta} \binom{\beta'}{\beta} k^{-|\beta'|}(D^{\beta'}\phi)(\xi/k) D^\alpha_z D^{\beta-\beta'}_\xi a(z,\xi). \tag{2.15}$$

On the support of $\phi(\xi/k)$, $|\xi| \le k$ so, using the symbol estimates on a, it follows that a_k is bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$. We easily conclude that

$$\left|D^\alpha_z D^\beta_\xi a_k(z,\xi)\right| \le C_{N,\alpha,\beta,k}(1+|\xi|)^{-N} \quad \forall\ \alpha,\beta,N,k. \tag{2.16}$$

Thus $a_k \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$.

So consider the difference

$$(a - a_k)(z,\xi) = (1-\phi)(\xi/k)\ a(z,\xi). \tag{2.17}$$

Now, $|(1-\phi)(\xi/k)| = 0$ in $|\xi| \le k$ so we only need estimate the difference in $|\xi| \ge k$ where this factor is bounded by 1. In this region $1 + |\xi| \ge 1 + k$ so, since $-m' + m < 0$,

$$\begin{aligned}(1+|\xi|)^{-m'}\left|(a-a_k)(z,\xi)\right| \le \\ (1+k)^{-m'+m}\sup_{z,\xi}|(1+|\xi|)^{-m}|a(z,\xi)| \le (1+k)^{-m'+m}\|a\|_{0,m} \longrightarrow 0.\end{aligned} \tag{2.18}$$

This is convergence with respect to the first symbol norm.

Next consider the ξ derivatives of (2.17). Using Leibniz' formula

$$\begin{aligned}D^\beta_\xi(a-a_k) &= \sum_{\gamma\le\beta}\binom{\beta}{\gamma} D^{\beta-\gamma}_\xi(1-\phi)(\frac{\xi}{k})\cdot D^\gamma_\xi a(z,\xi) \\ &= (1-\phi)(\frac{\xi}{k})\cdot D^\beta_\xi a(z,\xi) - \sum_{\gamma<\beta}\binom{\beta}{\gamma}\left(D^{\beta-\gamma}\phi\right)(\frac{\xi}{k})\cdot k^{-|\beta-\gamma|}D^\gamma_\xi a(z,\xi).\end{aligned}$$

In the first term, $D^\beta_\xi a(z,\xi)$ is a symbol of order $m - |\beta|$, so by the same argument as above

$$\sup_\xi(1+|\xi|)^{-m'+|\beta|}|(1-\phi)(\frac{\xi}{k})D^\beta_\xi a(x,\xi)| \longrightarrow 0 \tag{2.19}$$

as $k \longrightarrow \infty$ if $m' > m$. In all the other terms, $(D^{\beta-\gamma}\phi)(\zeta)$ has compact support, in fact $1 \le |\zeta| \le 2$ on the support. Thus for each term we get a bound

$$\sup_{k\le|\xi|\le 2k}(1+|\xi|)^{-m'+|\beta|}\cdot k^{-|\beta-\gamma|}C\cdot(1+|\xi|)^{m-|\gamma|} \le Ck^{-m'+m}. \tag{2.20}$$

The variables z play the rôle of parameters so we have in fact shown that

$$\sup_{\substack{z\in\Omega\\ \xi\in\mathbb{R}^n}} (1+|\xi|)^{-m'+|\beta|}\left|D_z^\alpha D_\xi^\beta(a-a_k)\right| \longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{2.21}$$

This means $a_k \longrightarrow a$ in each of the symbol norms, and hence in the topology of $S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ as desired. $\square$

In fact this proof suggests a couple of other 'obvious' results. Namely

$$S^m_\infty(\Omega;\mathbb{R}^n)\cdot S^{m'}_\infty(\Omega;\mathbb{R}^n)\subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n). \tag{2.22}$$

This can be proved directly using Leibniz' formula:

$$\begin{aligned}\sup_\xi(1+|\xi|)^{-m-m'+|\beta|}&\left|D_z^\alpha D_\xi^\beta\left(a(z,\xi)\cdot b(z,\xi)\right)\right|\\ \le\sum_{\substack{\mu\le\alpha\\ \gamma\le\beta}}\binom{\alpha}{\mu}\binom{\beta}{\gamma}&\sup_\xi(1+|\xi|)^{-m+|\gamma|}\left|D_z^\mu D_\xi^\gamma a(z,\xi)\right|\\ &\times\sup_\xi(1+|\xi|)^{-m'+|\beta-\gamma|}\left|D_z^{\alpha-\mu}D_\xi^{\beta-\gamma}b(z,\xi)\right|<\infty.\end{aligned}$$

We also note the action of differentiation:

$$\begin{aligned}D_z^\alpha &: S^m_\infty(\Omega;\mathbb{R}^n)\longrightarrow S^m_\infty(\Omega;\mathbb{R}^n)\text{ and}\\ D_\xi^\beta &: S^m_\infty(\Omega;\mathbb{R}^n)\longrightarrow S^{m-|\beta|}_\infty(\Omega;\mathbb{R}^n).\end{aligned} \tag{2.23}$$

In fact, while we are thinking about these things we might as well show the important consequence of *ellipticity*. A symbol $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ is said to be (globally) elliptic if

$$|a(z,\xi)|\ge\epsilon(1+|\xi|)^m - C(1+|\xi|)^{m-1},\ \epsilon>0 \tag{2.24}$$

or equivalently[1]

$$|a(z,\xi)|\ge\epsilon(1+|\xi|)^m \text{ in } |\xi|\ge C_\epsilon,\ \epsilon>0. \tag{2.25}$$

LEMMA 2.2. *If $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ is elliptic there exists $b\in S^{-m}_\infty(\Omega;\mathbb{R}^n)$ such that*

$$a\cdot b-1\in S^{-\infty}_\infty(\Omega;\mathbb{R}^n). \tag{2.26}$$

PROOF. Using (2.25) choose ϕ as in the proof of Lemma 2.1 and set

$$b(z,\xi)=\begin{cases}\frac{1-\phi(\xi/2C)}{a(z,\xi)} & |\xi|\ge C\\ 0 & |\xi|\le C.\end{cases} \tag{2.27}$$

Then b is $\mathcal{C}^\infty$ since $b=0$ in $C\le|\xi|\le C+\delta$ for some $\delta>0$. The symbol estimates follow by noting that, in $|\xi|\ge C$,

$$D_z^\alpha D_\xi^\beta b = a^{-1-|\alpha|-|\beta|}\cdot G_{\alpha\beta} \tag{2.28}$$

where $G_{\alpha\beta}$ is a symbol of order $(|\alpha|+|\beta|)m-|\beta|$. This may be proved by induction. Indeed, it is true when $\alpha=\beta=0$. Assuming (2.28) for some α and β, differentiation of (2.28) gives

$$\begin{aligned}D_{z_j}D_z^\alpha D_\xi^\beta b = D_{z_j}a^{-1-|\alpha|-|\beta|}\cdot G_{\alpha\beta} &= a^{-2-|\alpha|-|\beta|}G',\\ G' &= (-1-|\alpha|-|\beta|)(D_{z_j}a)G_{\alpha\beta}+aD_{z_j}G_{\alpha\beta}.\end{aligned}$$

[1]Note it is required that ϵ be chosen to be independent of z here, so this is a notion of uniform ellipticity.

By the inductive hypothesis, G' is a symbol of order $(|\alpha|+1+|\beta|)m-|\beta|$. A similar argument applies to derivatives with respect to the ξ variables. $\square$

2.2. Pseudodifferential operators

Now we proceed to discuss the formula (2.2) where we shall assume that, for some $w, m \in \mathbb{R}$,

$$a(x,y,\xi) = (1+|x-y|^2)^{w/2}\tilde{a}(x,y,\xi) \quad \tilde{a} \in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi). \tag{2.29}$$

The extra 'weight' factor (which allows polynomial growth in the direction of $x-y$) turns out, somewhat enigmatically, to both make no difference and be very useful! Notice[2] that if $a \in \mathcal{C}^\infty(\mathbb{R}^{2n}\times\mathbb{R}^n)$ then $a \in (1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|)^w(1+|\xi|)^{m-|\gamma|} \ \forall\ \alpha,\beta,\gamma \in \mathbb{N}^n_0. \tag{2.30}$$

If $m < -n$ then, for each $u \in \mathcal{S}(\mathbb{R}^n)$, the integral in (2.2) is absolutely convergent, locally uniformly in x, since

$$\begin{aligned}|a(x,y,\xi)u(y)| &\le C(1+|x-y|)^w(1+|\xi|)^m(1+|y|)^{-N}\\ &\le C(1+|x|)^w(1+|\xi|)^m(1+|y|)^m,\ m<-n.\end{aligned} \tag{2.31}$$

Here we have used the following simple consequence of the triangle inequality

$$(1+|x-y|) \le (1+|x|)(1+|y|)$$

from which it follows that

$$(1+|x-y|)^w \le \begin{cases}(1+|x|)^w(1+|y|)^w & \text{if } w>0\\ (1+|x|)^w(1+|y|)^{-w} & \text{if } w\le 0.\end{cases} \tag{2.32}$$

Thus we conclude that, provided $m<-n$,

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow (1+|x|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^n). \tag{2.33}$$

To show that, for general m, A exists as an operator, we prove that its Schwartz kernel exists.

Proposition 2.1. *The map, defined for $m<-n$ as a convergent integral,*

$$\begin{aligned}&(1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto I(a) =\\ &\qquad (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi \in (1+|x|^2+|y|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^{2n})\end{aligned} \tag{2.34}$$

extends by continuity to

$$I : (1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n}) \tag{2.35}$$

for each w, $m \in \mathbb{R}$ in the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $m'>m$.

[2] See Problem 2.5.

PROOF. Since we already have the density of $S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ in the toplogy of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $m'>m$, we only need to show the continuity of the map (2.34) on this residual subspace with respect to the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any m', which we may as well write as m. What we shall show is that, for each $w,m\in\mathbb{R}$, there are integers $N,k\in\mathbb{N}$ such that, in terms of the norms in (2.7) and (1.6)

$$\begin{aligned}\big|I(a)(\phi)\big| &\le C\|\tilde a\|_{N,m}\|\phi\|_k \quad \forall\ \phi\in\mathcal{S}(\mathbb{R}^{2n}),\\ &\qquad a=(1+|x-y|^2)^{w/2}\tilde a,\ \tilde a\in S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n).\end{aligned}\tag{2.36}$$

To see this we just use integration by parts.

Set $\tilde\phi(x,y)=(1+|x-y|^2)^{w/2}\phi(x,y)$. Observe that

$$\begin{aligned}(1+\xi\cdot D_x)e^{i(x-y)\cdot\xi}&=(1+|\xi|^2)e^{i(x-y)\cdot\xi}\\ (1-\xi\cdot D_y)e^{i(x-y)\cdot\xi}&=(1+|\xi|^2)e^{i(x-y)\cdot\xi}.\end{aligned}$$

Thus we can write, for $\tilde a\in S^{-\infty}_\infty$, with $a=(1+|x-y|^2)^{w/2}\tilde a$ and for any $q\in\mathbb{N}$

$$\begin{aligned}I(a)(\phi)&=\iint(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-2q}\\ &\qquad(1-\xi\cdot D_x)^q(1+\xi\cdot D_y)^q\big[\tilde a(x,y,\xi)\tilde\phi(x,y)\big]d\xi dxdy\\ &=\sum_{|\gamma|\le 2q}\iint\Big(\int e^{i(x-y)\cdot\xi}a^{(q)}_\gamma(x,y,\xi)d\xi\Big)D^\gamma_{(x,y)}\tilde\phi(x,y)dxdy.\end{aligned}\tag{2.37}$$

Here the $a^{(q)}_\gamma$ arise by expanding the powers of the operator

$$(1-\xi\cdot D_x)^q(1+\xi\cdot D_x)^q=\sum_{|\mu|,|\nu|\le q}C_{\mu,\nu}\xi^{\mu+\nu}D^\mu_xD^\nu_y$$

and applying Leibniz' formula. Thus $a^{(q)}_\gamma$ arises from terms in which $2q-|\gamma|$ derivatives act on $\tilde a$ so it is of the form

$$\begin{aligned}a_\gamma&=(1+|\xi|^2)^{-2q}\sum_{|\mu|\le|\gamma|,|\gamma|\le 2q}C_{\mu,\gamma}\xi^\gamma D^\mu_{(x,y)}\tilde a\\ &\Longrightarrow\|a_\gamma\|_{N,m}\le C_{m,q,N}\|\tilde a\|_{N+2q,m+2q}\ \forall\ m,N,q.\end{aligned}$$

So (for given m) if we take $-2q+m<-n$, e.g. $q>\max(\frac{n+m}{2},0)$ and use the integrability of $(1+|x|+|y|)^{-2n-1}$ on $\mathbb{R}^{2n}$, then

$$\big|I(a)(\phi)\big|\le C\|\tilde a\|_{2q,m}\|\tilde\phi\|_{2q+2n+1}\le C\|\tilde a\|_{2q,m}\|\phi\|_{2q+w+2n+1}.\tag{2.38}$$

This is the estimate (2.36), which proves the desired continuity. □

In showing the existence of the Schwartz' kernel in this proof we do *not* really need to integrate by parts in both x *and* y; either separately will do the trick. We can use this observation to show that these pseudodifferential operator act on $\mathcal{S}(\mathbb{R}^n)$.

LEMMA 2.3. *If* $a\in(1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ *then the operator* A, *with Schwartz kernel* $I(a)$, *is a continuous linear map*

$$A:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n).\tag{2.39}$$

We shall denote by $\Psi^m_\infty(\mathbb{R}^n)$ the linear space of operators (2.39), corresponding to $(1+|x-y|^2)^{-w/2}a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w. I call them *pseudodifferential operators 'of traditional type' – or type '1,0'.*[3]

Proof. Proceeding as in (2.37) but only integrating by parts in y we deduce that, for q large depending on m,

$$Au(\psi) = \sum_{\gamma\le 2q} (2\pi)^{-n} \iint\int e^{i(x-y)\cdot\xi} a_\gamma(x,y,\xi) D^\gamma_y u(y) d\xi \psi(x) dy dx,$$

$$a_\gamma \in (1+|x-y|^2)^{w/2} S^{m-q}(\mathbb{R}^{2n};\mathbb{R}^n) \text{ if } a \in (1+|x-y|^2)^{w/2} S^m(\mathbb{R}^{2n};\mathbb{R}^n).$$

The integration by parts is justified by continuity from $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. Taking $-q+m<-n-|w|$, this shows that Au is given by the convergent integral

$$Au(x) = \sum_{\gamma\le 2q} (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a_\gamma(x,y,\xi) D^\gamma_y u(y) d\xi dy, \tag{2.40}$$

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow (1+|x|^2)^{\frac{|w|}{2}} \mathcal{C}^0_\infty(\mathbb{R}^n)$$

which is really just (2.33) again. Here $\mathcal{C}^0_\infty(\mathbb{R}^n)$ is the Banach space of bounded continuous functions on $\mathbb{R}^n$, with the supremum norm. The important point is that the weight depends on w but not on m. Notice that

$$D_{x_j} Au(x) = (2\pi)^{-n} \sum_{|\gamma|\le 2q} \iint e^{i(x-y)\cdot\xi} \big(\xi_j + D_{x_j}\big) a_\gamma \cdot D^\gamma_y u(y) dy d\xi$$

and

$$x_j Au(x) = (2\pi)^{n} \sum_{|\gamma|\le 2q} \iint e^{i(x-y)\cdot\xi} \big(-D_{\xi_j} + y_j\big) a_\gamma \cdot D^\gamma_y u(y) dy d\xi.$$

Proceeding inductively (2.39) follows from (2.33) or (2.40) since we conclude that

$$x^\alpha D^\beta_x Au \in (1+|x|^2)^{\frac{|w|}{2}} \mathcal{C}^0_\infty(\mathbb{R}^n),\ \forall\ \alpha,\beta \in \mathbb{N}^n_0$$

and this implies that $Au \in \mathcal{S}(\mathbb{R}^n)$. □

2.3. Composition

There are two extreme cases of $I(a)$, namely where a is independent of either x or of y. Below we shall prove:

Theorem 2.1 (Reduction). *Each $A \in \Psi^m_\infty(\mathbb{R}^n)$ can be written uniquely as $I(a')$ where $a' \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$.*

This is the main step in proving the fundamental result of this Chapter, which is that two pseudodifferential operators can be composed to give a pseudodifferential operator and that the orders are additive. Thus our aim is to demonstrate the fundamental

Theorem 2.2. *[Composition] The space $\Psi^\infty_\infty(\mathbb{R}^n)$ is an order-filtered $*$-algebra on $\mathcal{S}(\mathbb{R}^n)$.*

[3]The meaning of which is explained in Problem 2.16.

We have already shown that each $A \in \Psi^{\infty}_{\infty}(\mathbb{R}^n)$ defines a continuous linear map (2.39). We now want to show that

$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A^* \in \Psi^m_\infty(\mathbb{R}^n) \tag{2.41}$$

$$A \in \Psi^m_\infty(\mathbb{R}^n),\ B \in \Psi^{m'}_\infty(\mathbb{R}^n) \Longrightarrow A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n), \tag{2.42}$$

since this is what is meant by an order-filtered (the orders add on composition) $*$-algebra (meaning (2.41) holds). In fact we will pick up some more information along the way.

2.4. Reduction

We proceed to prove Theorem 2.1, which we can restate as:

PROPOSITION 2.2. *The range of* (2.34) *(for any w) is the same as the range of I restricted to the image of the inclusion map*

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto a(x,\xi) \in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n).$$

PROOF. Suppose $a \in \big(1+|x-y|^2\big)^{w/2} S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w, then

$$I\big((x_j-y_j)a\big) = I\big(-D_{\xi_j}a\big)\ j=1,\dots,n. \tag{2.43}$$

Indeed this is just the result of inserting the identity

$$D_{\xi_j} e^{i(x-y)\cdot\xi} = (x_j-y_j)e^{i(x-y)\cdot\xi}$$

into (2.34) and integrating by parts. Since both sides of (2.43) are continuous on $\big(1+|x-y|^2\big)^{w/2} S^{\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ the identity holds in general. Notice that if a is of order m then $D_{\xi_j}a$ is of order $m-1$, so (2.43) shows that even though the operator with amplitude $(x_j-y_j)a(x,y,\xi)$ *appears* to have order m, it actually has order $m-1$.

To exploit (2.43) consider the Taylor series (with Legendre's remainder) for $a(x,y,\xi)$ around $x=y$:

$$\begin{aligned} a(x,y,\xi) = \sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \big(D^\alpha_y a\big)(x,x,\xi) \\ + \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \cdot R_{N,\alpha}(x,y,\xi). \end{aligned} \tag{2.44}$$

Here,

$$R_{N,\alpha}(x,y,\xi) = \int_0^1 (1-t)^{N-1}\big(D^\alpha_y a\big)(x,(1-t)x+ty,\xi)dt. \tag{2.45}$$

Now,

$$(x-y)^\alpha\big(D^\alpha_y a\big)(x,y,\xi) \in \big(1+|x-y|^2\big)^{\frac{(w+|\alpha|)}{2}} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n). \tag{2.46}$$

Applying (2.43) repeatedly we see that if A is the operator with kernel $I(a)$ then

$$A = \sum_{j=0}^{N-1} A_j + R_N, \quad A_j \in \Psi^{m-j}_\infty(\mathbb{R}^n),\ R_N \in \Psi^{m-N}_\infty(\mathbb{R}^n) \tag{2.47}$$

where the A_j have kernels

$$I\big(\sum_{|\alpha|=j}\frac{i^{|\alpha|}}{\alpha!}\big(D_y^\alpha D_\xi^\alpha a\big)(x,x,\xi)\big). \tag{2.48}$$

To proceed further we need somehow to *sum* this series. Of course we cannot really do this, but we can come close!

2.5. Asymptotic summation

Suppose $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. The fact that the *orders* are decreasing means that these symbols are getting very small, for $|\xi|$ large. The infinite series

$$\sum_j a_j(z,\xi) \tag{2.49}$$

need not converge. However we shall say that it converges asymptotically, or since it is a series we say it is 'asymptotically summable,' if there exists $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ such that,

$$\text{for every } N,\ a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n). \tag{2.50}$$

We write this relation as

$$a \sim \sum_{j=0}^{\infty} a_j. \tag{2.51}$$

Proposition 2.3. *Any series $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is asymptotically summable, in the sense of* (2.50)*, and the asymptotic sum is well defined up to an additive term in $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.*

Proof. The uniqueness part is easy. Suppose a and a' both satisfy (2.50). Taking the difference

$$a - a' = \big(a - \sum_{j=0}^{N-1} a_j\big) - \big(a' - \sum_{j=0}^{N-1} a_j\big) \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n). \tag{2.52}$$

Since $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is just the intersection of the $S^{-N}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ over N it follows that $a - a' \in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$, proving the uniqueness.

So to the existence of an asymptotic sum. To construct this (by Borel's method) we cut off each term 'near infinity in ξ'. Thus fix $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\phi(\xi) = 0$ in $|\xi| \le 1$, $\phi(\xi) = 1$ in $|\xi| \ge 2$, $0 \le \phi(\xi) \le 1$. Consider a decreasing sequence

$$\epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \downarrow 0. \tag{2.53}$$

We shall set

$$a(z,\xi) = \sum_{j=0}^{\infty} \phi(\epsilon_j\xi)a_j(z,\xi). \tag{2.54}$$

Since $\phi(\epsilon_j\xi) = 0$ in $|\xi| < 1/\epsilon_j \to \infty$ as $j \to \infty$, only finitely many of these terms are non-zero in any ball $|\xi| \le R$. Thus $a(z,\xi)$ is a well-defined $\mathcal{C}^\infty$ function. Of course we need to consider the seminorms, in $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, of each term.

The first of these is

$$\sup_z \sup_\xi (1+|\xi|)^{-m} \left|\phi(\epsilon_j \xi)\right| \left|a_j(z,\xi)\right|. \tag{2.55}$$

Now $|\xi| \geq \frac{1}{\epsilon_j}$ on the support of $\phi(\epsilon_j\xi)a_j(z,\xi)$ and since a_j is a symbol of order $m-j$ this allows us to estimate (2.55) by

$$\sup_z \sup_{|\xi|\geq\frac{1}{\epsilon_j}} (1+|\xi|)^{-j} \cdot \left[\bigl(1+|\xi|\bigr)^{-m+j}\left|a_j(z,\xi)\right|\right]$$
$$\leq \bigl(1+\frac{1}{\epsilon_j}\bigr)^{-j} \cdot C_j \leq \epsilon_j^j \cdot C_j$$

where the C_j's are *fixed* constants, independent of ϵ_j.

Let us look at the higher symbol estimates. As usual we can apply Leibniz' formula:

$$\sup_z \sup_\xi (1+|\xi|)^{-m+|\beta|}\left|D_z^\alpha D_\xi^\beta \phi(\epsilon_j\xi)a_j(z,\xi)\right|$$
$$\leq \sum_{\mu\leq\beta} \sup_z \sup_\xi (1+|\xi|)^{|\beta|-|\mu|-j}\epsilon_j^{|\beta|-|\mu|}\left|\bigl(D^{\beta-\mu}\phi\bigr)(\epsilon_j\xi)\right|$$
$$\times(1+|\xi|)^{-m+j+|\mu|}\left|D_z^\alpha D_\xi^\mu a_j(z,\xi)\right|.$$

The term with $\mu=\beta$ we estimate as before and the others, with $\mu\neq\beta$ are supported in $\frac{1}{\epsilon_j}\leq|\xi|\leq\frac{2}{\epsilon_j}$. Then we find that for all j

$$\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m} \leq C_{N,j}\epsilon_j^j \tag{2.56}$$

where $C_{N,j}$ is independent of ϵ_j.

So we see that for each given N we can arrange that, for instance,

$$\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m} \leq C_N\frac{1}{j^2}$$

by choosing the ϵ_j to satify

$$C_{N,j}\epsilon_j^j \leq \frac{1}{j^2}\ \forall\ j\geq j(N).$$

Notice the crucial point here, we can arrange that for *each* N the sequence of norms in (2.56) is dominated by $C_N j^{-2}$ by fixing $\epsilon_j < \epsilon_{j,N}$ for large j. Thus we can arrange convergence of *all* the sums

$$\sum_j \|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m}$$

by diagonalization, for example setting $\epsilon_j = \frac{1}{2}\epsilon_{j,j}$. Thus by choosing $\epsilon_j \downarrow 0$ rapidly enough we ensure that the series (2.54) converges. In fact the same argument allows us to ensure that for every N

$$\sum_{j\geq N}\phi(\epsilon_j\xi)a_j(z,\xi) \text{ converges in } S_\infty^{m-N}(\mathbb{R}^p;\mathbb{R}^n). \tag{2.57}$$

This certainly gives (2.50) with a defined by (2.54). $\square$

2.6. Residual terms

Now we can apply Proposition 2.3 to the series in (2.48), that is we can find $b \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$ satisfying

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\big(D^\alpha_y a\big)(x,x,\xi). \tag{2.58}$$

Let $B = I(b)$ be the operator defined by this amplitude (which is independent of y). Now (2.47) says that

$$A - B = \sum_{j=0}^{N-1} A_j + R_N - B$$

and from (2.50) applied to (2.58)

$$B = \sum_{j=0}^{N-1} A_j + R'_N,\ \ R'_N \in \Psi^{m-N}_\infty(\mathbb{R}^n)$$

Thus

$$A - B \in \Psi^{-\infty}_\infty(\mathbb{R}^n) = \bigcap_N \Psi^N_\infty(\mathbb{R}^n). \tag{2.59}$$

Notice that, at this stage, we do *not* know that $A-B$ has kernel $I(c)$ with $c \in S^{-\infty}_\infty(\mathbb{R}^{2n},\mathbb{R}^n)$, just that it has kernel $I(c_N)$ with $c_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for each N.

However:

PROPOSITION 2.4. *An operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an element of the space $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies the estimates*

$$\big|D^\alpha_x D^\beta_y K(x,y)\big| \le C_{N,\alpha,\beta}(1+|x-y|)^{-N}\ \forall\ \alpha,\beta,N. \tag{2.60}$$

PROOF. Suppose first that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$, which means that $A \in \Psi^N_\infty(\mathbb{R}^n)$ for every N. The Schwartz kernel, K_A, of A is therefore given by (2.34) with the amplitude $a_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. For $N << -n-1-p$ the integral converges absolutely and we can integrate by parts to show that

$$(x-y)^\alpha D^\beta_x D^\gamma_y K_A(x,y)$$
$$= (2\pi)^{-N}\int e^{i(x-y)\cdot\xi}(-D_\xi)^\alpha(D_x+i\xi)^\beta(D_y-i\xi)^\gamma a_N(x,y,\xi)d\xi$$

which converges absolutely, and uniformly in x,y, provided $|\beta|+|\gamma|+N-|\alpha| < -n$. Thus

$$\sup\big|(x-y)^\alpha D^\beta_x D^\gamma_y K\big| < \infty\ \forall\ \alpha,\beta,\gamma$$

which is another way of writing (2.60) i.e.

$$\sup\big(1+|x-y|^2\big)^N\big|D^\beta_x D^\gamma_y K\big| < \infty\ \forall\ \beta,\gamma,N.$$

Conversely suppose that (2.60) holds. Define

$$g(x,z) = K(x,x-z). \tag{2.61}$$

The estimates (2.60) become

$$\sup\big|D^\alpha_x z^\gamma D^\beta_z g(x,z)\big| < \infty\ \forall\ \alpha,\beta,\gamma. \tag{2.62}$$

That is, g is rapidly decreasing with all its derivatives in z. Taking the Fourier transform,

$$b(x,\xi) = \int e^{-iz\cdot\xi} g(x,z)dz \tag{2.63}$$

the estimate (2.62) translates to

$$\begin{gathered}\sup_{x,\xi} \left| D_x^\alpha \xi^\beta D_\xi^\gamma b(x,\xi)\right| < \infty \ \forall \ \alpha,\beta,\gamma \\ \Longleftrightarrow b \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n).\end{gathered} \tag{2.64}$$

Now the inverse Fourier transform in (2.63), combined with (2.61) gives

$$K(x,y) = g(x,x-y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x,\xi)d\xi \tag{2.65}$$

i.e. $K = I(b)$. This certainly proves the proposition and actually gives the stronger result.

$$A \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \Longleftrightarrow A = I(c),\ c \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n). \tag{2.66}$$

□

This also finishes the proof of Proposition 2.2 since in (2.58), (2.59) we have shown that

$$A = B + R,\ B = I(b), \quad R \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \tag{2.67}$$

so in fact

$$A = I(e),\ e \in S_\infty^m(\mathbb{R}_x^n;\mathbb{R}_\xi^n),\ e \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\big(D_y^\alpha D_\xi^\alpha a\big)(x,x,\xi). \tag{2.68}$$

□

2.7. Proof of Composition Theorem

First consider the adjoint formula. If

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

the adjoint is the operator

$$A^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

defined by duality:

$$A^*u(\bar\phi) = u(\overline{A\phi}) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{2.69}$$

Certainly $A^*u \in \mathcal{S}'(\mathbb{R}^n)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ since

$$A^*u(\psi) = u(\overline{A\bar\psi}) \text{ and } \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \overline{A\bar\psi} \in \mathcal{S}(\mathbb{R}^n) \tag{2.70}$$

is clearly continuous. In terms of Schwartz kernels,

$$\begin{aligned} A\phi(x) &= \int K_A(x,y)\phi(y)dy,\ \phi \in \mathcal{S}(\mathbb{R}^n) \\ A^*u(x) &= \int K_{A^*}(x,y)u(y)dy,\ u \in \mathcal{S}(\mathbb{R}^n).\end{aligned} \tag{2.71}$$

We then see that

$$\begin{aligned}\int K_{A^*}(x,y)u(y)\overline{\phi(x)}dydx &= \int \overline{K_A(x,y)\phi(y)}dyu(x)dx \\ \Longrightarrow K_{A^*}(x,y) &= \overline{K_A(y,x)}\end{aligned} \tag{2.72}$$

where we are using the uniqueness of Schwartz' kernels.

This proves (2.41) since

$$\begin{aligned}\overline{K_A(y,x)} &= \overline{\big[\frac{1}{(2\pi)^n}\int e^{i(y-x)\cdot\xi}a(y,x,\xi)d\xi\big]} \\ &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\bar{a}(y,x,\xi)d\xi\end{aligned} \tag{2.73}$$

i.e. $A^* = I(\bar{a}(y,x,\xi))$. Thus one advantage of allowing general operators (2.34) is that closure under the passage to adjoint is immediate.

For the composition formula we need to apply Proposition 2.2 twice. First to $A \in \Psi^m_\infty(\mathbb{R}^n)$, to write it with symbol $a(x,\xi)$

$$\begin{aligned}A\phi(x) &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\phi(y)dyd\xi \\ &= (2\pi)^{-n}\int e^{ix\cdot\xi}a(x,\xi)\hat{\phi}(\xi)d\xi.\end{aligned}$$

Then we also apply Proposition 2.2 to B^*,

$$B^*u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\bar{b}(x,\xi)\hat{u}(\xi)d\xi.$$

Integrating this against a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ gives

$$\begin{aligned}\langle B\phi, u\rangle = \langle\phi, B^*u\rangle &= (2\pi)^{-n}\int\int e^{-ix\cdot\xi}\phi(x)b(x,\xi)\overline{\hat{u}(\xi)}d\xi dx \\ \Longrightarrow \widehat{B\phi}(\xi) &= \int e^{-iy\cdot\xi}b(y,\xi)\phi(y)dy.\end{aligned} \tag{2.74}$$

Inserting this into the formula for $A\phi$ shows that

$$\Longrightarrow AB(u) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)u(y)dyd\xi.$$

Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty\big(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi\big)$ this shows that $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as claimed.

2.8. Quantization and symbols

So, we have now shown that there is an 'oscillatory integral' interpretation of

$$K(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi = I(a) \tag{2.75}$$

which defines, for any $w \in \mathbb{R}$, a continuous linear map

$$I : (1+|x-y|^2)^{\frac{w}{2}}S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

the range of which is the space of pseudodifferential operators on $\mathbb{R}^n$;

$$\begin{gathered}A \in \Psi^m_\infty(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ \exists\ w \text{ s.t. } K_A(x,y) = I(a),\ a \in \left(1+|x-y|^2\right)^{\frac{w}{2}}S^m_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right).\end{gathered} \tag{2.76}$$

Furthermore, we have shown in Proposition 2.2 that the special case, $w = 0$ and $\partial_y a \equiv 0$, gives an isomorphism

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_L}{\overset{\sigma_L}{\rightleftarrows}} S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.77}$$

The map here, $q_L = I$ on symbols independent of y, is the *left quantization map* and its inverse σ_L is the *left full symbol map.* Next we consider some more consequences of this reduction theorem.

As well as the left quantization map leading to the isomorphism (2.77) there is a right quantization map, similarly derived from (2.75):

$$q_R(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(y,\xi)d\xi, \quad a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{2.78}$$

In fact using the adjoint operator, $*$, on operators and writing as well $*$ for complex conjugation of symbols shows that

$$q_R = * \cdot q_L \cdot * \tag{2.79}$$

is also an isomorphism, with inverse σ_R[4]

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_R}{\overset{\sigma_R}{\rightleftarrows}} S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{2.80}$$

Using the proof of the reduction theorem we find:

LEMMA 2.4. *For any* $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)$,

$$\sigma_L\left(q_R(a)\right)(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}D^\alpha_x D^\alpha_\xi a(x,\xi) \sim e^{i<D_x,D_\xi>}a. \tag{2.81}$$

For the moment the last asymptotic equality is just to help in remembering the formula, which is the same as given by the formal Taylor series expansion at the origin of the exponential.

PROOF. This follows from the general formula (2.68). □

2.9. Principal symbol

One important thing to note from (2.81) is that

$$D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-|\alpha|}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right) \tag{2.82}$$

so that for *any* pseudodifferential operator

$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow \sigma_L(A) - \sigma_R(A) \in S^{m-1}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{2.83}$$

For this reason we consider the general quotient spaces

$$S^{m-[1]}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right) = S^m_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right)\big/S^{m-1}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right) \tag{2.84}$$

and, for $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, write $[a]$ for its image, i.e. equivalence class, in the quotient space $S^{m-[1]}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right)$. The 'principal symbol map'

$$\begin{gathered}\sigma_m : \Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^{m-[1]}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)\\ \text{is defined by } \sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)].\end{gathered} \tag{2.85}$$

[4]This involves the left and right symbols, see Problem 5.1 for another the more centrist 'Weyl' quantization.

As distinct from σ_L or σ_R, σ_m depends on m, i.e. one needs to know that the order is at most m before it is defined.

The isomorphism (2.77) is replaced by a weaker (but very useful) exact sequence.

LEMMA 2.5. *For every $m \in \mathbb{R}$*

$$0 \hookrightarrow \Psi^{m-1}_\infty(\mathbb{R}^n) \hookrightarrow \Psi^m_\infty(\mathbb{R}^n) \xrightarrow{\sigma_m} S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow 0$$

is a short exact sequence (the 'principal symbol sequence' or simply the 'symbol sequence').

PROOF. This is just the statement that the range of each map is the null space of the next i.e. that σ_m is surjective, which follows from (2.77), and that the null space of σ_m is just $\Psi^{m-1}_\infty(\mathbb{R}^n)$ and this is again (2.77) and the definition of σ_m. $\square$

The fundamental result proved above is that

$$\Psi^m_\infty(\mathbb{R}^n)\cdot\Psi^{m'}_\infty(\mathbb{R}^n) \subset \Psi^{m+m'}_\infty(\mathbb{R}^n). \tag{2.86}$$

In fact we showed that if $A = q_L(a)$, $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $B = q_R(b)$, $b \in S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ then the composite operator has Schwartz kernel

$$K_{A\cdot B}(x,y) = I\left(a(x,\xi)b(y,\xi)\right)$$

Using the formula (2.68) again we see that

$$\sigma_L(A\cdot B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\big[a(x,\xi) D^\alpha_x b(x,\xi)\big]. \tag{2.87}$$

Of course $b = \sigma_R(B)$ so we really want to rewrite (2.87) in terms of $\sigma_L(B)$.

LEMMA 2.6. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ then $A\circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ and*

$$\sigma_{m+m'}(A\circ B) = \sigma_m(A)\cdot\sigma_{m'}(B), \tag{2.88}$$

$$\sigma_L(A\circ B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi \sigma_L(A)\cdot D^\alpha_x\sigma_L(B). \tag{2.89}$$

PROOF. The simple formula (2.88) is already immediate from (2.87) since all terms with $|\alpha| \ge 1$ are of order $m+m'-|\alpha| \le m+m'-1$. To get the 'full' formula (2.89) we can insert into (2.87) the inverse of (2.81), namely

$$\sigma_R(x,\xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x \sigma_L(x,\xi) \sim e^{-i<D_x,D_\xi>}\sigma_L(x,\xi).$$

This gives the double sum (still asymptotically convergent)

$$\sigma_L(A\circ B) \sim \sum_\beta\sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\big[\sigma_L(A) D^\alpha_x \frac{i^{|\beta|}}{\beta!} D^\beta_x D^\beta_\xi \sigma_L(B)\big].$$

Setting $\gamma = \alpha+\beta$ this becomes

$$\sigma_L(A\circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} \sum_{0\le\alpha\le\gamma} \frac{\gamma!(-1)^{|\gamma-\alpha|}}{\alpha!(\gamma-\alpha)!} D^\alpha_\xi\big[\sigma_L(A)\times D^{\gamma-\alpha}_\xi D^\gamma_x \sigma_L(B)\big].$$

Then Leibniz' formula shows that this sum over α can be rewritten as

$$\sigma_L(A\circ B)\sim\sum_{\gamma}\frac{i^{|\gamma|}}{\gamma!}D_\xi^\gamma\sigma_L(A)\cdot D_x^\gamma\sigma_L(B)$$
$$\sim e^{i<D_y,D_\xi>}\sigma_L(A)(x,\xi)\sigma_L(B)(y,\eta)\big|_{y=x,\eta=\xi}.$$

This is just (2.89). $\square$

The simplicity of (2.88) over (2.89) is achieved at the expense of enormous loss of information. Still, many problems can be solved using (2.88) which we can think of as saying that the principal symbol maps give a homomorphism, for instance from the filtered algebra $\Psi^0_\infty(\mathbb{R}^n)$ to the *commutative* algebra $S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)$.

2.10. Ellipticity

We say that an element of $\Psi^m_\infty(\mathbb{R}^n)$ is *elliptic* if it is invertible modulo an error in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ with the approximate inverse of order $-m$ i.e.

$$\begin{gathered}A\in\Psi^m_\infty(\mathbb{R}^n)\text{ is elliptic}\\ \Longleftrightarrow\exists\quad B\in\Psi^{-m}_\infty(\mathbb{R}^n)\text{ s.t. }\ A\circ B-\mathrm{Id}\in\Psi^{-\infty}_\infty(\mathbb{R}^n).\end{gathered}\tag{2.90}$$

Thus ellipticity, here by definition, is invertibility in $\Psi^m_\infty(\mathbb{R}^n)\big/\Psi^{-\infty}_\infty(\mathbb{R}^n)$, so the inverse lies in $\Psi^{-m}_\infty(\mathbb{R}^n)\big/\Psi^{-\infty}_\infty(\mathbb{R}^n)$. The point about ellipticity is that it is a phenomenon of the *principal symbol.*

THEOREM 2.3. *The following conditions on $A\in\Psi^m_\infty(\mathbb{R}^n)$ are equivalent*

$$A\ \textit{is elliptic}\tag{2.91}$$

$$\exists\ [b]\in S^{-m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\ \textit{ s.t. }\ \sigma_m(A)\cdot[b]\equiv1\ \textit{ in }\ S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\tag{2.92}$$

$$\exists\ b\in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)\ \textit{ s.t. }\ \sigma_L(A)\cdot b-1\in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)\tag{2.93}$$

$$\exists\ \epsilon>0\ \textit{ s.t. }\ \big|\sigma_L(A)(x,\xi)\big|\ge\epsilon(1+|\xi|)^m\ \textit{ in }\ |\xi|>\frac{1}{\epsilon}.\tag{2.94}$$

PROOF. We shall show

$$(2.91)\Longrightarrow(2.92)\Longrightarrow(2.93)\Longleftrightarrow(2.94)\Longrightarrow(2.91).\tag{2.95}$$

In fact Lemma 2.2 shows the equivalence of (2.93) and (2.94). Since we know that $\sigma_0(\mathrm{Id})=1$ applying the identity (2.88) to the definition of ellipticity in (2.90) gives

$$\sigma_m(A)\cdot\sigma_{-m}(B)\equiv1\text{ in }S^{0-[1]}_\infty(\mathbb{R}^n,\mathbb{R}^n),\tag{2.96}$$

i.e. that (2.91) $\Longrightarrow$ (2.92).

Now assuming (2.96) (i.e. (2.92)), and recalling that $\sigma_m(A)=[\sigma_L(A)]$ we find that a representative b_1 of the class $[b]$ must satisfy

$$\sigma_L(A)\cdot b_1=1+e_1,\quad e_1\in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n),\tag{2.97}$$

this being the meaning of the equality of residue classes. Now for the remainder, $e_1\in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$, the Neumann series

$$f\sim\sum_{j\ge1}(-1)^je_1^j\tag{2.98}$$

is asymptotically convergent, so $f\in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ exists, and

$$(1+f)\cdot(1+e_1)=1+e_\infty,\quad e_\infty\in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n).\tag{2.99}$$

Then multiplying (2.97) by $(1+f)$ gives

$$\sigma_L(A)\cdot\{b_1(1+f)\} = 1 + e_\infty \tag{2.100}$$

which proves (2.93), since $b = b_1(1+f) \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Of course

$$\sup(1+|\xi|)^N|e_\infty| < \infty \quad \forall\ N \tag{2.101}$$

so

$$\exists\ C \text{ s.t. } \ |e_\infty(x,\xi)| < \frac{1}{2} \text{ in } |\xi| > C. \tag{2.102}$$

From (2.100) this means

$$\big|\sigma_L(A)(x,\xi)\big|\cdot\big|b(x,\xi)\big| \geq \frac{1}{2},\ |\xi| > C. \tag{2.103}$$

Since $|b(x,\xi)| \leq C(1+|\xi|)^{-m}$ (being a symbol of order $-m$), (2.103) implies

$$\inf_{|\xi|\geq C}\big|\sigma_L(A)(x,\xi)\big|(1+|\xi|)^{-m} \geq C > 0. \tag{2.104}$$

which shows that (2.93) implies (2.94).

Conversely, as already remarked, (2.94) implies (2.93).

Now suppose (2.93) holds. Set $B_1 = q_L(b)$ then from (2.88) again

$$\sigma_0(A\circ B_1) = [q_m(A)]\cdot[b] \equiv 1. \tag{2.105}$$

That is,

$$A\circ B_1 - \mathrm{Id} = E_1 \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{2.106}$$

Consider the Neumann series of operators

$$\sum_{j\geq 1}(-1)^j E_1^j. \tag{2.107}$$

The corresponding series of (left-reduced) symbols is asymptotically summable so we can choose $F \in \Psi^{-1}_\infty(\mathbb{R}^n)$ with

$$\sigma_L(F) \sim \sum_{j\geq 1}(-1)^j\sigma_L(E_1^j). \tag{2.108}$$

Then

$$(\mathrm{Id}+E_1)(\mathrm{Id}+F) = \mathrm{Id}+E_\infty,\ E_\infty \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.109}$$

Thus $B = B_1(\mathrm{Id}+F) \in \Psi^{-m}_\infty(\mathbb{R}^n)$ satisfies (2.90) and it follows that A is elliptic. □

In the definition of ellipticity in (2.90) we have taken B to be a 'right parametrix', i.e. a right inverse modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. We can just as well take it to be a *left* parametrix.

Lemma 2.7. *$A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic if and only if there exists $B' \in \Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

$$B'\circ A = \mathrm{Id}+E',\ E' \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \tag{2.110}$$

and then if B satisfies (2.90)*, $B - B' \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$.*

PROOF. Certainly (2.110) implies $\sigma_{-m}(B')\cdot\sigma_m(A) \equiv 1$, and the multiplication here is commutative so (2.92) holds and A is elliptic. Conversely if A is elliptic we get in place of (2.106)

$$B_1 \circ A - \mathrm{Id} = E_1' \in \Psi_\infty^{-1}(\mathbb{R}^n).$$

Then defining F' as in (2.108) with E_1' in place of E_1 we get $(\mathrm{Id}+F')(\mathrm{Id}+E_1') = \mathrm{Id}+E_\infty'$ and then $B' = (\mathrm{Id}+F')\circ B_1$ satisfies (2.110). Thus 'left' ellipticity as in (2.110) is equivalent to right ellipticity. Applying B to (2.110) gives

$$B' \circ (\mathrm{Id}+E) = B' \circ (A \circ B) = (\mathrm{Id}+E') \circ B \tag{2.111}$$

which shows that $B - B' \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. □

Thus a left parametrix of an elliptic element of $\Psi_\infty^m(\mathbb{R}^n)$ is always a right, hence two-sided, parametrix and such a parametrix is unique up to an additive term in $\Psi_\infty^{-\infty}(\mathbb{R}^n)$.

2.11. Elliptic regularity and the Laplacian

One of the main reasons that the 'residual' terms *are* residual is that they are smoothing operators.

LEMMA 2.8. *If $E \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ then*

$$E : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n). \tag{2.112}$$

PROOF. This follows from Proposition 2.4 since we can regard the kernel as a $\mathcal{C}^\infty$ function of x taking values in $\mathcal{S}(\mathbb{R}^n_y)$. □

Directly from the existence of parametrices for elliptic operators we can deduce the regularity of solutions to elliptic (pseudodifferential) equations.

PROPOSITION 2.5. *If $A \in \Psi_\infty^m(\mathbb{R}^n)$ is elliptic and $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $Au = 0$ then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

PROOF. Let $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$ be a parametrix for A. Then $B \circ A = \mathrm{Id}+E$, $E \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. Thus,

$$u = (BA - E)u = -Eu \tag{2.113}$$

and the conclusion follows from Lemma 2.8. □

Suppose that $g_{ij}(x)$ are the components of an '∞-metric' on $\mathbb{R}^n$, i.e.

$$\begin{gathered} g_{ij}(x) \in \mathcal{C}_\infty^\infty(\mathbb{R}^n), i,j = 1,\dots,n \\ \Big|\sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j\Big| \ge \epsilon|\xi|^2 \quad \forall\ x \in \mathbb{R}^n, \xi \in \mathbb{R}^n,\ \epsilon > 0. \end{gathered} \tag{2.114}$$

The *Laplacian* of the metric is the second order differential operator

$$\Delta_g = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} D_{x_i} g^{ij}\sqrt{g} D_{x_j} \tag{2.115}$$

where

$$g(x) = \det g^{ij}(x),\ g^{ij}(x) = (g_{ij}(x))^{-1}.$$

The Laplacian is determined by the integration by parts formula

$$\int_{\mathbb{R}^n} \sum_{i,j} g^{ij}(x) D_{x_i}\phi \cdot \overline{D_{x_j}\psi} dg = \int \Delta_g \phi \cdot \overline{\psi} dg \ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \tag{2.116}$$

where

$$dg = \sqrt{g} dx. \tag{2.117}$$

Our assumption in (2.114) shows that $\Delta = \Delta_g \in \mathrm{Diff}^2_\infty(\mathbb{R}^n) \subset \Psi^2_\infty(\mathbb{R}^n)$ is in fact elliptic, since

$$\sigma_2(\Delta) = \sum_{i,j=1} g^{ij}\xi_i\xi_j. \tag{2.118}$$

Thus Δ has a two-sided parametrix $B \in \Psi^{-2}_\infty(\mathbb{R}^n)$

$$\Delta \circ B \equiv B \circ \Delta \equiv \mathrm{Id} \qquad \mod \quad \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.119}$$

In particular we see from Proposition 2.5 that $\Delta u = 0$, $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

2.12. L^2 boundedness

So far we have thought of pseudodifferential operators, the elements of $\Psi^m_\infty(\mathbb{R}^n)$ for some m, as defining continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$ and, by duality, on $\mathcal{S}'(\mathbb{R}^n)$. Now that we have proved the composition formula we can use it to prove other 'finite order' regularity results. The basic one of these is L^2 boundedness:

PROPOSITION 2.6. *[Boundedness] If $A \in \Psi^0_\infty(\mathbb{R}^n)$ then, by continuity from $\mathcal{S}(\mathbb{R}^n)$, A defines a bounded linear operator*

$$A : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n). \tag{2.120}$$

Our proof will be in two stages, the first part is by direct estimation. Namely, *Schur's lemma* gives a useful criterion for an integral operator to be bounded on L^2.

LEMMA 2.9 (Schur). *If $K(x,y)$ is locally integrable on $\mathbb{R}^{2n}$ and is such that*

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|dy,\ \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|dx < \infty \tag{2.121}$$

then the operator $K : \phi \longmapsto \int_{\mathbb{R}^n} K(x,y)\phi(y)dy$ is bounded on $L^2(\mathbb{R}^n)$.

PROOF. Since $\mathcal{S}(\mathbb{R}^n)$ is dense[5] in $L^2(\mathbb{R}^n)$ we only need to show the existence of a constant, C, such that

$$\int \left|K\phi(x)\right|^2 dx \le C \int |\phi|^2 \quad \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{2.122}$$

Writing out the integral on the left

$$\begin{aligned} &\int \Big| \int K(x,y)\phi(y)dy\Big|^2 dx \\ &= \iiint K(x,y)\overline{K(x,z)}\phi(y)\overline{\phi(z)}\ dydzdx \end{aligned} \tag{2.123}$$

[5]See Problem 2.18

is certainly absolutely convergent and

$$\begin{aligned}&\int \left|K\phi(x)\right|^2 dx\\ &\leq \left(\iiint \left|K(x,y)K(x,z)\right|\left|\phi(y)\right|^2 dydxdz\right)^{\frac{1}{2}}\\ &\times \left(\iiint \left|K(x,y)K(x,z)\right|\left|\phi(z)\right|^2 dzdxdy\right)^{\frac{1}{2}}.\end{aligned}$$

These two factors are the same. Since

$$\int \left|K(x,y)\right|\left|K(x,z)\right| dxdz \leq \sup_{x\in\mathbb{R}^n} \int \left|K(x,z)\right| dz \cdot \sup_{y\in\mathbb{R}^n} \int \left|K(x,y\right| dx$$

(2.122) follows. Thus (2.121) gives (2.122). □

This standard lemma immediately implies the L^2 boundedness of the 'residual terms.' Thus, if $K \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n)$ then its kernel satisfies (2.60). This in particular implies

$$\left|K(x,y)\right| \leq C\left(1+|x-y|\right)^{-n-1}$$

and hence that K satisfies (2.121). Thus

$$\text{each } K \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n) \text{ is bounded on } L^2(\mathbb{R}^n). \tag{2.124}$$

2.13. Square root and boundedness

To prove the general result, (2.120), we shall use the clever idea, due to Hörmander, of using the (approximate) square root of an operator. We shall say that an element $[a] \in S^{m-[1]}_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$ is positive if there is some $0 < a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ in the equivalence class.

Proposition 2.7. *Suppose $A \in \Psi^m_{\infty}(\mathbb{R}^n)$, $m > 0$, is self-adjoint, $A = A^*$, and elliptic with a positive principal symbol, then there exists $B \in \Psi^{m/2}_{\infty}(\mathbb{R}^n)$, $B = B^*$, such that*

$$A = B^2 + G, \quad G \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n). \tag{2.125}$$

Proof. This is a good exercise in the use of the symbol calculus. Let $a \in S^m_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$, $a > 0$, be a positive representative of the principal symbol of A. Now[6]

$$b_0 = a^{\frac{1}{2}} \in S^{m/2}_{\infty}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.126}$$

Let $B_0 \in \Psi^{m/2}_{\infty}(\mathbb{R}^n)$ have principal symbol b_0. We can assume that $B_0 = B_0^*$, since if not we just replace B_0 by $\frac{1}{2}(B_0 + B_0^*)$ which has the same principal symbol.

The symbol calculus shows that $B_0^2 \in \Psi^m_{\infty}(\mathbb{R}^n)$ and

$$\sigma_m(B_0^2) = \left(\sigma_{m/2}(B_0)\right)^2 = b_0^2 = a_0 \mod S^{m-1}_{\infty}.$$

Thus

$$A - B_0^2 = E_1 \in \Psi^{m-1}_{\infty}(\mathbb{R}^n). \tag{2.127}$$

[6]See Problem 2.19 for an outline of the proof

Then we proceed inductively. Suppose we have chosen $B_j \in \Psi_\infty^{m/2-j}(\mathbb{R}^n)$, with $B_j^* = B_j$, for $j \le N$ such that

$$A - \left(\sum_{j=0}^{N} B_j\right)^2 = E_{N+1} \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \tag{2.128}$$

Of course we *have* done this for $N = 0$. Then see the effect of adding $B_{N+1} \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$:

$$A - \left(\sum_{j=0}^{N+1} B_j\right)^2 = E_{N+1} - \left(\sum_{j=0}^{N} B_j\right) B_{N+1} - B_{N+1}\left(\sum_{j=0}^{N} B_j\right) - B_{N+1}^2. \tag{2.129}$$

On the right side all terms are of order $m - N - 2$, except for

$$E_{N+1} - B_0 B_{N+1} - B_{N+1} B_0 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \tag{2.130}$$

The principal symbol, of order $m - N - 1$, of this is just

$$\sigma_{m-N-1}(E_{N+1}) - 2\, b_0 \cdot \sigma_{\frac{m}{2}-N-1}(B_{N+1}). \tag{2.131}$$

Thus if we choose $B_{N+1} \in \Psi_\infty^{\frac{m}{2}-N-1}(\mathbb{R}^n)$ with

$$\sigma_{m/2-N-1}(B_{N+1}) = \frac{1}{2}\frac{1}{b_0} \cdot \sigma_{m-N-1}(E_{N+1})$$

and replace B_{N+1} by $\frac{1}{2}(B_{N+1} + B_{N+1}^*)$, we get the inductive hypothesis for $N+1$. Thus we have arranged (2.128) for every N. Now define $B = \frac{1}{2}(B' + (B')^*)$ where

$$\sigma_L(B') \sim \sum_{j=0}^{\infty} \sigma_L(B_j). \tag{2.132}$$

Since all the B_j are self-adjoint B also satisfies (2.132) and from (2.128)

$$A - B^2 = A - \left(\sum_{j=0}^{N} B_j + B_{(N+1)}\right)^2 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n) \tag{2.133}$$

for every N, since $B_{(N+1)} = B - \sum_{j=0}^{N} B_j \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$. Thus $A - B^2 \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and we have proved (2.125), and so Proposition 2.7. □

Here is Hörmander's argument to prove Proposition 2.6. We want to show that

$$\|A\phi\| \le C\|\phi\| \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{2.134}$$

where $A \in \Psi_\infty^0(\mathbb{R}^n)$. The square of the left side can be written

$$\int A\phi \cdot \overline{A\phi} dx = \int \phi \cdot \overline{(A^* A\phi)} dx.$$

So it suffices to show that

$$\langle \phi, A^* A\phi \rangle \le C\|\phi\|^2. \tag{2.135}$$

Now $A^*A \in \Psi^0_\infty(\mathbb{R}^n)$ with $\sigma_0(A^*A) = \overline{\sigma_0(A)}\sigma_0(A) \in \mathbb{R}$. If $C > 0$ is a large constant,

$$C > \sup_{x,\xi} \left|\sigma_L(A^*A)(x,\xi)\right|$$

then $C - A^*A$ has a positive representative of its principal symbol. We can therefore apply Proposition 2.7 to it:

$$C - A^*A = B^*B + G, \quad G \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.136}$$

This gives

$$\begin{aligned} \langle \phi, A^*A\phi\rangle =& C\langle \phi,\phi\rangle - \langle \phi, B^*B\phi\rangle - \langle \phi, G\phi\rangle \\ =& C\|\phi\|^2 - \|B\phi\|^2 - \langle \phi, G\phi\rangle. \end{aligned} \tag{2.137}$$

The second term on the right is negative and, since $G \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$, we can use the residual case in (2.124) to conclude that

$$\left|\langle \phi, G\phi\rangle\right| \le C'\|\phi\|^2 \Longrightarrow \|A\phi\|^2 \le C\|\phi\|^2 + C'\|\phi\|^2,$$

so (2.120) holds and Proposition 2.6 is proved.

2.14. Sobolev boundedness

Using the basic boundedness result, Proposition 2.6, and the calculus of pseudodifferential operators we can prove more general results on the action of pseudodifferential operators on Sobolev spaces.

Recall that for any positive integer, k,

$$H^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n);\ D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k\right\}. \tag{2.138}$$

Using the Fourier transform we find

$$u \in H^k(\mathbb{R}^n) \Longrightarrow \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n) \quad \forall\ |\alpha| \le k. \tag{2.139}$$

Now these finitely many conditions can be written as just the one condition

$$\left(1 + |\xi|^2\right)^{k/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n). \tag{2.140}$$

Notice that $a(\xi) = (1 + |\xi|^2)^{k/2} = \langle\xi\rangle^k \in S^k_\infty(\mathbb{R}^n)$. Here we use the notation

$$\langle\xi\rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}} \tag{2.141}$$

for a smooth (symbol) of the size of $1 + |\xi|$, thus (2.140) just says

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \langle D\rangle^k u \in L^2(\mathbb{R}^n). \tag{2.142}$$

For *negative* integers

$$H^k(\mathbb{R}^n) = \Big\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\beta| \le -k} D^\beta u_\beta,\ u_\beta \in L^2(\mathbb{R}^n)\Big\},\ -k \in \mathbb{N}. \tag{2.143}$$

The same sort of discussion applies, showing that

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \langle D\rangle^k u \in L^2(\mathbb{R}^n),\ k \in \mathbb{Z}. \tag{2.144}$$

In view of this we define the Sobolev space $H^m(\mathbb{R}^n)$, for any real order, by

$$u \in H^m(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \langle D\rangle^m u \in L^2(\mathbb{R}^n). \tag{2.145}$$

It is a Hilbert space with

$$\|u\|^2_m = \|\langle D\rangle^m u\|^2_{L^2} = \int (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 d\xi. \tag{2.146}$$

Clearly we have

$$H^m(\mathbb{R}^n) \supseteq H^{m'}(\mathbb{R}^n) \text{ if } m' \geq m. \tag{2.147}$$

Notice that it is rather unfortunate that these spaces get *smaller* as m gets bigger, as opposed to the spaces $\Psi^m_\infty(\mathbb{R}^n)$ which get *bigger* with m. Anyway that's life and we have to think of

$$\begin{cases} H^{\infty}(\mathbb{R}^n) = \bigcap\limits_m H^m(\mathbb{R}^n) & \text{as the residual space} \\ H^{-\infty}(\mathbb{R}^n) = \bigcup\limits_m H^m(\mathbb{R}^n) & \text{as the big space.} \end{cases} \tag{2.148}$$

It is important to note that

$$\mathcal{S}(\mathbb{R}^n) \subsetneq H^{\infty}(\mathbb{R}^n) \subsetneq H^{-\infty}(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n). \tag{2.149}$$

In particular we do *not* capture all the tempered distributions in $H^{-\infty}(\mathbb{R}^n)$. We therefore consider *weighted* versions of these Sobolev spaces:

$$\langle x\rangle^q H^m(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle x\rangle^{-q} u \in H^m(\mathbb{R}^n)\big\}. \tag{2.150}$$

THEOREM 2.4. *For each $q, m, M \in \mathbb{R}$ each $A \in \Psi^M_\infty(\mathbb{R}^n)$ defines a continuous linear map*

$$A : \langle x\rangle^q H^m(\mathbb{R}^n) \longrightarrow \langle x\rangle^q H^{m-M}(\mathbb{R}^n). \tag{2.151}$$

PROOF. Let us start off with $q = 0$, so we want to show that

$$A : H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n),\ A \in \Psi^M_\infty(\mathbb{R}^n). \tag{2.152}$$

Now from (2.145) we see that

$$\begin{aligned} u \in H^m(\mathbb{R}^n) &\Longleftrightarrow \langle D\rangle^m u \in L^2(\mathbb{R}^n) \\ &\Longleftrightarrow \langle D\rangle^{m-M} \langle D\rangle^M u \in L^2(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^M u \in H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M. \end{aligned} \tag{2.153}$$

That is,

$$\langle D\rangle^M : H^m(\mathbb{R}^n) \longleftrightarrow H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M. \tag{2.154}$$

To prove (2.152) it suffices to show that

$$B = \langle D\rangle^{-M+m} \cdot A \cdot \langle D\rangle^{-m} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \tag{2.155}$$

since then $A = \langle D\rangle^{-m+M} \cdot B \cdot \langle D\rangle^m$ maps $H^m(\mathbb{R}^n)$ to $H^{m-M}(\mathbb{R}^n)$:

$$\begin{tikzcd} H^m(\mathbb{R}^n) \arrow[r, "A"] \arrow[d, "\langle D\rangle^m"'] & H^{m-M}(\mathbb{R}^n) \arrow[d, "\langle D\rangle^{m-M}"] \\ L^2(\mathbb{R}^n) \arrow[r, "B"'] & L^2(\mathbb{R}^n). \end{tikzcd} \tag{2.156}$$

Since $B \in \Psi^0_\infty(\mathbb{R}^n)$, by the composition theorem, we already know (2.155).

Thus we have proved (2.152). To prove the general case, (2.151), we proceed in the same spirit. Thus $\langle x\rangle^q$ is an isomorphism from $H^m(\mathbb{R}^n)$ to $\langle x\rangle^q H^m(\mathbb{R}^n)$, by definition. So to get (2.151) we need to show that

$$Q = \langle x\rangle^{-q} \cdot A \cdot \langle x\rangle^q :\ H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n), \tag{2.157}$$

i.e. satisfies (2.152). Consider the Schwartz kernel of Q. Writing A in left-reduced form, with symbol a,

$$K_Q(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}\langle x\rangle^{-q}a(x,\xi)d\xi\cdot\langle y\rangle^q. \tag{2.158}$$

Now if we check that

$$\langle x\rangle^{-q}\langle y\rangle^q a(x,\xi) \in \left(1+|x-y|^2\right)^{\frac{|q|}{2}} S^M_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right) \tag{2.159}$$

then we know that $Q \in \Psi^M_\infty(\mathbb{R}^n)$ and we get (2.157) from (2.152). Thus we want to show that

$$\langle x-y\rangle^{-|q|}\frac{\langle y\rangle^q}{\langle x\rangle^q}a(x,\xi) \in S^M_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \tag{2.160}$$

assuming of course that $a(x,\xi) \in S^M_\infty(\mathbb{R}^n;\mathbb{R}^n)$. By interchanging the variables x and y if necessary we can assume that $q<0$. Consider separately the two regions

$$\begin{aligned}\left\{(x,y);|x-y|<\frac{1}{4}(|x|+|y|)\right\} &= \Omega_1\\ \left\{(x,y);|x-y|>\frac{1}{8}(|x|+|y|)\right\} &= \Omega_2.\end{aligned} \tag{2.161}$$

In Ω_1, x is "close" to y, in the sense that

$$|x| \le |x-y|+|y| \le \frac{1}{4}(|x|+|y|)+|y| \Longrightarrow |x| \le \frac{4}{3}\cdot\frac{5}{4}|y| \le 2|y|. \tag{2.162}$$

Thus

$$\langle x-y\rangle^{-q}\cdot\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}} \le C \ \text{ in } \Omega_1. \tag{2.163}$$

On the other hand in Ω_2,

$$|x|+|y| < 8|x-y| \Longrightarrow |x| < 8|x-y| \tag{2.164}$$

so again

$$\langle x-y\rangle^{-q}\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}} \le C. \tag{2.165}$$

In fact we easily conclude that

$$\langle x-y\rangle^{-q}\frac{\langle y\rangle^q}{\langle x\rangle^q} \in \mathcal{C}^\infty_\infty(\mathbb{R}^n) \quad \forall\ q, \tag{2.166}$$

since differentiation by x or y makes all terms "smaller". This proves (2.160), hence (2.159) and (2.157) and therefore (2.151), i.e. the theorem is proved. □

We can capture any tempered distribution in a weighted Sobolev space; this is really Schwartz' representation theorem which says that any $u \in \mathcal{S}'(\mathbb{R}^n)$ is of the form

$$u = \sum_{\text{finite}} x^\alpha D^\beta_x u_{\alpha\beta},\ u_{\alpha\beta} \text{ bounded and continuous.} \tag{2.167}$$

Clearly $\mathcal{C}^0_\infty(\mathbb{R}^n) \subset \langle x\rangle^{1+n}L^2(\mathbb{R}^n)$. Thus as a special case of Theorem 2.4,

$$D^\beta_x : \langle x\rangle^{1+n}L^2(\mathbb{R}^n) \longrightarrow \langle x\rangle^{1+n}H^{-|\beta|}(\mathbb{R}^n)$$

so

LEMMA 2.10.

$$\mathcal{S}'(\mathbb{R}^n) = \bigcup_M \langle x\rangle^M H^{-M}(\mathbb{R}^n). \tag{2.168}$$

The elliptic regularity result we found before can now be refined:

PROPOSITION 2.8. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic then*

$$\begin{gathered} Au \in \langle x\rangle^p H^q(\mathbb{R}^n),\ u \in \langle x\rangle^{p'} H^{q'}(\mathbb{R}^n) \\ \Longrightarrow u \in \langle x\rangle^{p''} H^{q''}(\mathbb{R}^n),\ p'' = \max(p, p'),\ q'' = \max(q + m, q'). \end{gathered} \tag{2.169}$$

PROOF. The existence of a left parametrix for A, $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$,

$$B \cdot A = \operatorname{Id} + G,\ G \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

means that

$$u = B(Au) + Gu \in \langle x\rangle^p H^{q+m}(\mathbb{R}^n) + \langle x\rangle^{p'} H^\infty(\mathbb{R}^n) \subset \langle x\rangle^{p''} H^{q+m}(\mathbb{R}^n). \tag{2.170}$$

□

2.15. Polyhomogeneity

So far we have been considering operators $A \in \Psi^m_\infty(\mathbb{R}^n)$ which correspond, via (2.2), to amplitudes satisfying the symbol estimates (2.6), i.e. in $S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$. As already remarked, there are many variants of these estimates and corresponding spaces of pseudodifferential operators. Some *weakening* of the estimates is discussed in the problems below, starting with Problem 2.16. Here we consider a restriction of the spaces, in that we define

$$S^m_{\mathrm{ph}}(\mathbb{R}^p; \mathbb{R}) \subset S^m_\infty(\mathbb{R}^p; \mathbb{R}^n). \tag{2.171}$$

The definition of the subspace (2.171) is straightforward. First we note that if $a \in \mathcal{C}^\infty(\mathbb{R}^p; \mathbb{R}^n)$ is homogeneous of degree $m \in \mathbb{C}$ in $|\xi| \geq 1$, then

$$a(z, t\xi) = t^m a(z, \xi),\ |t|, |\xi| \geq 1 \tag{2.172}$$

where for complex m we always mean the principal branch of t^m for $t > 0$. If it also satisfies the uniform regularity estimates

$$\sup_{z \in \mathbb{R}^n,\ |\xi| \leq 2} |D^\alpha_z D^\beta_\xi a(z, \xi)| < \infty\ \forall\ \alpha, \beta, \tag{2.173}$$

then in fact

$$a \in S^{\Re m}_\infty(\mathbb{R}^p; \mathbb{R}^n). \tag{2.174}$$

Indeed, (2.173) is exactly the restriction of the symbol estimates to $z \in \mathbb{R}^p$, $|\xi| \leq 2$. On the other hand, in $|\xi| \geq 1$, $a(z, \xi)$ is homogeneous so

$$|D^\alpha_z D^\beta_\xi a(z, \xi)| = |\xi|^{m - |\beta|} |D^\alpha_z D^\beta_\xi a(z, \hat{\xi})|, \quad \hat{\xi} = \frac{\xi}{|\xi|}$$

from which the symbol estimates follow.

DEFINITION 2.2. *For any $m \in \mathbb{C}$, the subspace of (one-step)*[7] *polyhomogeneous symbols is defined as a subset (2.171) by the requirement that $a \in S^m_{\mathrm{ph}}(\mathbb{R}^p; \mathbb{R}^n)$ if*

[7]For a somewhat more general class of polyhomogeneous symbols, see problem 2.8.

and only if there exist elements $a_{m-j}(z,\xi) \in S^{\Re m}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ *which are homogeneous of degree* $m-j$ *in* $|\xi| \geq 1$*, for* $j \in \mathbb{N}_0$*, such that*

$$a \sim \sum_j a_{m-j}. \tag{2.175}$$

Clearly

$$S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \cdot S^{m'}_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \subset S^{m+m'}_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n), \tag{2.176}$$

since the asymptotic expansion of the product is given by the formal product of the asymtotic expansion. In fact there is equality here, because

$$(1+|\xi|^2)^{m/2} \in S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \tag{2.177}$$

and multiplication by $(1+|\xi|^2)^{m/2}$ is an isomorphism of the space $S^0_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ onto $S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n)$. Furthermore differentiation with respect to z_j or ξ_l preserves asymptotic homogeneity so

$$\begin{matrix} D_{x_j} : S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \longrightarrow S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \\ D_{\xi_l} : S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \longrightarrow S^{m-1}_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n) \end{matrix} \quad \forall j = 1,\dots,n.$$

It is therefore no great surprise that the polyhomogeneous operators form a subalgebra.

Proposition 2.9. *The spaces* $\Psi^m_{\text{ph}}(\mathbb{R}^n) \subset \Psi^m_{\infty}(\mathbb{R}^n)$ *defined by the condition that the kernel of* $A \in \Psi^m_{\text{ph}}(\mathbb{R}^n)$ *should be of the form* $I(a)$ *for some*

$$a \in (1+|x-y|^2)^{w/2} S^m_{\text{ph}}(\mathbb{R}^{2n};\mathbb{R}^n), \tag{2.178}$$

are such that

$$\Psi^m_{\text{ph}}(\mathbb{R}^n) \circ \Psi^{m'}_{\text{ph}}(\mathbb{R}^n) = \Psi^{m+m'}_{\text{ph}}(\mathbb{R}^n),\ (\Psi^m_{\text{ph}}(\mathbb{R}^n))^* = \Psi^{\bar{m}}_{\text{ph}}(\mathbb{R}^n) \tag{2.179}$$

for all $m, m' \in \mathbb{C}$.

Proof. Since the definition shows that

$$\Psi^m_{\text{ph}}(\mathbb{R}^n) \subset \Psi^{\Re m}_{\infty}(\mathbb{R}^n)$$

we know already that

$$\Psi^m_{\text{ph}}(\mathbb{R}^n) \cdot \Psi^{m'}_{\text{ph}}(\mathbb{R}^n) \subset \Psi^{\Re(m+m')}_{\infty}(\mathbb{R}^n).$$

To see that products are polyhomogeneous it suffices to use (2.176) and (2.178) which together show that the asymptotic formulæ describing the left symbols of $A \in \Psi^m_{\text{ph}}(\mathbb{R}^n)$ and $B \in \Psi^{m'}_{\text{ph}}(\mathbb{R}^m)$, e.g.

$$\sigma_L(A) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_y a(x,y,\xi)|_{y=x}$$

imply that $\sigma_L(A) \in S^m_{\text{ph}}(\mathbb{R}^n;\mathbb{R}^n)$, $\sigma_L(B) \in S^m_{\text{ph}}(\mathbb{R}^n;\mathbb{R}^n)$. Then the asymptotic formula for the product shows that $\sigma_L(A \cdot B) \in S^{m+m'}_{\text{ph}}(\mathbb{R}^n;\mathbb{R}^n)$.

The proof of $*$-covariance is similarly elementary, since if $A = I(a)$ then $A^* = I(b)$ with $b(x,y,z) = \overline{a(y,x,\xi)} \in S^m_{\text{ph}}(\mathbb{R}^{2n};\mathbb{R}^n)$. $\square$

In case m is real this subspace is usually denoted simply $\Psi^m(\mathbb{R}^n)$ and its elements are often said to be 'classical' pseudodifferential operators. As a small exercise in the use of the principal symbol map we shall show that

$$\begin{gathered} A \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n),\ A \text{ (uniformly) elliptic} \Longrightarrow \exists \text{ a parametrix} \\ B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n),\ A \cdot B - \mathrm{Id},\ B \cdot A - \mathrm{Id} \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n). \end{gathered} \tag{2.180}$$

In fact we already know that $B \in \Psi^{-m}_{\infty}(\mathbb{R}^n)$ exists with these properties, and even that it is unique modulo $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$. To show that $B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n)$ we can use the principal symbol map.

For elements $A \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ the principal symbol $\sigma_m(A) \in S^{\Re m-[1]}_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$ has a preferred class of representatives, namely the leading term in the expansion of $\sigma_L(A)$

$$\sigma_m(A) = \sigma(\xi) a_m(x,\xi) \qquad \operatorname{mod} S^{m-1}_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$$

where $\sigma|\xi| = 1$ in $|\xi| \geq 1$, $\sigma|\xi| = 0$ in $|\xi| \leq 1/2$. It is even natural to identify the principal symbol with $a_m(x,\xi)$ as a *homogeneous* function. Then we can see that

$$\begin{aligned} &A \in \Psi^{\Re m}_{\infty}(\mathbb{R}^n),\ \sigma_{\Re m}(A) \text{ homogeneous of degree m} \\ &\qquad\qquad \Longleftrightarrow \Psi^m_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{\Re m-1}_{\infty}(\mathbb{R}^n). \end{aligned} \tag{2.181}$$

Indeed, we just subtract from A an element $A_1 \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ with $\sigma_{\Re m}(A_1) = \sigma_{\Re m}(A)$, then $\sigma_{\Re m}(A - A_1) = 0$ so $A - A_1 \in \Psi^{m-1}_{\infty}(\mathbb{R}^n)$.

So, returning to the proof of (2.180) note straight away that

$$\sigma_{-\Re m}(B) = \sigma_{\Re m}(A)^{-1}$$

has a homogeneous representative, namely $a_m(x,\xi)^{-1}$. Thus we have shown that for $j = 1$

$$B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{-m-j}_{\infty}(\mathbb{R}^n). \tag{2.182}$$

We take (2.182) as an inductive hypthesis for general j. Writing this decomposition $B = B' + B_j$ it follows from the identity (2.180) that

$$A \cdot B = A \cdot B' + AB_j = \mathrm{Id} \quad \operatorname{mod} \Psi^{-\infty}_{\infty}(\mathbb{R}^n)$$

so

$$A \cdot B_j = \mathrm{Id} - AB' \in \Psi^0_{\mathrm{ph}}(\mathbb{R}^n) \cap \Psi^{-j}_{\infty}(\mathbb{R}^n) = \Psi^{-j}_{\mathrm{ph}}(\mathbb{R}^n).$$

Now applying B on the left, or using the principal symbol map, it follows that $B_j \in \Psi^{-m-j}_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{-m-j-1}_{\infty}(\mathbb{R}^n)$ which gives the inductive hypothesis (2.182) for $j+1$.

It is usually the case that a construction in $\Psi^*_{\infty}(\mathbb{R}^n)$, applied to an element of $\Psi^*_{\mathrm{ph}}(\mathbb{R}^n)$ will yield an element of $\Psi^*_{\mathrm{ph}}(\mathbb{R}^n)$ and when this is the case it can generally be confirmed by an inductive argument like that used above to check (2.180).

2.16. Topologies and continuity of the product

As a subspace[8]

$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \subset S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$$

---

[8]Polyhomogeneous symbols may seem to be quite sophisticated objects but they are really smooth functions on manifolds with boundary; see Problems 2.8–2.7.

is not closed. Indeed, since it contains $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, its closure contains all of $S^{m'}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ for $m' < m$. In fact it is a dense subspace.[9] To capture its properties we can strengthen the topology $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ inherits from $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$.

As well as the symbol norms $\|\cdot\|_{N,m}$ in (2.7) we can add norms on the terms in the expansions in (2.175)

$$\|D^\alpha_x D^\beta_\xi a_{m-j}(x,\xi)\|_{L^\infty(G)},\ G = \mathbb{R}^p \times \{1 \le |\xi| \le 2\}. \tag{2.183}$$

We can further add the symbol norms ensuring (2.175), i.e.,

$$\|a - \sum_{j=0}^{k} a_{m-j}\|_{m-k-1,N} \quad \forall\ k, N. \tag{2.184}$$

Together these give a countable number of norms on $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$. With respect to the metric topology defined as in (2.8) the spaces $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ are then *complete*.[10].

Since we have shown that the left symbol map is a linear isomorphism $\Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ we give $\Psi^m_\infty(\mathbb{R}^n)$ a topology by declaring this to be a topological isomorphism. Similarly we declare

$$\sigma_L : \Psi^m_{\mathrm{ph}}(\mathbb{R}^n) \longleftrightarrow S^m_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n) \tag{2.185}$$

to be a topological isomorphism.

Having given the spaces $\Psi^m_\infty(\mathbb{R}^n)$ and $\Psi^m_\infty(\mathbb{R}^n)$ topologies it is natural to ask about the continuity of the operations on them.

PROPOSITION 2.10. *The adjoint and product maps are continuous*

$$\begin{gathered}\Psi^m_\infty(\mathbb{R}^n) \xrightarrow{\ *\ } \Psi^m_\infty(\mathbb{R}^n),\\ \Psi^m_\infty(\mathbb{R}^n) \times \Psi^{m'}_\infty(\mathbb{R}^n) \longrightarrow \Psi^{m+m'}_\infty(\mathbb{R}^n)\end{gathered} \tag{2.186}$$

and similarly for the polyhomogeneous spaces.

PROOF. Note that we have put metric topologies on these spaces so it suffices to check sequential continuity. Now the commutative product is continuous, as follows from direct estimation,

$$S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \times S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) \longrightarrow S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) \tag{2.187}$$

as is the 'commutative adjoint', $a(x,y,\xi) \longmapsto \overline{a(y,x\xi)}$ on S^m. The same is true for the polyhomogeneous spaces. From this it follows that it is only necessary to show the continuity of the reduction map

$$S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto \sigma_L(I(a)) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.188}$$

Recall that this map is accomplished in two steps, first taking the Taylor series at $y = x$, integrating by parts and taking an asymptotic sum. This constructs $b \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ so that $q_L(b) - I(a) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Then the case $m = -\infty$ is done directly by estimation. Given a convergent sequence in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$, each of the terms in the Taylor series converges and it follows that the asymptotic sums can be arranged to converge, that is if $a_n \to a$ in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ then there exists $b_n \to b \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ such that $q_L(b_n) - I(a_n) \to q_L(b) - I(a) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Combined with the case $m = -\infty$ this shows that reduction to the left symbol is continuous. □

[9] See Problem 2.9.

[10] See Problem 2.10.

A result which will be useful later follows from the same argument.

LEMMA 2.11. *Suppose* $\phi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $i = 1, 2$, *and* $\phi_1 = 1$ *on* $\operatorname{supp}(\phi_2)$ *then*

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto \sigma_L(\phi_1 q_L(a)(1-\phi_2)) \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n) \tag{2.189}$$

is continuous.

Since we have given topologies to the spaces of pseudodifferential operators the notion of continuous dependence on parameters is well defined. Indeed the same is true of smooth dependence on parameters, since a map $a : [0,1] \longrightarrow \Psi^m_\infty(\mathbb{R}^n)$ is $\mathcal{C}^1$ if it is continuous, the difference quotients $(a(t+s)-a(t))/s$ are continuous down to $s = 0$, and the resulting derivative is smooth. Then smoothness is just iterative regularity in this sense. Essentially by definition this means that $A \in \mathcal{C}^\infty([0,1]_\epsilon; \Psi^m_\infty(\mathbb{R}^n))$ is the left-reduced symbol $a = \sigma_L(A(\epsilon)) \in \mathcal{C}^\infty([0,1]; S^m_\infty(\mathbb{R}^n;\mathbb{R}^n))$.

2.17. Linear invariance

It is rather straightforward to see that the algebra $\Psi^\infty_\infty(\mathbb{R}^n)$ is invariant under affine transformations of $\mathbb{R}^n$. In particular if $T_a x = x + a$, for $a \in \mathbb{R}^n$, is translation by a and

$$T_a^* f(x) = f(x+a),\ T_a^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

is the isomorphism on functions then a new operator is defined by

$$T_a^* A_a f = A T_a^* f \text{ and } A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A_a \in \Psi^m_\infty(\mathbb{R}^n).$$

In fact the left-reduced symbols satisfy

$$\sigma_L(A_a)(x,\xi) = \sigma_L(A)(x+a,\xi),\ A_a = T^*_{-a} A T^*_a.$$

Similarly if $T \in \mathrm{GL}(n)$ is an invertible linear transformation of $\mathbb{R}^n$ and $A_T f = T^* A (T^*)^{-1} f$ then

$$\begin{aligned} A_T f(x) &= (2\pi)^{-n} \int e^{i(Tx-y)\cdot\xi} a(Tx,\xi) f(T^{-1}y) d\xi dy \\ &= (2\pi)^{-n} \int e^{i(Tx-Ty)\cdot\xi} a(Tx,\xi) f(y) |\det(T)| d\xi dy \end{aligned} \tag{2.190}$$

so changing dual variable to $(T^t)^{-1}\xi$ shows that

$$\begin{aligned} &A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A_T \in \Psi^m_\infty(\mathbb{R}^n) \\ &\qquad\qquad\text{and } \sigma_L(A_T)(x,\xi) = \sigma_L(A)(Tx,(T^t)^{-1}\xi) \end{aligned} \tag{2.191}$$

where T^t is the transpose of T (so $Tx \cdot \xi = x \cdot T^t\xi$) and the determinant factors cancel. Thus it suffices to check that

$$S^m_\infty(\mathbb{R}^q;\mathbb{R}^n) \ni a \longmapsto a' = a(Tx, A\xi) \in S^m(\mathbb{R}^q;\mathbb{R}^n) \tag{2.192}$$

for any linear tranformation T on $\mathbb{R}^q$ and invertible linear tranformation A on $\mathbb{R}^n$. Clearly the derivatives of a' are linear combinations of derivatives of a at the image point so it the symbol estimates for a' follow from those for a and the invertibility of A which implies that

$$c|\xi| \le |A\xi| \le C|\xi|,\ c, C > 0. \tag{2.193}$$

This invariance means that we can define the spaces $\Psi^m_\infty(V)$ and $\Psi^m_{\mathrm{ph}}(V)$ for any vector space V (or even affine space) as operators on $\mathcal{S}(V)$.

2.18. Local coordinate invariance

To transfer the definition of pseudodifferential operators to manifolds we need to show not only invariance under linear transformations but also under a diffemorphism $F : \Omega \longrightarrow \Omega'$ between open subsets of $\mathbb{R}^n$. For this to make sense we need to consider an operator on $\mathbb{R}^n$ which acts on functions defined in Ω'. Thus, consider

$$\Psi^m_{\rm c}(\Omega') = \{A \in \Psi^m_\infty(\mathbb{R}^n) \text{ has kernel satisfying } \operatorname{supp}(A) \Subset \Omega' \times \Omega'\}\,. \tag{2.194}$$

There are plenty of such operators if $\Omega' \neq \emptyset$ since if $\phi, \psi \in \mathcal{C}^\infty_c(\Omega')$ and $B \in \Psi^m_\infty(\mathbb{R}^n)$ then $A = \phi B \psi \in \Psi^m_{\rm c}(\Omega')$ since it satisfies (2.194). It follows that if $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ has support in $K \times \mathbb{R}^n$ for some $K \Subset \Omega'$ then there exists $A \in \Psi^m_{\rm c}(\Omega')$ such that $\sigma_L(A) \equiv a$ modulo $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$ – simply take some B with this symbol and then set $A = \phi B \phi$ where $\phi \in \mathcal{C}^\infty_c(\Omega')$ but $\phi = 1$ in a neighbourhood of K.

Proposition 2.11. *If $F : \Omega \longrightarrow \Omega'$ is a diffeomorphism then for $A \in \Psi^m_c(\Omega')$,*

$$A_F u = F^* A (F^{-1})^* (u\big|_\Omega) \text{ defines an isomorphism } \Psi^m_c(\Omega') \longrightarrow \Psi^m_c(\Omega). \tag{2.195}$$

Proof. Since $A \in \Psi^m_\infty(\mathbb{R}^n)$,

$$K_A(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, \xi) d\xi \tag{2.196}$$

for some $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$. Now choose $\psi \in \mathcal{C}^\infty_c(\Omega)$ such that $\psi(x)\psi(y) = 1$ on $\operatorname{supp}(K_A)$, which is possible by (2.194). Then

$$K_A = I\left(\psi(x)\psi(y)a(x, \xi)\right). \tag{2.197}$$

In fact suppose $\mu_\epsilon(x, y) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ and $\mu \equiv 1$ in $|x - y| < \epsilon$ for $\epsilon > 0$, $\mu(x, y) = 0$ in $|x - y| > 2\epsilon$. Then if

$$K_{A_\epsilon} = I\left(\mu_\epsilon(x, y)\psi(x)\psi(y)a(x, \xi)\right) \tag{2.198}$$

we know that if

$$A'_\epsilon = A - A_\epsilon \text{ then } K_{A'_\epsilon} = \left(1 - \mu_\epsilon(x, y)\right) K_A \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.199}$$

Then $A'_\epsilon \in \Psi^{-\infty}_{\rm c}(\Omega')$ and

$$(A'_\epsilon)_F \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.200}$$

So we only need to consider A_ϵ defined by (2.198). Now

$$K_{(A_\epsilon)_F}(x, y) = (2\pi)^{-n} \int e^{i(G(x)-G(y))\cdot\xi} b(G(x), G(y), \xi) \Big|\frac{\partial G}{\partial y}\Big| d\xi \tag{2.201}$$

where $b(x, y, \xi) = \mu_\epsilon(x - y)\psi(x)\psi(y)a(x, \xi)$. Applying Taylor's formula,

$$G(x) - G(y) = (x - y) \cdot T(x, y) \tag{2.202}$$

where $T(x, y)$ is an invertible $\mathcal{C}^\infty$ matrix on $K \times K \cap \{|x - y| < \epsilon\}$ for $\epsilon < \epsilon(K)$, where $\epsilon(K) > 0$ depends on the compact set $K \Subset \Omega'$. Thus we can set

$$\eta = T^t(x, y) \cdot \xi \tag{2.203}$$

and rewrite (2.201) as

$$\begin{gathered} K_{(A_\epsilon)_F}(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\eta} c(x, y, \eta) d\eta \\ c(x, y, \eta) = b\left(G(x), G(y), (T^t)^{-1}(x, y)\eta\right) \Big|\frac{\partial G}{\partial y}\Big| \cdot \big|\det T(x, y)\big|^{-1}. \end{gathered} \tag{2.204}$$

So it only remains to show that $c \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and the proof is complete. We can drop all the $\mathcal{C}^\infty$ factors, given by $\left|\partial G/\partial y\right|$ etc. and proceed to show that

$$\left|D^\alpha_x D^\beta_y D^\gamma_\xi a\left(G(x), G(y), S(x,y)\xi\right)\right| \le C(1+|\xi|)^{m-|\gamma|} \quad \text{on } K\times K\times\mathbb{R}^n \tag{2.205}$$

where $K \subset\subset \Omega'$ and S is $\mathcal{C}^\infty$ with $|\det S| \ge \epsilon$. The estimates with $\alpha=\beta=0$ follow easily and the general case by induction:

$$\begin{aligned} &D^\alpha_x D^\beta_y D^\gamma_\xi a\left(G(x), G(y), S(x,y)\xi\right) \\ &\qquad = \sum_{\substack{|\mu|\le|\alpha|+|\beta|+|\gamma| \\ |\alpha'|\le|\alpha|,|\beta|\le|\beta| \\ |\nu|+|\gamma|\le|\mu|}} M^{\alpha',\rho',\mu'}_{\alpha,\beta,\gamma,\nu}(x,y)\xi^\nu \left(D^{\alpha'}D^{\beta'}D^\mu a\right)(G(x),G(y),S\xi) \end{aligned}$$

where the coefficients are $\mathcal{C}^\infty$ and the main point is that $|\nu| \le |\mu|$. □

2.19. Semiclassical limit

Let us at least pretend to go back to the beginning once more in order to understand the following 'problem'. From the origins of quantum mechanics the relationship between the quantum and related classical system has always been a primary interest. In classical Hamiltonian mechanics the 'energy' (I will keep to one dimension for the moment in the interest only of simplicity) is the sum of the kinetic and potential energies,

$$E(x,\xi) = \frac{1}{2}\hbar\xi^2 + V(x) \tag{2.206}$$

Here $\hbar$ is a 'small parameter' which represents either a coupling constant (the fine structure constant relating the energy change in an atom to the frequence of the light emitted) or else a small 'mass'. The 'corresponding' (one has to be careful about this, the process of quantization does not really work this way) quantum system is

$$q_L(E) = -\frac{1}{2}\hbar\frac{d^2}{dx^2} + V(x). \tag{2.207}$$

For $\hbar > 0$ – which is really the case – this is a perfectly good elliptic (at least locally) differential operator. However something singular clearly happens as $\hbar \downarrow 0$ (although you might ask how a constant is supposed to go to zero; fortunately we have other less frivolous reasons for looking at this).

If we simply set $\hbar = \epsilon^2$ then we can rewrite (2.207) in the form

$$-\frac{1}{2}(\epsilon\frac{d}{dx})^2 + V(x). \tag{2.208}$$

This suggests that to generalize the structure in (2.208) to 'arbitrary symbols' in place of (2.206) we should simply consider operators of the form

$$\begin{aligned} A_\epsilon u(x) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(\epsilon,x,y,\epsilon\xi)u(y)dyd\xi \\ &= (2\pi\epsilon)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\eta/\epsilon} a(\epsilon,x,y,\eta)u(y)dyd\eta \end{aligned} \tag{2.209}$$

where the second version follows from the first by changing variable to $\eta = \epsilon\xi$ and $a \in \mathcal{C}^\infty([0,1]_\epsilon; S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ is a symbol in the usual sense which may also depend smoothly on ϵ.

Definition 2.3. *Let* $\Psi^m_{\text{sl-}\infty}(\mathbb{R}^n) \subset \mathcal{C}^\infty((0,1];\Psi^m_\infty(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\text{sl}}(\mathbb{R}^n)) \subset \mathcal{C}^\infty((0,1];\Psi^m_{\text{ph}}(\mathbb{R}^n))$ *be the subspace consisting of those 1-parameter families which are of the form* (2.209) *for some* $a \in \mathcal{C}^\infty([0,1];S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n))$ *(resp.* $a \in \mathcal{C}^\infty([0,1];S^m_{\text{ph}}(\mathbb{R}^{2n};\mathbb{R}^n))$.

There is no question about the form of the kernels of these operators. Namely, directly from the second form of the definition

$$A_\epsilon \text{ has kernel of the form } \epsilon^{-n}K_\epsilon(x,\frac{x-y}{\epsilon}) \tag{2.210}$$

where $K_\epsilon(x,x-y)$ is the kernel of a smooth family of pseudodifferential operators in the usual sense, namely

$$K_\epsilon(x,x-y) \text{ is the kernel of } I(a_\epsilon). \tag{2.211}$$

So, as $\epsilon \downarrow 0$ the kernel very much 'bunches up' around the diagonal. This rather explicit description does not tell us directly about the composition properties of these 1-parameter families of operators. However we can work this out fairly easily. First check what happens for the operators of order $-\infty$.

Proposition 2.12. *The space* $\Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n) = \Psi^{-\infty}_{\text{sl-}\infty}(\mathbb{R}^n)$ *is closed under composition and adjoints and there is a short exact multiplicative sequence*

$$\epsilon\Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n) \xrightarrow{\sigma_{\text{sl}}} S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.212}$$

Proof. Already from (2.209) it follows directly that the residual algebra is given by symbols of order $-\infty$, that is

$$\begin{gathered} A_\epsilon \in \bigcap_m \Psi^m_{\text{sl-}\infty}(\mathbb{R}^n) \Longleftrightarrow \\ A_\epsilon \text{ is of the form (2.209) with } a \in \mathcal{C}^\infty([0,1];S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)) \end{gathered} \tag{2.213}$$

since the kernel $K_\epsilon(x,x-y)$ is uniquely determined by A_ϵ. This also shows that the 'residual space' is the same for the classical and non-classical cases.

Thus if $A_\epsilon \in \Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n)$ then there exists $K_\epsilon \in \mathcal{C}^\infty_\infty([0,1]\times\mathbb{R}^n;\mathcal{S}(\mathbb{R}^n))$ such that

$$A_\epsilon \text{ has kernel } \epsilon^{-n}K_\epsilon(\epsilon,x,\frac{x-y}{\epsilon}). \tag{2.214}$$

So the composite – really only for $\epsilon > 0$ – of two such (families of) operators A_ϵ and B_ϵ, where the kernel of B_ϵ is given by (2.214) for a different function L_ϵ, has kernel

$$\begin{aligned} \epsilon^{-n}J_\epsilon(x,\frac{x-y}{\epsilon}) &= \epsilon^{-2n}\int_{\mathbb{R}^n} K(x,\frac{x-z}{\epsilon})L_\epsilon(z,\frac{z-y}{\epsilon})dz \\ &= \epsilon^{-n}\int_{\mathbb{R}^n} K(x,t)L_\epsilon(x-\epsilon t,\frac{x-y}{\epsilon}+t)dt \end{aligned} \tag{2.215}$$

where $t = (x-z)/\epsilon$. Thus changing independent variable to $Z = (x-y)/\epsilon$ the kernel of the product (for $\epsilon > 0$) becomes

$$J_\epsilon(x,Z) = \int_{\mathbb{R}^n} K_\epsilon(x,t)L_\epsilon(x-\epsilon t,Z+t)dt. \tag{2.216}$$

Now, it is easy to see that $J_\epsilon(x,Z) \in \mathcal{C}^\infty_\infty([0,1]_\epsilon \times \mathbb{R}^n;\mathcal{S}(\mathbb{R}^n))$. The rapid decay in t in the first factor in the integrand gives rapid convergence of the integral and overall boundness of J_ϵ. Rapid decay in Z follows from the estimate

$$|Z| \le |t| + |Z+t| \tag{2.217}$$

and differentiating with respect to any of the independent variables gives a similar integral with similar bounds.

This shows that the composite is also in $\Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n)$. Notice that at $\epsilon = 0$,

$$J_0(x,Z) = \int_{\mathbb{R}^n} K_0(x,t)L_0(x,Z+t)dt \Longrightarrow c(0,x,\xi) = a(0,x,\xi)b(0,x,\xi). \tag{2.218}$$

by taking the Fourier transform in Z. Thus (2.212) is satisfied by the map

$$\sigma_{\mathrm{sl}}(A_\epsilon) = a(0,x,\xi) \in S_\infty^{-\infty}(\mathbb{R}^n;\mathbb{R}^n) = \mathcal{C}_\infty^\infty(\mathbb{R}^n;\mathcal{S}(\mathbb{R}^n)). \tag{2.219}$$

$\square$

It is important to contrast the behaviour of this 'semiclassical symbol' with the usual symbol – with which it is closely related of course. Namely the semiclassical symbol describes in rather complete detail the leading behaviour of the operator at $\epsilon = 0$ and is multiplicative. What this really shows is the basic property of the semiclassical limit, namely that these operators 'become commutative' at $\epsilon = 0$ (where they also fail to exist in the usual sense).[11] As with the principal symbol rather fine results can be proved by iteration. Thus

$$A_\epsilon \in \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n) \text{ and } \sigma_{\mathrm{sl}}(A_\epsilon) = 0 \Longrightarrow A_\epsilon = \epsilon A_\epsilon^{(1)},\ A_\epsilon^{(1)} \in \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n). \tag{2.220}$$

Then if one can arrange repeatedly that $\sigma_{\mathrm{sl}}(A_\epsilon^{(1)}) = 0$ and so on, one may finally conclude that[12]

$$A_\epsilon \in \bigcap_N \epsilon^N \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n) \Longleftrightarrow A_\epsilon \in \mathcal{C}^\infty([0,1];\Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n)) \text{ and } \frac{d^k}{d\epsilon^k}A_\epsilon\big|_{\epsilon=0} = 0. \tag{2.221}$$

Now we proceed to show that this result extends directly to the operators of finite order.

Theorem 2.5. *The semiclassical families in* $\Psi_{\mathrm{sl}\text{-}\infty}^m(\mathbb{R}^n)$ *(or* $\Psi_{\mathrm{sl}}^m(\mathbb{R}^n)$*) form an order-filtered $*$-algebra with two multiplicative symbol maps, one a uniform (perhaps better to say 'rescaled') version of the usual symbol and the second a finite order version of the semiclassical symbol in* (2.219)*:*

$$\begin{gathered}\tilde\sigma_m : \Psi_{\mathrm{sl}}^m(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty([0,1]\times\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)),\ \tilde\sigma_m(A_\epsilon)(x,\eta) = \sigma_m(A_\epsilon)(x,\eta/\epsilon),\\ \sigma_{\mathrm{sl}} : \Psi_{\mathrm{sl}}^m(\mathbb{R}^n) \longrightarrow S_{\mathrm{ph}}^m(\mathbb{R}^n\times\mathbb{R}^n);\end{gathered} \tag{2.222}$$

they are separately surjective and are jointly subject only to the compatibility condition

$$\sigma_{\mathrm{sl}}(A_\epsilon) = \tilde\sigma_m(A_\epsilon)\big|_{\epsilon=0} \text{ in } S_\infty^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.223}$$

Proof. By definition $A_\epsilon \in \Psi_{\mathrm{sl}\text{-}\infty}^m(\mathbb{R}^n)$ means precisely that there is a smooth family $a_\epsilon \in \mathcal{C}^\infty([0,1];S_\infty^m(\mathbb{R}^n;\mathbb{R}^n))$ such that if $K_\epsilon(x,x-y)$ is the family of kernels of $q_L(a_\epsilon)$ then (2.210) holds. Thus the two maps in the statement of the theorem, with

$$\begin{gathered}\tilde\sigma_m(A_\epsilon) = [a_\epsilon] \in \mathcal{C}^\infty([0,1];S_\infty^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)) \text{ and}\\ \sigma_{\mathrm{sl}}(A_\epsilon) = a_0 \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)\end{gathered} \tag{2.224}$$

are certainly well-defined and subject only to the stated compatibility condition.

[11] See Problem 2.22.

[12] See Problem 2.24 for an outline of the proof.

Thus the main issue is multiplicativity. Since a_ϵ can be smoothly approximated by symbols of order $-\infty$ we can use continuity in the symbol topology and start from (2.216). For $\epsilon = 1$

$$
\begin{aligned}
J(x,Z) &= \int_{\mathbb{R}^n} K(x,t)L(x-t,Z+t)dt,\\
K(x,t) &= (2\pi)^{-n}\int e^{it\cdot\xi}b(x,\xi)d\xi,\\
L(x,t) &= (2\pi)^{-n}\int e^{it\cdot\xi}a(x,\xi)d\xi,\\
c(x,\xi) &= \int e^{-iZ\cdot\xi}J(x,Z)dZ
\end{aligned}
\tag{2.225}
$$

reproduces the usual composition formula. Thus we know that this formula extends by continuity to define the jointly continuous product map

$$
S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \times S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow S^{m+m'}_\infty(\mathbb{R}^n;\mathbb{R}^n).
\tag{2.226}
$$

Now, we can simplify this by assuming that a is constant-coefficient, i.e. is independent of the base variable. The to evaluate $c(0,\xi)$ we only need to know $J(0,Z)$ which is given by the (extension by continuity of) the simplified formula, which therefore, by restriction, defines a continuous map

$$
J(0,Z) = \int_{\mathbb{R}^n} K(t)L(-t,Z+t)dt,\ S^m_\infty(\mathbb{R}^n) \times S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow S^{m+m'}_\infty(\mathbb{R}^n).
\tag{2.227}
$$

Now, from (2.225)

$$
L(-t,Z) = (2\pi)^{-n}\int e^{iZ\cdot\xi}a(-t,\xi)d\xi
\tag{2.228}
$$

so in the corresponding formula with ϵ varying

$$
J(0,Z) = \int_{\mathbb{R}^n} K(t)L(-\epsilon t,Z+t)dt
\tag{2.229}
$$

$L(-\epsilon t, Z+t)$ corresponds to the symbol $a(-\epsilon t,\xi) \in \mathcal{C}^\infty([0,1]; S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n))$ as follows easily by direct differentiation. Thus if we fix x in (2.216) at any point in $\mathbb{R}^n$ this shows that the product extends by continuity to the finite order symbol spaces. Then, using the bilinearity, the smooth dependence on x as a parameter can be restored. Thus in fact the same results on composition follow as in the smoothing case, that

$$
\begin{aligned}
\tilde\sigma_{m+m'}(A_\epsilon B_\epsilon) &= \tilde\sigma_m(A_\epsilon)\tilde\sigma_{m'}(B_\epsilon) \text{ and}\\
\sigma_{\text{sl-}\infty}(A_\epsilon B_\epsilon) &= \sigma_{\text{sl-}\infty}(A_\epsilon)\sigma_{\text{sl-}\infty}(B_\epsilon).
\end{aligned}
\tag{2.230}
$$

□

Of course the uniform symbol $\tilde\sigma_m(A)$ is not quite the usual symbol precisely because of rescaling but is equivalent to it for $\epsilon > 0$. Namely

$$
\sigma_m(A_\epsilon)(x,\xi) = \tilde\sigma_m(A_\epsilon)(\epsilon,x,\epsilon\xi).
\tag{2.231}
$$

Maybe you like to have things written out explicitly as short exact sequences. There are in fact three such (or more if you allow polyhomogeneous/∞ variants), all of which are also multiplicative. Thus

(2.232)

$$\Psi_{\mathrm{sl}}^{m-1}(\mathbb{R}^n) \longrightarrow \Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \xrightarrow{\tilde{\sigma}_m} \mathcal{C}^\infty([0,1]; S_{\mathrm{ph}}^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)),$$

$$\epsilon\Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \longrightarrow \Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \xrightarrow{\sigma_{\mathrm{sl}}} S_{\mathrm{ph}}^{m}(\mathbb{R}^n;\mathbb{R}^n),$$

$$\epsilon\Psi_{\mathrm{sl}}^{m-1}(\mathbb{R}^n) \longrightarrow$$

$$\Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \overset{(\tilde{\sigma}_m,\sigma_{\mathrm{sl}})}{\longrightarrow}$$

$$\big\{(\tilde{a},a) \in S_{\mathrm{ph}}^{m}(\mathbb{R}^n;\mathbb{R}^n) \oplus \mathcal{C}^\infty([0,1]; S_{\mathrm{ph}}^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)); \tilde{a} = a\big|_{\epsilon=0} \text{ in } S_{\mathrm{ph}}^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)\big\}.$$

We also want to check coordinate invariance. Note that the semiclassical algebras are mapped into themselves by multiplication of the kernel by an element of $\mathcal{C}^\infty_\infty(\mathbb{R}^{2n}_{x,y})$. In particular we may freely localize on the left or the right by a smooth function of compact support and stay in the algebra. The coordinate invariance of the semiclassical algebra then follows from that of the usual algebra.

PROPOSITION 2.13. *If $A_\epsilon \in \Psi_{\text{sl-}\infty}^{m}(\mathbb{R}^n)$ has kernel with compact support in $\Omega\times\Omega$ for some open $\Omega \subset \mathbb{R}^n$ and $F : \Omega \longrightarrow \Omega'$ is a diffeomorphism then $A_{F,\epsilon} = (F^{-1})^* A_\epsilon F^* \in \Psi_{\text{sl-}\infty}^{m}(\mathbb{R}^n)$ and*

$$\begin{aligned}\tilde{\sigma}_m(A_{F,\epsilon}) &= (F^*)^*\tilde{\sigma}_m(A_\epsilon)\\ \sigma_{\mathrm{sl}}(A_{F,\epsilon}) &= (F^*)^*\sigma_{\mathrm{sl}}(A_\epsilon).\end{aligned} \tag{2.233}$$

Note that F^* is linear on the fibres so commutes with the rescaling map.

We will also need some boundedness properties of semiclassical families. The following will suffice for our purposes.

PROPOSITION 2.14. *For $A_\epsilon \in \Psi_{\text{sl-}\infty}^{0}(\mathbb{R}^n)$,*

$$\sup_{0<\epsilon\le 1} \|A_\epsilon\|_{L^2(\mathbb{R}^n)} < \infty. \tag{2.234}$$

PROOF. It is only the uniformity in (2.234) that is at issue, since we know the boundedness for $1 \ge \epsilon \ge \delta$ for any $\delta > 0$. The argument we give is essentially the same as for boundedness. Namely for $C > 0$ large enough we can extract an approximate square-root

$$C - A_\epsilon^* A_\epsilon = B_\epsilon^2 + E_\epsilon,\ B \in \Psi_{\mathrm{sl}}^{0}(\mathbb{R}^n),\ E \in \epsilon^\infty\mathcal{C}^\infty([0,1;\Psi^{-\infty}(\mathbb{R}^n)). \tag{2.235}$$

This can be seen using essentially the same symbolic computation as before but now for both symbols. Thus if $C > \sigma_0(A)^*\sigma_0(A)$ and $C > \sigma_{\mathrm{sl}}(A)^*\sigma_{\mathrm{sl}}(A)$ (and note that the second can well be larger than the first) then be can choose $B \in \Psi_{\mathrm{sl}}^{0}(\mathbb{R}^n)$ with $B^* = B$, $\sigma_0(B)^2 = C - \sigma_0(A)^*\sigma_0(A)$, $\sigma_{\mathrm{sl}}(B)^2 = C\sigma_{\mathrm{sl}}(A)^*\sigma_{\mathrm{sl}}(A)$ (because the consistency condition is satisfied) and hence

$$C - A^*A = B^2 + E_1,\ E_1 \in \epsilon\Psi_{\mathrm{sl}}^{-1}(\mathbb{R}^n). \tag{2.236}$$

Then the construction can be iterated as before to construct a solution to (2.235). The uniform boundedness of E_ϵ is clear – in fact its norm vanishes rapidly as $\epsilon \downarrow 0$ so the uniform boundedness follows. $\square$

2.20. Adiabatic and semiclassical families

In the preceeding section semiclassical families of smoothing operators were discussed. Later we need to consider similar families with two parameters. So, here the local case is analysed. Consider a decomposition of Euclidean space into two factors,

$$\mathbb{R}^{n+\tilde{n}} = \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}. \tag{2.237}$$

It is straightforward to consider an 'adiabatic' analogue of the semiclassical calculus above. Namely if consider smooth families of kernels of smoothing operators in $\Psi^{-\infty}_{\infty}(\mathbb{R}^{n+\tilde{n}}$ as before. Now however then 'compress' them as for the semiclassical calculus, but only in the second set of variables and consider the families of smoothing operators for $\delta > 0$,

$$\begin{gathered} B : \mathcal{S}(\mathbb{R}^{n+\tilde{n}}) \longrightarrow \mathcal{C}^\infty((0,1]; \mathcal{S}(\mathbb{R}^{n+\tilde{n}}), \\ Bf(\delta, z, \tilde{z}) = \delta^{-\tilde{n}} \int_{\mathbb{R}^{n+\tilde{n}}} B(\delta, z, z - z', \tilde{z}, \frac{\tilde{z} - \tilde{z}'}{\delta}) f(z', \tilde{z}') dz' d\tilde{z}', \\ B(\delta, z, z - z', \tilde{z}, \tilde{z} - \tilde{z}') \in \mathcal{C}^\infty([0,1]; \Psi^{-\infty}_{\infty}(\mathbb{R}^{n+\tilde{n}})). \end{gathered} \tag{2.238}$$

Proposition 2.15. *The collection of families of operators of the form* (2.238) *forms an algebra, denoted* $\Psi^{-\infty}_{\infty\,\mathrm{ad}}(\mathbb{R}^n; \mathbb{R}^{\tilde{n}})$ *under composition with adiabatic symbol map*

$$B \longmapsto \sigma_{\mathrm{ad}}(B)(z, z - z', \tilde{z}, \zeta) = \int_{\mathbb{R}^{\tilde{n}}} e^{-iZ\zeta} B(z, z - z', \tilde{z}, Z) dZ, \tag{2.239}$$

giving a multiplicative short exact sequence

$$\delta \Psi^{-\infty}_{\infty\,\mathrm{ad}}(\mathbb{R}^n; \mathbb{R}^{\tilde{n}}) \longrightarrow \Psi^{-\infty}_{\infty\,\mathrm{ad}}(\mathbb{R}^n; \mathbb{R}^{\tilde{n}}) \xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{S}^{-\infty}_{\infty}(\mathbb{R}^{\tilde{n}}; \mathbb{R}^{\tilde{n}}; \Psi^{-\infty}_{\infty}(\mathbb{R}^n)) \tag{2.240}$$

Of course the algebra depends on set of variables in which the 'adiabatic limit' is taken. The semiclassical calculus corresponds to $n = 0$, meaning no 'noncommutative' variables survive.

Proof. Following the discussion above of the semiclassical limit, simply change variables in the composition formula which holds in $\delta > 0$ defining the left side

$$\begin{aligned} &\delta^{-\tilde{n}} C(\delta; z, z - z'', \tilde{z}, \frac{\tilde{z} - \tilde{z}''}{\delta}) \\ &\quad = \delta^{-2\tilde{n}} \int_{\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}} A(\delta; z, z - z', \tilde{z}, \frac{\tilde{z} - \tilde{z}'}{\delta}) B(\delta; z', z' - z'', \tilde{z}', \frac{\tilde{z}' - \tilde{z}''}{\delta}) dz' d\tilde{z}' \end{aligned} \tag{2.241}$$

by introducing $\tilde{Z} = (\tilde{z} - \tilde{z}'')/\delta$ and $\tilde{Z}' = (\tilde{z} - \tilde{z}')/\delta$ so that

$$C(\delta; z, z - z'', \tilde{z}, Z) = \int_{\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}} A(\delta; z, z - z', \tilde{z}, \tilde{Z}') B(\delta; z', z' - z'', \tilde{z} - \delta \tilde{Z}', \tilde{Z} - \tilde{Z}') dz' d\tilde{Z}'. \tag{2.242}$$

In this form the same argument as in the semiclassical case shows that the composite is of the same type. Moreover, when $\delta = 0$ the composite kernel is given just by convolution in the second variables, with $\tilde{z}$ just a parameter, and still by operator composition in the first variables. Thus gives the multiplicativity of the adiabatic symbol map in (2.240). $\square$

As well as this adiabatic calculus we need to consider a two parameter calculus in which both the overall semiclassical limit and the adiabatic limit just considered occur. Thus, still starting with the same types of kernels, but now with two parameters,

$$A \in \mathcal{C}^\infty([0,1]_\epsilon \times [0,1]_\delta; \Psi^{-\infty}_{\infty}(\mathbb{R}^{n+\tilde{n}}) \tag{2.243}$$

we consider the families of operators with kernels

$$\epsilon^{-n-\tilde{n}}\delta^{-\tilde{n}}B(z,\frac{z-z}{\epsilon},\tilde{z},\frac{\tilde{z}-\tilde{z}'}{\epsilon\delta}). \tag{2.244}$$

Proposition 2.16. *The space of operators two-parameter familes of operators with kernels of the form* (2.244) *forms an algebra under composition, denoted* $\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})$ *which has two multiplicative 'symbol' maps*

$$\begin{gathered}\sigma_{\mathrm{sl}}\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})\longrightarrow\mathcal{C}^\infty([0,1]_\delta;S^{-\infty}_\infty(\mathbb{R}^{n+\tilde{n}};\mathbb{R}^{n+\tilde{n}})),\\ \sigma_{\mathrm{sl}}(B)=\int e^{-iZ\zeta-i\tilde{Z}\tilde{\zeta}}B(0,\delta,z,Z,\tilde{z},\tilde{Z}),\\ \sigma_{\mathrm{ad}}\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})\longrightarrow S^{-\infty}_\infty(\mathbb{R}^{\tilde{n}};\mathbb{R}^{\tilde{n}};\Psi^{-\infty}_{\infty\,\mathrm{sl}}(\mathbb{R}^n))\\ \sigma_{\mathrm{ad}}(B)=\epsilon^{-\tilde{n}}\int e^{-i\tilde{z}\tilde{\zeta}}B(\epsilon,0,z,\frac{z-z'}{\epsilon},\tilde{z},\tilde{Z}).\end{gathered} \tag{2.245}$$

The proof of this is easy, as before, the main problem is to take in what it actually means! Passing to $\delta=0$ we get a family depending on ϵ which is just on $\mathbb{R}^n$, the first variables, and is undergoing an adiabatic limit as $\epsilon\downarrow 0$. Passing to $\epsilon=0$ for $\delta>0$ we are simply doing a semiclassical limit in which δ appears as a parameter and in an appropriate sense is uniform down to $\delta=0$. Of course the limits at $\epsilon=\delta=0$ should match up independently of the order in which the variables go to zero. This is encapsulated in the identity

$$\sigma_{\mathrm{sl}}(B)\big|_{\delta=0}=\sigma_{\mathrm{sl}}(\sigma_{\mathrm{ad}}(B)). \tag{2.246}$$

Proof. Writing down the composition formula for $C=A\circ B$ as before, when $\epsilon>0$ and $\delta>0$ and changing variables we find that

$$\begin{gathered}C(\epsilon,\delta,z,Z,\tilde{z},\tilde{Z})=\\ \int_{\mathbb{R}^n\times\mathbb{R}^{\tilde{n}}}A(\epsilon,\delta;z,Z',\tilde{z},\tilde{Z}')B(\epsilon,\delta;z-\epsilon Z',Z-Z',\tilde{z}-\epsilon\delta\tilde{Z}',\tilde{Z}-\tilde{Z}')dZd\tilde{Z}'.\end{gathered} \tag{2.247}$$

Again it if straightforward to check that C is a family of kernels of smoothing operators. Moreover setting $\delta=0$ gives the adiabatic symbol, which from (2.247) undergoes the composition law for the the semiclassical composition in the first variables under composition of operators. On the other hand, setting $\epsilon=0$ gives the same semiclassical composition formula as before, although the scaling of variables involved in the definitions is different. □

The definitions of the symbols show that there are two short exact sequences

$$\begin{gathered}\delta\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})\longrightarrow\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})\xrightarrow{\sigma_{\mathrm{ad}}}S^{-\infty}_\infty(\mathbb{R}^{\tilde{n}};\mathbb{R}^{\tilde{n}};\Psi^{-\infty}_{\infty\,\mathrm{sl}}(\mathbb{R}^n))\\ \epsilon\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})\longrightarrow\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})\xrightarrow{\sigma_{\mathrm{sl}}}\mathcal{C}^\infty([0,1]_\delta;S^{-\infty}_\infty(\mathbb{R}^{n+\tilde{n}};\mathbb{R}^{n+\tilde{n}}))\end{gathered} \tag{2.248}$$

Moreover, the combined symbol map $\sigma_{\mathrm{sl}}\oplus\sigma_{\mathrm{ad}}$ has null space $\epsilon\delta\Psi^{-\infty}_{\infty\,\mathrm{sl\,ad}}(\mathbb{R}^n;\mathbb{R}^{\tilde{n}})$ and range the direct sum of the ranges, subject just to the compatibility condition (2.246).

2.21. Smooth and holomorphic families

I have gone through the description of 'classical' pseudodifferential operators of complex order here, even though it might seem rather strange – I want to emphasize that these really do arise in practice. In particular we will want to consider the notion of a holomorphic family of complex order $f(z)$ where f is holomorphic.

First consider the issue of continuous or smooth dependence on parameters. Since we have at least implicitly given $\Psi^m_\infty(\mathbb{R}^n)$ and $\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ topologies, this is already defined. In fact of course it is just the continuous or smooth dependence of the left-reduced symbol on the parameters, say in some open or smoothly-bounded subset of $\mathbb{R}^p$. Tracking back through the arguments above, it can be seen that the product theorem actually gives continuous dependence of the symbol of a product on the symbols of the factors, although a little thought is needed here because of the asymptotic summation involved see Problem 2.25 for a little more on this point. It is important that the product is unique. For homolormophy say of an element of $\Psi^m_\infty(\mathbb{R}^n)$ in terms of a complex variable $s \in U \subset \mathbb{C}$ open the discussion is essentially the same. Namely a (strongly) holomorphic function into a fixed topological vector space is just a continuous function which satisfies Cauchy criterion, that it integrates to zero around any closed contour. This is actually equivalent to smoothness in s and

$$\overline{\partial} A(s) = 0. \tag{2.249}$$

So, there is nothing very interesting going on here. For polyhomogeneous operators of a fixed order the story is the same, with the spaces of operators and symbols altered appropriately. However if the order itself is allowed to vary then a different notion of 'holomorphy' arises. Namely if $F : U \longrightarrow \mathbb{C}$ is itself a holomorphic function, we may consider polyhomogeneous symbols which are of order $f(s)$. As noted above this can be simplified by writing the (left-reduced) symbol in the form

$$a(s, x, \xi) =< \xi >^{f(s)} b(s, x, \xi) \tag{2.250}$$

where $b \in S^0_{\mathrm{ph}}(\mathbb{R}^n; \mathbb{R}^b)$. Then by holomorphy in this new sense we mean holomorphy of b in the usual sense, as a polyhomogeneous symbol of order 0. We can write $\Psi^f_{\mathrm{hol}}(\mathbb{R}^n)$ for this linear space of operators. Note that we drop the 'ph' since this does not make much sense without it!

Proposition 2.17. *If $A(s) \in \Psi^f_{\mathrm{hol}}(\mathbb{R}^n)$ and $B \in \Psi^g_{\mathrm{hol}}(\mathbb{R}^n)$ for two holomorphic functions f, $g : U \longrightarrow \mathbb{C}$,*

$$A \circ B \in \Psi^{f+g}_{\mathrm{hol}}(\mathbb{R}^n). \tag{2.251}$$

Proof. I suppose I should write one! □

Why bother with such operators? Globally in this sense on $\mathbb{R}^n$ it is difficult to come up with sensible examples but on a compact manifold or for the better 'global' calculi on $\mathbb{R}^n$ discussed below there are natural examples. For instance, getting very much ahead of myself here, if $A \in \Psi^1_{\mathrm{ph}}(M)$ is self-adjoint and elliptic on a compact manifold M then the complex powers A^z for an entire family, so complex in the sense above for $z \in \mathbb{C}$. This was first proved by Seeley and is the starting point for many interesting developments, see Chapters 4, 6 and 7 below.

2.22. Problems

PROBLEM 2.1. Show, in detail, that for each $m \in \mathbb{R}$

$$(1+|\xi|^2)^{\frac{1}{2}m} \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \tag{2.252}$$

for any p. Use this to show that

$$S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

PROBLEM 2.2. Consider $w=0$ and $n=2$ in the definition of symbols and show that if $a \in S^1_\infty(\mathbb{R}^2)$ is elliptic then for $r>0$ sufficiently large the integral

$$\int\limits_0^{2\pi} \frac{1}{2\pi}\frac{1}{a(re^{i\theta})}\frac{d}{d\theta}a(re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{d\theta}\log a(re^{i\theta})d\theta,$$

exists and is an integer independent of r, where $z=\xi_1+i\xi_2$ is the complex variable in $\mathbb{R}^2=\mathbb{C}$. Conclude that there is an elliptic symbol, a on $\mathbb{R}^2$, such that there does not exist b, a symbol with

$$b \neq 0 \text{ on } \mathbb{R}^2 \text{ and } a(\xi)=b(\xi) \text{ for } |\xi|>r \tag{2.253}$$

for any r.

PROBLEM 2.3. Show that a symbol $a \in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ which satisfies an estimate

$$|a(z,\xi)| \le C(1+|\xi|)^{m'},\ m'<m \tag{2.254}$$

is necessarily in the space $S^{m'+\epsilon}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ for all $\epsilon>0$.

PROBLEM 2.4. Show that if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^p_z\times\mathbb{R}^n)$ and $\psi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\psi(\xi)=1$ in $|\xi|<1$ then

$$c_\phi(z,\xi)=\phi(z,\frac{\xi}{|\xi|})(1-\psi)(\xi)\in S^0(\mathbb{R}^p_z;\mathbb{R}^n_\xi). \tag{2.255}$$

If $a\in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ define the cone support of a in terms of its complement

$$\begin{aligned}\operatorname{cone\,supp}(a)^{\complement} = \{(\bar z,\bar\xi)\in\mathbb{R}^p_z\times(\mathbb{R}^n_\xi\setminus\{0\});\exists\\ \phi\in\mathcal{C}^\infty_c(\mathbb{R}^p_z;\mathbb{R}^n),\ \phi(\bar z,\bar\xi)\neq 0,\ \text{such that } c_\phi a\in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)\}.\end{aligned} \tag{2.256}$$

Show that if $a\in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ and $b\in S^{m'}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ then

$$\operatorname{cone\,supp}(ab)\subset\operatorname{cone\,supp}(a)\cap\operatorname{cone\,supp}(b). \tag{2.257}$$

If $a\in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ and $\operatorname{cone\,supp}(a)\emptyset$ does it follow that $a\in S^{-\infty}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$?

PROBLEM 2.5. Prove that (2.30) is a characterization of functions $a\in(1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$. [Hint: Use Liebniz' formula to show instead that the equivalent estimates

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)|\le C_{\alpha,\beta,\gamma}(1+|x-y|^2)^{w/2}(1+|\xi|)^{m-|\gamma|}\ \forall\ \alpha,\beta,\gamma\in\mathbb{N}^n_0$$

characterize this space.]

PROBLEM 2.6. Show that $A\in\Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies all the estimates

$$|D^\alpha_x D^\beta_y a(x,y)|<C_{\alpha,\beta,N}(1+|x-y|)^{-N} \tag{2.258}$$

for multiindices $\alpha,\beta\in\mathbb{N}^n_0$ and $N\in\mathbb{N}_0$.

PROBLEM 2.7. Polyhomogeneous symbols as smooth functions.

PROBLEM 2.8. General polyhomogeneous symbols and operators.

PROBLEM 2.9. Density of polyhomogeneous symbols in L^∞ symbols of the same order.

PROBLEM 2.10. Completeness of the spaces of polyhomogeneous symbols.

PROBLEM 2.11. Fourier transform??

PROBLEM 2.12. Show that the kernel of any element of $\Psi^{\infty}_{\infty}(\mathbb{R}^n)$ is $\mathcal{C}^\infty$ away from the diagonal. Hint: Prove that $(x-y)^\alpha K(x,y)$ becomes increasingly smooth as $|\alpha|$ increases.

PROBLEM 2.13. Show that for any $m \geq 0$ the unit ball in $H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is *not* precompact, i.e. there is a sequence $f_j \in H^m(\mathbb{R}^n)$ which has $\|f_j\|_m \leq 1$ and has no subsequence convergent in $L^2(\mathbb{R}^n)$.

PROBLEM 2.14. Show that for any $R > 0$ there exists $N > 0$ such that the Hilbert subspace of $H^N(\mathbb{R}^n)$

$$\{u \in H^N(\mathbb{R}^n); u(x) = 0 \text{ in } |x| > R\} \tag{2.259}$$

is compactly included in $L^2(\mathbb{R}^n)$, i.e. the intersection of the unit ball in $H^N(\mathbb{R}^n)$ with the subspace (2.259) is precompact in $L^2(\mathbb{R}^n)$. Hint: This is true for *any* $N > 0$, taking $N >> 0$ will allow you to use the Sobolev embedding theorem and Arzela-Ascoli.

PROBLEM 2.15. Using Problem 2.14 (or otherwise) show that for any $\epsilon > 0$

$$(1+|x|)^\epsilon H^\epsilon(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$$

is a compact inclusion, i.e. any infinite sequence f_n such that $(1+|x|^2)^{-\epsilon}$ is bounded in $H^\epsilon(\mathbb{R}^n)$ has a subsequence convergent in $L^2(\mathbb{R}^n)$. Hint: Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(x) = 1$ in $|x| < 1$ and, for each k, consider the sequence $\phi(x/k)f_j$. Show that the Fourier transform converts this into a sequence which is bounded in $(1+|\xi|^2)^{-\frac{1}{2}\epsilon}H^N(\mathbb{R}^n_\xi)$ for any N. Deduce that it has a convergent subsequence in $L^2(\mathbb{R}^n)$. By diagonalization (and using the rest of the assumption) show that f_j itself has a convergent subsequence.

PROBLEM 2.16. About ρ and δ.

PROBLEM 2.17. Prove the formula (2.191) for the left-reduced symbol of the operator A_T obtained from the pseudodifferential operator A by linear change of variables. How does the right-reduced symbol transform?

PROBLEM 2.18. Density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$.

PROBLEM 2.19. Square-root of a positive elliptic symbol is a symbol.

PROBLEM 2.20. Write out a proof to Proposition 4.2. Hint (just to do it elegantly, it is straightforward enough): Write A in right-reduced form as in (2.74) and apply it to $\hat{u}$; this gives a formula for $\hat{A}u$.

PROBLEM 2.21. Show that any continuous linear operator

$$\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

has Schwartz kernel in $\mathcal{S}(\mathbb{R}^{2n})$.

PROBLEM 2.22. Show that if A_ϵ and B_ϵ are as in Proposition 2.12 then they have unique representations as in (2.209) with left-reduced symbols, respectively a, b and for the composite c all in $\mathcal{C}^\infty_\infty([0,1]\times\mathbb{R}^n;\mathcal{S}(\mathbb{R}^n))$ and where in the sense of Taylor series at $\epsilon = 0$,

$$c(\epsilon, x, \eta) \simeq \sum_\alpha \frac{\epsilon^{|\alpha|}}{\alpha!}\partial^\alpha_\eta a(\epsilon, x, \xi)\cdot \partial^\alpha_x b(\epsilon, x, \eta). \tag{2.260}$$

PROBLEM 2.23. Give the details of the reduction argument in the semiclassical setting. Here are some suggestions. First use integration by parts based on the identity

$$\epsilon^2 \Delta_\eta e^{i(x-y)\cdot\eta/\epsilon} = |x-y|^2 e^{i(x-y)\cdot\eta/\epsilon} \tag{2.261}$$

to show that the kernel of a semiclassical family A_ϵ is smooth in $|x-y|>\delta>0$ in all variables, including ϵ, as a funtion of x and $x-y$, with all x derivatives bounded and rapidly decaying in $x-y$ – that is smoothly cut off in $|x-y>\delta>0$ it is in $\mathcal{C}^\infty([0,1]_\epsilon;\Psi^{-\infty}_\infty(\mathbb{R}^n)$ and vanishes with all its derivatives at $\epsilon=0$. Next use the left reduction argument and asymptotic summation to treat the part of the kernel supported in $|x-y|<\delta$.

PROBLEM 2.24. Proof of (2.221).

PROBLEM 2.25. Asymptotic summation of holomorphic families of symbols.

CHAPTER 3

Schwartz and smoothing algebras

The standard algebra of operators discussed in the previous chapter is not really representative, in its global behaviour, of the algebra of pseudodifferential operators on a compact manifold. Of course this can be attributed to the non-compactness of $\mathbb{R}^n$. However, as we shall see below in the discussion of the isotropic algebra, and then again in the later discussion of the scattering algebra, there are closely related global algebras of pseudodifferential operators on $\mathbb{R}^n$ which behave much more as in the compact case.

The 'non-compactness' of the algebra $\Psi^{\infty}_{\infty}(\mathbb{R}^n)$ is evidenced by the fact the the elements of the 'residual' algebra $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$ are not all compact as operators on $L^2(\mathbb{R}^n)$, or any other interesting space on which they act. In this chapter we consider a smaller algebra of operators in place of $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$. Namely

$$(3.1)\quad A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),$$
$$A\phi(x) = \int_{\mathbb{R}^n} A(x,y)\phi(y)dy,\ A \in \mathcal{S}(\mathbb{R}^{2n}).$$

The notation here, as the residual part of the isotropic algebra – which has not yet been defined – is rather arbitrary but it seems better than introducing a notation which will be retired later; it might be better to think of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ as the 'Schwartz algebra.'

After discussing this 'Schwartz algebra' at some length we will turn to the corresponding algebra of smoothing operators on a compact manifold (even with corners). This requires a brief introduction to manifolds, with which however I will assume some familiarity, including integration of densities. Then essentially all the results discussed here for operators on $\mathbb{R}^n$ are extended to the more general case, and indeed the Schwartz algebra itself is realized as one version of this generalization.

By definition then, $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is the algebra which corresponds to the non-commutative product on $\mathcal{S}(\mathbb{R}^{2n})$ given by

$$(3.2)\qquad A \circ B(x,y) = \int_{\mathbb{R}^n} A(x,z)B(z,y)dz.$$

The properties we discuss here have little direct relation to the 'microlocal' concepts which are discussed in the preceeding chapter. Rather they are more elementary, or at least familiar, results which are needed (and in particular are generalized) later in the discussion of global properties. This formula, (3.2) extends to smoothing operators on manifolds and gives $\mathcal{C}^\infty(M^2)$, where M is a compact manifold, the structure of a non-commutative algebra.

In the discussion of the semiclassical limit of smoothing operators at the end of this chapter the relationship between this non-commutative product and the commutative product on T^*M is discussed. This is used extensively later.

3.1. The residual algebra

The residual algebra in both the isotropic and scattering calculi, discussed below, has two important properties not shared by the residual algebra $\Psi_\infty^{-\infty}(\mathbb{R}^n)$, of which it is a subalgebra (and in fact in which it is an ideal). The first is that as operators on $L^2(\mathbb{R}^n)$ the residual isotropic operators are compact.

PROPOSITION 3.1. *Elements of $\Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ are characterized amongst continuous operators on $\mathcal{S}(\mathbb{R}^n)$ by the fact that they extend by continuity to define continuous linear maps*

$$A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{3.3}$$

In particular the image of a bounded subset of $L^2(\mathbb{R}^n)$ under an element of $\Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ is contained in a compact subset.

PROOF. The kernels of elements of $\Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ are in $\mathcal{S}(\mathbb{R}^{2n})$ so the mapping property (3.3) follows.

The norm $\sup_{|\alpha|\le 1} |\langle x\rangle^{n+1} D^\alpha u(x)|$ is continuous on $\mathcal{S}(\mathbb{R}^n)$. Thus if $S \subset L^2(\mathbb{R}^n)$ is bounded and $A \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ the continuity of $A : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ implies that $A(S)$ is bounded with respect to this norm. The theorem of Arzela-Ascoli shows that any sequence in $A(S)$ has a strongly convergent subsequence in $\langle x\rangle^n \mathcal{C}_\infty^0(\mathbb{R}^n)$ and such a sequence converges in $L^2(\mathbb{R}^n)$. Thus $A(S)$ has compact closure in $L^2(\mathbb{R}^n)$ which means that A is compact. □

The second important property of the residual algebra is that it is 'bi-ideal' or a 'corner' in the bounded operators on $L^2(\mathbb{R}^n)$. Note that it is not an ideal.

LEMMA 3.1. *If $A_1,\ A_2 \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ and B is a bounded operator on $L^2(\mathbb{R}^n)$ then $A_1BA_2 \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$.*

PROOF. The kernel of the composite $C = A_1BA_2$ can be written as a distributional pairing

$$C(x,y) = \int_{\mathbb{R}^{2n}} B(x',y')A_1(x,x')A_2(y',y)dx'dy' = (B, A_1(x,\cdot)A_2(\cdot,y)) \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.4}$$

Thus the result follows from the continuity of the exterior product, $\mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{4n})$. □

In fact the same conclusion, with essentially the same proof, holds for any continuous linear operator B from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

3.2. The augmented residual algebra

Recall that a bounded operator is said to have finite rank if its range is finite dimensional. If we consider a bounded operator B on $L^2(\mathbb{R}^n)$ which is of finite rank then we may choose an orthonormal basis f_j, $j = 1, \dots, N$ of the range $BL^2(\mathbb{R}^n)$. The functionals $u \longmapsto \langle Bu, f_j\rangle$ are continuous and so define non-vanishing elements $g_j \in L^2(\mathbb{R}^n)$. It follows that the Schwartz kernel of B is

$$B = \sum_{j=1}^N f_j(x)\overline{g_j(y)}. \tag{3.5}$$

If $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then the range must lie in $\mathcal{S}(\mathbb{R}^n)$ and similarly for the range of the adjoint, so the functions f_j are linearly dependent on some finite collection of functions $f'_j \in \mathcal{S}(\mathbb{R}^n)$ and similarly for the g_j. Thus it can be arranged that the f_j and g_j are in $\mathcal{S}(\mathbb{R}^n)$.

Proposition 3.2. *If $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then $\mathrm{Id}+A$ has, as an operator on $L^2(\mathbb{R}^n)$, finite dimensional null space and closed range which is the orthocomplement of the null space of $\mathrm{Id}+A^*$. There is an element $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that*

$$(\mathrm{Id}+A)(\mathrm{Id}+B) = \mathrm{Id}-\Pi_1,\ \ (\mathrm{Id}+B)(\mathrm{Id}+A) = \mathrm{Id}-\Pi_0 \tag{3.6}$$

where Π_0, $\Pi_1 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ are the orthogonal projections onto the null spaces of $\mathrm{Id}+A$ and $\mathrm{Id}+A^*$ and furthermore, there is an element $A' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ of rank equal to the dimension of the null space such that $\mathrm{Id}+A+sA'$ is an invertible operator on $L^2(\mathbb{R}^n)$ for all $s \neq 0$.

Proof. Most of these properties are a direct consequence of the fact that A is compact as an operator on $L^2(\mathbb{R}^n)$.

We have shown, in Proposition 3.1 that each $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is compact. It follows that

$$N_0 = \mathrm{Nul}(\mathrm{Id}+A) \subset L^2(\mathbb{R}^n) \tag{3.7}$$

has compact unit ball. Indeed the unit ball, $B = \{u \in \mathrm{Nul}(\mathrm{Id}+A)\}$ satisfies $B = A(B)$, since $u = -Au$ on B. Thus B is closed (as the null space of a continuous operator) and precompact, hence compact. Any Hilbert space with a compact unit ball is finite dimensional, so $\mathrm{Nul}(\mathrm{Id}+A)$ is finite dimensional.

Now, let $R_1 = \mathrm{Ran}(\mathrm{Id}+A)$ be the range of $\mathrm{Id}+A$; we wish to show that this is a closed subspace of $L^2(\mathbb{R}^n)$. Let $f_k \to f$ be a sequence in R_1, converging in $L^2(\mathbb{R}^n)$. For each k there exists a unique $u_k \in L^2(\mathbb{R}^n)$ with $u_k \perp N_0$ and $(\mathrm{Id}+A)u_k = f_k$. We wish to show that $u_k \to u$. First we show that $\|u_k\|$ is bounded. If not, then along a subsequent $v_j = u_{k(j)}$, $\|v_j\| \to \infty$. Set $w_j = v_j/\|v_j\|$. Using the compactness of A, $w_j = -Aw_j + f_{k(j)}/\|v_j\|$ must have a convergent subsequence, $w_j \to w$. Then $(\mathrm{Id}+A)w = 0$ but $w \perp N_0$ and $\|w\| = 1$ which are contradictory. Thus the sequence u_k is bounded in $L^2(\mathbb{R}^n)$. Then again $u_k = -Au_k + f_k$ has a convergent subsequence with limit u which is a solution of $(\mathrm{Id}+A)u = f$; hence R_1 is closed. The orthocomplement of the range of a bounded operator is always the null space of its adjoint, so R_1 has a finite-dimensional complement $N_1 = \mathrm{Nul}(\mathrm{Id}+A^*)$. The same argument applies to $\mathrm{Id}+A^*$ so gives the orthogonal decompositions

$$\begin{aligned} &L^2(\mathbb{R}^n) = N_0 \oplus R_0,\ \ N_0 = \mathrm{Nul}(\mathrm{Id}+A),\ \ R_0 = \mathrm{Ran}(\mathrm{Id}+A^*)\\ &L^2(\mathbb{R}^n) = N_1 \oplus R_1,\ \ N_1 = \mathrm{Nul}(\mathrm{Id}+A^*),\ \ R_1 = \mathrm{Ran}(\mathrm{Id}+A). \end{aligned} \tag{3.8}$$

Thus we have shown that $\mathrm{Id}+A$ induces a continuous bijection $\tilde{A} : R_0 \longrightarrow R_1$. From the closed graph theorem the inverse is a bounded operator $\tilde{B} : R_1 \longrightarrow R_0$. In this case continuity also follows from the argument above.[1] Thus $\tilde{B}$ is the generalized inverse of $\mathrm{Id}+A$ in the sense that $B = \tilde{B} - \mathrm{Id}$ satisfies (3.6). It only remains to show that $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. This follows from (3.6), the identities in which

[1] We need to show that $\|\tilde{B}f\|$ is bounded when $f \in R_1$ and $\|f\| = 1$. This is just the boundedness of $u \in R_0$ when $f = (\mathrm{Id}+A)u$ is bounded in R_1.

show that

$$B = -A - AB - \Pi_1,\ -B = A + BA + \Pi_0 \\ \Longrightarrow B = -A + A^2 + ABA - \Pi_1 + A\Pi_0. \tag{3.9}$$

All terms here are in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$; for ABA this follows from Proposition 3.1.

It remains to show the existence of the finite rank perturbation A'. This is equivalent to the vanishing of the index, that is

$$\operatorname{Ind}(\operatorname{Id} + A) = \dim \operatorname{Nul}(\operatorname{Id} + A) - \dim \operatorname{Nul}(\operatorname{Id} + A^*) = 0. \tag{3.10}$$

Indeed, let f_j and g_j, $j = 1, \dots, N$, be respective bases of the two finite dimensional spaces $\operatorname{Nul}(\operatorname{Id} + A)$ and $\operatorname{Nul}(\operatorname{Id} + A^*)$. Then

$$A' = \sum_{j=1}^{N} g_j(x)\overline{f_j(y)} \tag{3.11}$$

is an isomorphism of N_0 onto N_1 which vanishes on R_0. Thus $\operatorname{Id} + A + sA'$ is the direct sum of $\operatorname{Id} + A$ as an operator from R_0 to R_1 and sA' as an operator from N_0 to N_1, invertible when $s \neq 0$.

There is a very simple proof[2] of the equality (3.10) if we use the trace functional discussed in Section 3.5 below; this however is logically suspect as we use (although not crucially) approximation by finite rank operators in the discussion of the trace and this in turn might appear to use the present result via the discussion of ellipticity and the harmonic oscillator. Even though this is not really the case we give a clearly independent, but less elegant proof.

Consider the one-parameter family of operators $\operatorname{Id} + tA$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. We shall see that the index, the difference in dimension between $\operatorname{Nul}(\operatorname{Id} + tA)$ and $\operatorname{Nul}(\operatorname{Id} + tA^*)$ is locally constant. To see this it is enough to consider a general A near the point $t = 1$. Consider the pieces of A with respect to the decompositions $L^2(\mathbb{R}^n) = N_i \oplus R_i$, $i = 0, 1$, of domain and range. Thus A is the sum of four terms which we write as a 2×2 matrix

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}.$$

Since $\operatorname{Id} + A$ has only one term in such a decomposition, $\tilde{A}$ in the lower right, the solution of the equation $(\operatorname{Id} + tA)u = f$ can be written

$$(t-1)A_{00}u_0 + (t-1)A_{01}u_\perp = f_1,\ (t-1)A_{10}u_0 + (A' + (t-1)A_{11})u_\perp = f_\perp \tag{3.12}$$

Since $\tilde{A}$ is invertible, for $t - 1$ small enough the second equation can be solved uniquely for $u_\perp$. Inserted into the first equation this gives

$$G(t)u_0 = f_1 + H(t)f_\perp, \\ G(t) = (t-1)A_{00} - (t-1)^2 A_{01}(A' + (t-1)A_{11})^{-1}A_{10}, \\ H(t) = -(t-1)A_{01}(A' + (t-1)A_{11})^{-1}. \tag{3.13}$$

[2]Namely the trace of a finite rank projection, such as either Π_0 or Π_1, is its rank, hence the dimension of the space onto which it projects. From the identity satisfied by the generalized inverse we see that

$$\operatorname{Ind}(\operatorname{Id} + A) = \operatorname{Tr}(\Pi_0) - \operatorname{Tr}(\Pi_1) = \operatorname{Tr}\left((\operatorname{Id} + B)(\operatorname{Id} + A) - (\operatorname{Id} + A)(\operatorname{Id} + B)\right) = \operatorname{Tr}([B, A]) = 0$$

from the basic property of the trace.

The null space is therefore isomorphic to the null space of $G(t)$ and a complement to the range is isomorphic to a complement to the range of $G(t)$. Since $G(t)$ is a finite rank operator acting from N_0 to N_1 the difference of these dimensions is constant in t, namely equal to $\dim N_0 - \dim N_1$, near $t = 1$ where it is defined.

This argument can be applied to tA so the index is actually constant in $t \in [0, 1]$ and since it certainly vanishes at $t = 0$ it vanishes for all t. In fact, as we shall note below, $\mathrm{Id} + tA$ is invertible outside a discrete set of $t \in \mathbb{C}$. □

COROLLARY 3.1. *If* $\mathrm{Id} + A$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *is injective or surjective on* $L^2(\mathbb{R}^n)$, *in particular if it is invertible as a bounded operator, then it has an inverse of the form* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

COROLLARY 3.2. *If* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *then as an operator on* $\mathcal{S}(\mathbb{R}^n)$ *or* $\mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Id} + A$ *is Fredholm in the sense that its null space is finite dimensional and its range is closed with a finite dimensional complement.*

PROOF. This follows from the existence of the generalized inverse of the form $\mathrm{Id} + B$, $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. □

3.3. Exponential and logarithm

PROPOSITION 3.3. *The exponential*

$$\exp(A) = \sum_j \frac{1}{j!} A^j : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.14}$$

is a globally defined, entire, function with range containing a neighbourhood of the identity and with inverse on such a neighbourhood given by the analytic function

$$\log(\mathrm{Id} + A) = \sum_j \frac{(-1)^j}{j} A^j,\ A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \|A\|_{L^2} < 1 \tag{3.15}$$

3.4. The residual group

By definition, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the set (if you want to be concrete you can think of them as operators on $L^2(\mathbb{R}^n)$) of invertible operators in $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. If we identify this topologically with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then, as follows from Corollary 3.1, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is open. We will think of it as an infinite-dimensional manifold modeled, of course, on the linear space $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n})$. Since I have no desire to get too deeply into the general theory of such Fréchet manifolds I will keep the discussion as elementary as possible.

The dual space of $\mathcal{S}(\mathbb{R}^p)$ is $\mathcal{S}'(\mathbb{R}^p)$. If we want to think of $\mathcal{S}(\mathbb{R}^p)$ as a manifold we need to consider smooth functions and forms on it. In the finite-dimensional case, the exterior bundles are the antisymmetric parts of the tensor powers of the dual. Since we are in infinite dimensions the tensor power needs to be completed and the usual choice is the 'projective' tensor product. In our case this is something quite simple, namely the k-fold completed tensor power of $\mathcal{S}'(\mathbb{R}^p)$ is just $\mathcal{S}'(\mathbb{R}^{kp})$. Thus we set

$$\begin{aligned} \Lambda^k \mathcal{S}(\mathbb{R}^p) = \{u \in \mathcal{S}'(\mathbb{R}^{kp});\ &\text{for any permutation} \\ &e,\ u(x_{e(1)}, \dots x_{e(h)}) = \operatorname{sgn}(e) u(x_1, \dots x_k)\}. \end{aligned} \tag{3.16}$$

In view of this it is enough for us to consider smooth functions on open sets $F \subset \mathcal{S}(\mathbb{R}^p)$ with values in $\mathcal{S}'(\mathbb{R}^p)$ for general p. Thus

$$v : F \longrightarrow \mathcal{S}'(\mathbb{R}^p),\ F \subset \mathcal{S}(\mathbb{R}^n) \text{ open} \tag{3.17}$$

is continuously differentiable on F if there exists a continuous map

$$v' : F \longrightarrow \mathcal{S}'(\mathbb{R}^{n+p}) \text{ and each } u \in F \text{ has a neighbourhood } U \\ \text{such that for each } N \ \exists\ M \quad \text{with} \\ \|v(u+u') - v(u) - v'(u;u')\|_N \le C\|u'\|_M^2,\ \forall\ u, u+u' \in U.$$

Then, as usual we define smoothness as infinite differentiability by iterating this definition. The smoothness of v in this sense certainly implies that if $f : X \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a smooth from a finite dimensional manifold then $v \circ F$ is smooth.

Thus we define the notion of a smooth *form* on $F \subset \mathcal{S}(\mathbb{R}^n)$, an open set, as a smooth map

$$\alpha : F \to \Lambda^k \mathcal{S}(\mathbb{R}^p) \subset \mathcal{S}'(\mathbb{R}^{kp}). \tag{3.18}$$

In particular we know what smooth forms are on $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

The de Rham differential acts on forms as usual. If $v : F \to \mathbb{C}$ is a function then its differential at $f \in F$ is $dv : F \longrightarrow \mathcal{S}'(\mathbb{R}^n) = \Lambda^1\mathcal{S}(\mathbb{R}^n)$, just the derivative. As in the finite-dimensional case d extends to forms by enforcing the condition that $dv = 0$ for constant forms and the distribution identity over exterior products

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta. \tag{3.19}$$

3.5. Traces on the residual algebra

The algebras we are studying are topological algebras, so it makes sense to consider continuous linear functionals on them. The most important of these is the *trace*. To remind you what it is we consider first its properties for matrix algebras.

Let $M(N;\mathbb{C})$ denote the algebra of $N \times N$ complex matrices. We can simply define

$$\mathrm{Tr} : M(N;\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} \tag{3.20}$$

as the sum of the diagonal entries. The fundamental property of this functional is that

$$\mathrm{Tr}([A,B]) = 0\ \forall\ A, B \in M(N;\mathbb{C}). \tag{3.21}$$

To check this it is only necessary to write down the definition of the composition in the algebra. Thus

$$(AB)_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}.$$

It follows that

$$\begin{aligned} \mathrm{Tr}(AB) &= \sum_{i=1}^{N} (AB)_{ii} = \sum_{i,k=1}^{N} A_{ik}B_{ki} \\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} B_{ki}A_{ik} = \sum_{k=1}^{N} (BA)_{kk} = \mathrm{Tr}(BA) \end{aligned}$$

which is just (3.21).

Of course any multiple of Tr has the same property (3.21) but the normalization condition

$$\operatorname{Tr}(\mathrm{Id}) = N \tag{3.22}$$

distinguishes it from its multiples. In fact (3.21) and (3.22) together distinguish $\operatorname{Tr} \in M(N;\mathbb{C})'$ as a point in the N^2 dimensional linear space which is the dual of $M(N;\mathbb{C})$.

LEMMA 3.2. *If $F : M(N;\mathbb{C}) \to \mathbb{C}$ is a linear functional satisfying* (3.21) *and $B \in M(N;\mathbb{C})$ is any matrix such that $F(B) \neq 0$ then $F(A) = \frac{F(B)}{\operatorname{Tr}(B)} \operatorname{Tr}(A)$.*

PROOF. Consider the basis of $M(N;\mathbb{C})$ given by the elementary matrices E_{jk}, where E_{jk} has jk-th entry 1 and all others zero. Thus

$$E_{jk}E_{pq} = \delta_{kp}E_{jq}.$$

If $j \neq k$ it follows that

$$E_{jj}E_{jk} = E_{jk},\ E_{jk}E_{jj} = 0.$$

Thus

$$F([E_{jj}, E_{jk}]) = F(E_{jk}) = 0 \text{ if } j \neq k.$$

On the other hand, for any i and j

$$E_{ji}E_{ij} = E_{jj},\ E_{ij}E_{ji} = E_{ii}$$

so

$$F(E_{jj}) = F(E_{11})\ \forall\ j.$$

Since the E_{jk} are a basis,

$$\begin{aligned} F(A) &= F(\sum_{j,k=1}^{N} A_{ij}E_{ij}) \\ &= \sum_{j,l=1}^{N} A_{jj}F(E_{ij}) \\ &= F(E_{11})\sum_{j=1}^{N} A_{jj} = F(E_{11}) \operatorname{Tr}(A). \end{aligned}$$

This proves the lemma. □

For the isotropic smoothing algebra we have a similar result.

PROPOSITION 3.4. *If $F : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathbb{C}$ is a continuous linear functional satisfying*

$$F([A, B]) = 0\ \forall\ A, B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.23}$$

then F is a constant multiple of the functional

$$\operatorname{Tr}(A) = \int_{\mathbb{R}^n} A(x,x)dx. \tag{3.24}$$

PROOF. Recall that $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \subset \Psi^{\infty}_{\text{iso}}(\mathbb{R}^n)$ is an ideal so $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{\infty}_{\text{iso}}(\mathbb{R}^n)$ implies that AB, $BA \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and it follows that the equality $F(AB) = F(BA)$, or $F([A, B]) = 0$, is meaningful. To see that it holds we just use the continuity of F. We know that if $B \in \Psi^{\infty}_{\text{iso}}(\mathbb{R}^n)$ then there is a sequence $B_n \to B$ in the topology of $\Psi^{m}_{\text{iso}}(\mathbb{R}^n)$ for some m. Since this implies $AB_n \to AB$, $B_nA \to BA$ in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ we see that

$$F([A, B]) = \lim_{n\to\infty} F([A, B_n]) = 0.$$

We use this identity to prove (3.24). Take $B = x_j$ or D_j, $j = 1, \dots, n$. Thus for any $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$

$$F([A, x_j]) = F([A, D_j]) = 0.$$

Now consider F as a distribution acting on the kernel $A \in \mathcal{S}(\mathbb{R}^{2n})$. Since the kernel of $[A, x_j]$ is $A(x, y)(y_j - x_j)$ and the kernel of (A, D_j) is $-(D_{y_j} + D_{x_j})A(x, y)$ we conclude that, as an element of $\mathcal{S}'(\mathbb{R}^{2n})$, F satisfies

$$(x_j - y_j)F(x, y) = 0,\ (D_{x_j} + D_{y_j})F(x, y) = 0.$$

If we make the linear change of variables to $p_i = \frac{x_i+y_i}{2}$, $q_i = x_i - y_i$ and set $\tilde{F}(p, q) = F(x, y)$ these conditions become

$$D_{q_i}\tilde{F} = 0,\ p_i\tilde{F} = 0,\ i = 1, \dots, N.$$

As we know from Lemmas 1.2 and 1.3, this implies that $\tilde{F} = c\delta(p)$ so

$$F(x, y) = c\delta(x - y)$$

as a distribution. Clearly $\delta(x - y)$ gives the functional Tr defined by (3.24), so the proposition is proved. $\square$

We still need to justify the use of the same notation, Tr, for these two functionals. However, if $L \subset \mathcal{S}(\mathbb{R}^n)$ is any finite dimensional subspace we may choose an orthonal basis $\varphi_i \in L$, $i = 1, \dots, l$,

$$\int_{\mathbb{R}^n} |\varphi_i(x)|^2 dx = 0,\ \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)dx = 0,\ i \neq j.$$

Then if a_{ij} is an $l \times l$ matrix,

$$A = \sum_{i,j=1}^{\ell} a_{ij}\varphi_i(x)\overline{\varphi_j(y)} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n).$$

From (3.24) we see that

$$\begin{aligned}
\operatorname{Tr}(A) &= \sum_{ij} a_{ij} \operatorname{Tr}(\varphi_i\bar{\varphi}_j) \\
&= \sum_{ij} a_{ij} \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)dx \\
&= \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(a).
\end{aligned}$$

Thus the two notions of trace coincide. In any case this already follows, up to a constant, from the uniqueness in Lemma 3.2.

3.6. Fredholm determinant

For $N \times N$ matrices, the determinant is a multiplicative polynomial map

$$\det : M(N;\mathbb{C}) \longrightarrow \mathbb{C},\ \det(AB) = \det(A)\det(B),\ \det(\mathrm{Id}) = 1. \tag{3.25}$$

It is not quite determined by these conditions, since $\det(A)^k$ also satisfies then. The fundamental property of the determinant is that it defines the group of invertible elements

$$\mathrm{GL}(N,\mathbb{C}) = \{A \in M(N;\mathbb{C}); \det(A) \neq 0\}. \tag{3.26}$$

A reminder of a direct definition is given in Problem 4.7.

The Fredholm determinant is an extension of this definition to a function on the ring $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. This can be done in several ways using the density of finite rank operators, as shown in Corollary 4.2. We proceed by generalizing the formula relating the determinant to the trace. Thus, for any smooth curve with values in $\mathrm{GL}(N;\mathbb{C})$ for any N,

$$\frac{d}{ds}\det(A_s) = \det(A_s)\operatorname{tr}(A_s^{-1}\frac{A_s}{ds}). \tag{3.27}$$

In particular if (3.25) is augmented by the normalization condition

$$\frac{d}{ds}\det(\mathrm{Id} + sA)\big|_{s=0} = \operatorname{tr}(A)\ \forall\ A \in M(N;\mathbb{C}) \tag{3.28}$$

then it is determined.

A branch of the logarithm can be introduced along any curve, smoothly in the parameter, and then (3.27) can be rewritten

$$d\log\det(A) = \operatorname{tr}(A^{-1}dA). \tag{3.29}$$

Here $\mathrm{GL}(N;\mathbb{C})$ is regarded as a subset of the linear space $M(N;\mathbb{C})$ and dA is the canonical identification, at the point A, of the tangent space to $M(N,\mathbb{C})$ with $M(N,\mathbb{C})$ itself. This just arises from the fact that $M(N,\mathbb{C})$ is a linear space. Thus $dA(\frac{d}{ds}(A + sB)\big|_{s=0} = B$. This allows the expression on the right in (3.29) to be interpreted as a smooth 1-form on the manifold $\mathrm{GL}(N;\mathbb{C})$. Note that it is independent of the local choice of logarithm.

To define the Fredholm determinant we shall extend the 1-form

$$\alpha = \operatorname{Tr}(A^{-1}dA) \tag{3.30}$$

to the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Here dA has essentially the same meaning as before, given that Id is fixed. Thus at any point $A = \mathrm{Id} + B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ it is the identification of the tangent space with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ using the linear structure:

$$dA(\frac{d}{ds}(\mathrm{Id} + B + sE)\big|_{s=0}) = E,\ E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Since dA takes values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the trace functional in (3.30) is well defined.

The 1-form α is closed. In the finite-dimensional case this follows from (3.29). For (3.30) we can compute directly. Since $d(dA) = 0$, essentially by definition, and

$$dA^{-1} = -A^{-1}dAA^{-1} \tag{3.31}$$

we see that

$$d\alpha = -\operatorname{Tr}(A^{-1}(dA)A^{-1}(dA)) = 0. \tag{3.32}$$

Here we have used the trace identity, and the antisymmetry of the implicit wedge product in (3.32), to conlcude that $d\alpha = 0$. For a more detailed discussion of this point see Problem 4.8.

From the fact that $d\alpha = 0$ we can be confident that there is, locally near any point of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, a function f such that $df = \alpha$; then we will define the Fredholm determinant by $\det_{\mathrm{Fr}}(A) = \exp(f)$. To define $\det_{\mathrm{Fr}}$ globally we need to see that this is well defined.

Lemma 3.3. *For any smooth closed curve $\gamma : \mathbb{S}^1 \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the integral*

$$\int_\gamma \alpha = \int_{\mathbb{S}^1} \gamma^* \alpha \in 2\pi i \mathbb{Z}. \tag{3.33}$$

That is, α defines an integral cohomology class, $[\frac{\alpha}{2\pi i}] \in H^1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n); \mathbb{Z})$.

Proof. This is where we use the approximability by finite rank operators. If π_N is the orthogonal projection onto the span of the eigenspaces of the smallest N eigenvalues of the harmonic oscillator then we know from Section 4.3 that $\pi_N E \pi_N \to E$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for any element. In fact it follows that for the smooth curve that $\gamma(s) = \mathrm{Id} + E(s)$ and $E_N(s) = \pi_N E(s) \pi_N$ converges uniformly with all s derivatives. Thus, for some N_0 and all $N > N_0$, $\mathrm{Id} + E_N(s)$ is a smooth curve in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and hence $\gamma_N(s) = \mathrm{Id}_N + E_N(s)$ is a smooth curve in $\mathrm{GL}(N; \mathbb{C})$. Clearly

$$\int_{\gamma_N} \alpha \longrightarrow \int_\gamma \alpha \text{ as } N \to \infty, \tag{3.34}$$

and for finite N it follows from the identity of the trace with the matrix trace (see Section 3.5) that $\int_N \gamma_N^* \alpha$ is the variation of $\arg \log \det(\gamma_N)$ around the curve. This gives (3.33). $\square$

Now, once we have (3.33) and the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we may define

$$\det_{\mathrm{Fr}}(A) = \exp(\int_\gamma \alpha),\ \gamma : [0,1] \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \gamma(0) = \mathrm{Id},\ \gamma(1) = A. \tag{3.35}$$

Indeed, Lemma 3.3 shows that this is independent of the path chosen from the identity to A. Notice that the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ follows from the connectedness of the $\mathrm{GL}(N, \mathbb{C})$ and the density argument above.

The same arguments and results apply to $G^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ using the fact that the trace functional extends continuously to $\Psi^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for any $\epsilon > 0$.

Proposition 3.5. *The Fredholm determinant, defined by* (3.35) *on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ (or $G^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$ for $\epsilon > 0$) and to be zero on the complement in $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ (or $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$) is an entire function satisfying*

$$\begin{aligned}\det_{\mathrm{Fr}}(AB) = \det_{\mathrm{Fr}}(A) \det_{\mathrm{Fr}}(B),\ A, B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \\ (\mathit{or}\ \mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)),\ \det_{\mathrm{Fr}}(\mathrm{Id}) = 1.\end{aligned} \tag{3.36}$$

Proof. We start with the multiplicative property of $\det_{\mathrm{Fr}}$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus is $\gamma_1(s)$ is a smooth curve from Id to A_1 and $\gamma_2(s)$ is a smooth curve from Id to A_2 then $\gamma(s) = \gamma_1(s)\gamma_2(s)$ is a smooth curve from Id to $A_1 A_2$. Consider the differential on this curve. Since

$$\frac{d(A_1(s)A_2(s))}{ds} = \frac{dA_1(s)}{ds} A_2(s) + A_1(s) \frac{dA_2(s)}{ds}$$

the 1-form becomes

$$\gamma^*(s)\alpha(s) = \operatorname{Tr}(A_2(s)^{-1}\frac{dA_2(s)}{ds}) + \operatorname{Tr}(A_2(s)^{-1}A_1(s)^{-1}\frac{dA_2(s)}{ds}A_2(s)). \tag{3.37}$$

In the second term on the right we can use the trace identity, since $\operatorname{Tr}(GA) = \operatorname{Tr}(AG)$ if $G \in \Psi^{\mathbb{Z}}_{\text{iso}}(\mathbb{R}^n)$ and $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus (3.37) becomes

$$\gamma^*(s)\alpha(s) = \gamma_1^*\alpha + \gamma_2^*\alpha.$$

Inserting this into the definition of $\det_{\text{Fr}}$ gives (3.36) when both factors are in $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Of course if either factor is not invertible, then so is the product and hence both $\det_{\text{Fr}}(AB)$ and at least one of $\det_{\text{Fr}}(A)$ and $\det_{\text{Fr}}(B)$ vanishes. Thus (3.36) holds in general when $\det_{\text{Fr}}$ is extended to be zero on the non-invertible elements.

Thus it remains to establish the smoothness. That $\det_{\text{Fr}}(A)$ is smooth in any real parameters in which $A \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ depends, or indeed is holomorphic in holomorphic parameters, follows from the definition since α clearly depends smoothly, or holomorphically, on parameters. In fact the same follows if holomorphy is examined as a function of E, $A = \operatorname{Id} + E$, for $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus it is only smoothness across the non-invertibles that is at issue. To prove this we use the multiplicativity just established.

If $A = \operatorname{Id} + E$ is not invertible, $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then it has a generalized inverse $\operatorname{Id} + E'$ as in Proposition 4.3. Since A has index zero, we may actually replace E' by $E' + E''$, where E'' is an invertible linear map from the orthocomplement of the range of A to its null space. Then $\operatorname{Id} + E' + E'' \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and $(\operatorname{Id} + E' + E'')A = \operatorname{Id} - \Pi_0$. To prove the smoothness of $\det_{\text{Fr}}$ on a neighbourhood of A it is enough to prove the smoothness on a neighbourhood of $\operatorname{Id} - \Pi_0$ since $\operatorname{Id} + E' + E''$ maps a neighbourhood of the first to a neighbourhood of the second and $\det_{\text{Fr}}$ is multiplicative. Thus consider $\det_{\text{Fr}}$ on a set $\operatorname{Id} - \Pi_0 + E$ where E is near 0 in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$, in particular we may assume that $\operatorname{Id} + E \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus

$$\det_{\text{Fr}}(\operatorname{Id} + E - \Pi_0) = \det(\operatorname{Id} + E)\det(\operatorname{Id} - \Pi_0 + (G_E - \operatorname{Id})\Pi_0)$$

were $G_E = (\operatorname{Id} + E)^{-1}$ depends holomorphically on E. Thus it suffices to prove the smoothness of $\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H\Pi_0)$ where $H \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$

Consider the deformation $H_s = \Pi_0 H \Pi_0 + s(\operatorname{Id} - \Pi_0)H\Pi_0$, $s \in [0, 1]$. If $\operatorname{Id} - \Pi_0 + H_s$ is invertible for one value of s it is invertible for all, since its range is always the range of $\operatorname{Id} - \Pi_0$ plus the range of $\Pi_0 H \Pi_0$. It follows that $\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H_s)$ is smooth in s; in fact it is constant. If the family is not invertible this follows immediately and if it is invertible then

$$\begin{aligned}&\frac{d\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H_s)}{ds}\\ &\qquad = \det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H_s)\operatorname{Tr}\left((\operatorname{Id} - \Pi_0 + H_s)^{-1}(\operatorname{Id} - Pi_0)H\Pi_0)\right) = 0\end{aligned}$$

since the argument of the trace is finite rank and off-diagonal with respect to the decomposition by Π_0.

Thus finally it is enough to consider the smoothness of $\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + \Pi_0 H \Pi_0)$ as a function of $H \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Since this is just $\det(\Pi_0 H \Pi_0)$, interpreted as a finite rank map on the range of Π_0 the result follows from the finite dimensional case. $\square$

3.7. Fredholm alternative

Since we have shown that $\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is an entire function, we see that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the complement of a (singular) holomorphic hypersurface, namely the surface $\{\mathrm{Id} + E; \det_{\mathrm{Fr}}(\mathrm{Id} + E) = 0\}$. This has the following consequence, which is sometimes call the 'Fredholm alternative' and also part of 'analytic Fredholm theory'.

Lemma 3.4. *If $\Omega \subset \mathbb{C}$ is an open, connected set and $A : \Omega \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic function then either $\mathrm{Id} + A(z)$ is invertible on all but a discrete subset of Ω and $(\mathrm{Id} + A(z))$ is meromorphic on Ω with all residues of finite rank, or else it is invertible at no point of Ω.*

Proof. Of course the point here is that $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z))$ is a holomorphic function on Ω. Thus, either $\det_{\mathrm{Fr}}(A(z)) = 0$ is a discrete set, $D \subset \Omega$ or else $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z)) \equiv 0$ on Ω; this uses the connectedness of Ω. Since this corresponds exactly to the invertibility of $\mathrm{Id} + A(z)$ the main part of the lemma is proved. It remains only to show that, in the former case, $(\mathrm{Id} + A(z))^{-1}$ is meromorphic. Thus consider a point $p \in D$. Thus the claim is that near p

$$(\mathrm{Id} + A(z))^{-1} = \mathrm{Id} + E(z) + \sum_{j=1}^{N} z^{-j} E_j,\ E_j \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ of finite rank} \tag{3.38}$$

and where $E(z)$ is locally holomorphic with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

If N is sufficiently large and Π_N is the projection onto the first N eigenspaces of the harmonic oscillator then $B(z) = \mathrm{Id} + E(z) - \Pi_N E(z) \Pi_N$ is invertible near p with the inverse being of the form $\mathrm{Id} + F(z)$ with $F(z)$ locally holomorphic. Now

$$\begin{aligned}(\mathrm{Id} + F(z))(\mathrm{Id} + E(z)) &= \mathrm{Id} + (\mathrm{Id} + F(z))\Pi_N E(z) \Pi_N \\ &= (\mathrm{Id} - \Pi_N) + \Pi_N M(z) \Pi_N + (\mathrm{Id} - \Pi_N) M'(z) \Pi_N.\end{aligned}$$

It follows that this is invertible if and only if $M(z)$ is invertible as a matrix on the range of Π_N. Since it must be invertible near, but not at, p, its inverse is a meromorphic matrix $K(z)$. It follows that the inverse of the product above can be written

$$\mathrm{Id} - \Pi_N + \Pi_N K(z) \Pi_N - (\mathrm{Id} - \Pi_N) M'(z) \Pi_N K(z) \Pi_N. \tag{3.39}$$

This is meromorphic and has finite rank residues, so it follows that the same is true of $A(z)^{-1}$. □

3.8. Manifolds and functions

Here is a version of the standard definition of a manifold (with corners). First let M be a Hausdorff topological space. That is, we already have the 'topology' of open subsets of M, closed under arbitrary intersections and finite unions. We then know which real-functions on M are continuous – namely those $f : M \longrightarrow \mathbb{R}$ such that $f^{-1}(a, b) \subset M$ is open for every $a < b$. The Hausdorff condition is that these continuous functions separate points, so if $p_1 \neq p_2$ are two points in M then there is a continuous function f on M such that $f(p_1) \neq f(p_2)$. We also assume that M is second countable, that the topology has a countable basis – there is a countable collection of open subsets such that every open subset is a union of these particular open subsets.

A $\mathcal{C}^\infty$ structure on M can be taken to be a subset $\mathcal{C}^\infty(M) \subset \mathcal{C}^0(M)$ of the space of continuous functions which has the following properties. First, it is a subalgebra. Second it generates (product) coordinate systems. That is there is a countable open cover of M by subsets U_i for each of which there are n elements $f_{i,j} \in \mathcal{C}^\infty(M)$ such that $F_i = (f_{1,1}, \dots, f_{i,n})$ restricts to U_i to give a topological isomorphism

$$F_i\big|_{U_i} : U_i \longrightarrow [0,1)^k \times (-1,1)^{n-k} \subset \mathbb{R}^n \tag{3.40}$$

and such that if $g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ has support in $(-1,1)^n$ then

$$g' = \begin{cases} F_i^* g & \text{on } U_i \\ 0 & \text{on } M \setminus U_i \end{cases} \in \mathcal{C}^\infty(M), \tag{3.41}$$

and that these functions form an ideal in $\mathcal{C}^\infty(M)$. Thirdly we require that $\mathcal{C}^\infty(M)$ is maximal in the sense that if $g : M \longmapsto \mathbb{R}$ and for each i, $g\big|_{U_i} = F_i^* h_i$ for some $h_i \in \mathcal{C}^\infty((-1)^n)$ then $g \in \mathcal{C}^\infty(M)$.

In fact I would call a manifold as defined in the preceeding paragraph a t-manifold. It has various problems. One is that I have not insisted that the local dimension n is not fixed. This is not a serious problem, but it means that M may be up to even a countable union of compoents, each of which is a connected manifold, in the same sense, and hence has fixed dimension. Often this is required anyway, at at least it is how most people think – that a manifold is connected. Apart from that there are more serious problems with the boundary when k, which is the local boundary codimension, takes the value 2 or greater. This is not really imortant here but I usually insist on an additional condition, that the boundary faces be embedded. This is actually a combinatorial condition and means that each boundary hypersurface, defined as the closure of a component of the set of boundary points of 'codimension one' (meaning the union of the the inverse images of the subsets, in the coordinate patches, of $[0,1)^k \times (-1,1)^{n-k}$ where exactly one of the first k variables vanishes), is embedded. One way of thinking about this is that some neighbourhood of each point in the closure of such a boundary point meets the component of the codimension one boundary in a connected set.

A map between manifolds, $f : M \longrightarrow N$ is smooth if and only if the composite $u \circ f \in \mathcal{C}^\infty(M)$ for every $y \in \mathcal{C}^\infty(N)$. It is usual to write this as a pull-back map

$$f^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M),\ f^*u = u \circ f. \tag{3.42}$$

The discussion above is not a good way to learn about manifolds – I am assuming you will look things up somewhere if you don't know about them. The only real virtue of this definition is that it is short. [3]

3.9. Tangent and cotangent bundles

From one manifold we can make others. The most basic examples of this is the passage to a boundary face of a manifold with corners and taking products of manifolds. A more sophisticated example, blow up, is discussed briefly below and we have already described to compactification of Euclidean space to a ball. However the most frequently encountered 'derived' manifold below is the cotangent

[3] In case you, gentle reader, really want to learn the elementary theory of manifolds for yourself and are unable to pick up an appropriate book I have added (or will add) lots of 'problems' to guide, or remind, you a little.

bundle. Once again the approach I give here is not really introductory, its main virtue is brevity.

On Euclidean space of a smooth function near a point, $\bar{z}$, can always be decomposed in terms of coordinate functions

$$f(z) = f(\bar{z}) + \sum_{j=1}^{n} f_j(z)(z_j - \bar{z}_j) \tag{3.43}$$

where the coefficient functions f_j are smooth near $\bar{z}$. The f_j are not determined by this Taylor expansion but their values at $\bar{z}$, namely the derivatives of f at $\bar{z}$, are determined. We can capture these derivatives, collectively, as elements of the vector space

$$\mathcal{J}(\bar{z})/\mathcal{J}(\bar{z})^2,\ \mathcal{J}(\bar{z}) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n); f(\bar{z}) = 0\},\ \mathcal{J}(\bar{z})^2 = \{\sum_{\text{finite}} f_i g_i,\ f_i,\ g_i \in \mathcal{J}(\bar{z})\}. \tag{3.44}$$

Thus $f(z) - f(\bar{z}) \in \mathcal{J}(\bar{z})$ and $\mathcal{J}(\bar{z})/\mathcal{J}(\bar{z})^2$ is an n-dimensional vector space. In fact it is only necessary for f to be defined and smooth in some neighbourhood of $\bar{z}$ for this to be well defined since if ϕ is a cutoff, supported sufficiently close to $\bar{z}$ and equal to 1 in some neighbourhood, then the class of $f\phi - f(\bar{z})$ in $\mathcal{J}(\bar{z})/\mathcal{J}(\bar{z})^2$ is independent of the choice of ϕ. Of course this is the deRham differential. Moreover the discussion extends immediately to smooth manifold and defines

$$df(p) \in T^*_p M = \mathcal{J}(p)/\mathcal{J}(p)^2, \tag{3.45}$$

the cotangent space at each point $p \in M$. This is a vector space of dimension n which is spanned by the differentials of any coordinate system in a neighbourhood of p.

The union of the cotangent fibres has a natural structure as a manifold

$$T^*M = \bigcup_{p \in M} T^*_p M \xrightarrow{\pi} M. \tag{3.46}$$

Namely a coordinate system on an open set $U \subset M$ gives a global coordinate system on the open subset $\pi^{-1}(U)$ identifying it (by definition smoothly) with $U \times \mathbb{R}^n$.

The tangent bundle can be defined as the dual of T^*M or directly in terms of vector fields; taking the first approach

$$T_p M = \{v : T^*_p M \longrightarrow \mathbb{R},\ \text{linear}\},\ TM = \bigcup_{p \in M} T_p M \xrightarrow{\pi} M. \tag{3.47}$$

Coordinate systems on M again give coordinate systems on TM.

3.10. Integration and densities

There is no natural notion equivalent to the Lebesgue integral on a manifold, the problem being that the 'measure' part is changes by a positive smooth multiple under coordinate transformations, namely by the Jacobian determinant. It is therefore necessary either to make a choice of 'density' or else to include the density in the integrand, and integrate only densities. The latter approach is taken here and this requires the introduction of the density bundle, which is a simple example of a trivial line bundle which is not canonically trivial.

PROBLEM 3.1. Show that the smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are 'positively' homogeneous of some complex degree s, meaning the satify

$$f(rz) = r^s f(z),\ \forall\ r > 0,\ z \in \mathbb{R}^n \setminus \{0\} \tag{3.48}$$

(where r^s is the standard branch) is a trivial, but not canonically trivial, line bundle over $\mathbb{S}^{n-1}$, except in the case $s = 0$ when it *is* canonically trivial.

At each point of a manifold consider the 1-dimensional, real, vector space of totally antisymmetric absolutely homogeneous n-multilinear functions

(3.49)

$$\Omega_p M = \{\nu : T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R},\ \nu(v_{e(1)}, \dots, v_{e(n)}) = \operatorname{sgn} e \nu(v_1, \dots, v_n),\ \nu(tv_1, \dots, v_n) = |t|\nu(v_1, \dots, v_n),\ t \in$$

where $v_i \in T_pM$, $i = 1, \dots, n$ are arbitrary and e is any permutation. It is straightforward to check that this is a linear space (it seems a little strange if view of the absolute value of t in the last identity but it is true). If z_i are local coordinates in a neighbourhood of p then the differentials dz_i define a density

$$\nu(v_1, \dots, v_n) = |\det dz_i(v_j)|. \tag{3.50}$$

This is the local coordinate representative of Lebesgue measure at the point.

As for the tangent bundle above, the union of the fibres Ω_p form a manifold,

$$\Omega M = \bigcup_{p \in M} \Omega_p M \xrightarrow{\pi} M. \tag{3.51}$$

A section of ΩM, meaning a smooth map $\nu : M \longrightarrow \Omega M$ such that $\pi\nu = \mathrm{Id}_M$, is by definition a *smooth density* on M. The linear space of such sections is denoted $\mathcal{C}^\infty(M; \Omega)$ and the behaviour of integrals under coordinate transformation reduces directly to the existence of a well defined integral:

$$\int_M : \mathcal{C}^\infty(M; \Omega) \longrightarrow \mathbb{R}. \tag{3.52}$$

Checking that this is well-defined reduces to the usual change-of-variable formula fo Lebesgue (or Riemann) integral in local coordinates.

3.11. Smoothing operators

Now, we come to the point of interest in this chapter. If M is a compact manifold then the algebra of smoothing operators on M behaves in very much the same was as the Schwartz algebra on $\mathbb{R}^n$. In fact it is isomorphic to it as an algebra (if the dimension of M is positive) although there is no natural isomorphism. As we shall see later, the smoothing operators form the residual part of the pseudodifferential algebra on a manifold and are important for that reason. However they also play a crucial role in the index theorem as presented here.

By definition we can take a smoothing operator to be an integral operator with smooth kernel:-

$$A : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M),\ Au(z) = \int_M A(z, z')u(z'),\ A \in \mathcal{C}^\infty(M^2; \pi_R^*\Omega). \tag{3.53}$$

Here $\pi :_R M^2 \ni (z, z') \longmapsto z' \in M$ is the 'right' projection. Thus A, the kernel (where we use the same letter for kernel and operator because they determine each other and so to use a separate notation is rather wasteful) is just a smooth function on M^2 which 'carries along with it' a smooth denisty on the right factor of M. If one prefers to do so, one can simply choose a positive denisty $0 < \nu \in \mathcal{C}^\infty(M; \Omega)$

and then the kernel becomes $A = A'\nu(z')$ where $A' \in \mathcal{C}^\infty(M^2)$. I prefer the more invariant approach of hiding the density in the kernel.

PROPOSITION 3.6. *The smoothing operators on a compact manifold form an algebra, denoted $\Psi^{-\infty}(M)$, under operator composition.*

PROOF. Indeed if A and B are smoothing operators on M with kernels having the same names then, by Fubini's theorem,

(3.54)
$$(AB)u(x) = A(Bu)(z) = \int_M A(z,z'')(Bu)(z'') = \int_M A(z,z'')\int_M B(z'',z')u(z')Mso$$
$$(AB)(z,z') = \int_M A(z,z'')B(z'',z').$$

Thus this formula defines an associative algebra structure (because composition of operators is associative) on $\Psi^{-\infty}(M) = \mathcal{C}^\infty(M^2;\pi_R^*\Omega)$ as claimed. □

A moments thought will show that this argument, and the composition law , carry over perfectly well to any compact manifold with corners. This more general case is interesting in part because of the subalgebras (but not ideals) that then arise in $\Psi^{-\infty}(M)$.

PROPOSITION 3.7. *If M is a compact manifold with corners and $H \subset M$ is a boundary face then the subspace of $\Psi^{-\infty}(M)$ consisting of kernels which vanish to order k at $H \times M$ and $M \times H$ is a subalgebra.*

The case of $k = \infty$ and $H = \partial\mathbb{B}^n$ for a ball is of particular interest since if the ball is interpreted as the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, then

$$\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \{A \in \Psi^{-\infty}(\overline{\mathbb{R}^n}); A \equiv 0 \text{ at } (\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) \cup (\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}).\} \tag{3.55}$$

Here $\equiv$ stands for equality in Taylor series.

PROBLEM 3.2. Prove the equality in (3.55). Let me use the notation

$$\dot{\mathcal{C}}^\infty(M) = \{u \in \mathcal{C}^\infty(M); u \equiv 0 \text{ at } \partial M\} \subset \mathcal{C}^\infty(M)$$

for the space of smooth functions on a manifold with corners which vanish to infinite order at each boundary point. Then the identity (3.55) becomes

$$\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$$

under radial compactification. First check the single space version

$$\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n}) = \mathcal{S}(\mathbb{R}^n) \tag{3.56}$$

and then generalize (or use a clever argument) to pass to (3.2).

We remark on some related simple properties of smoothing operators. If $U \subset M$ is a coordinate neighbourhood, with coordinate map $F : U \longrightarrow U' \subset \mathbb{R}^n$ and ψ, $\psi' \in \mathcal{C}^\infty(M)$ has $\operatorname{supp}(\psi) \cup \operatorname{supp}(\psi') \subset U$ then

$$\begin{gathered} A_{\psi,\psi',F} : \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto (F^{-1})^*\left(\psi A(F^*((F^{-1})^*\psi' \cdot f))\right) \in \mathcal{S}(\mathbb{R}^n) \\ \text{is an element of } \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \end{gathered} \tag{3.57}$$

Indeed, the kernel of $A_{\psi,F}$ is

(3.58)
$$(F^{-1})^*\psi(z)((F^{-1})^*\times(F^{-1})^*A)(z,z')(F^{-1})^*\psi'(z') = B|dz'|,\ B \in \mathcal{C}^\infty_c(\mathbb{R}^{2n}) \subset \mathcal{S}(\mathbb{R}^{2n}).$$

*****

Extension of the results above for the residual isotropic algebra on Euclidean space to smoothing operators on compact manifolds.

*****

3.12. Semiclassical limit algebra

Now we next want to extend the discussion of semiclassical smoothing operators on $\mathbb{R}^n$, in §2.19 , to smoothing operators on compact manifolds; later we will extend this to pseudodifferential operators. Initially at least let M be a compact manifold without boundary. Let $\Delta \subset M^2$ be the diagonal,

$$\Delta = \{(z,z) \in M^2; z \in M\}. \tag{3.59}$$

DEFINITION 3.1. *An element of $\Psi_{\mathrm{sl}}^{-\infty}(M)$, the space of semiclassical families of smoothing operators on a compact manifold (without boundary) M, is a smooth family of smoothing operators $A_\epsilon \in \mathcal{C}^\infty((0,1] \times M^2; \pi_L^*\Omega)$ such that as $\epsilon \downarrow 0$ the kernel satisfies the two conditions:*

$$\begin{gathered} A_\epsilon \phi(z,z') \in \dot{\mathcal{C}}^\infty([0,1) \times M^2; \pi_R^*\Omega) \text{ if } \phi \in \mathcal{C}^\infty(M^2),\ \operatorname{supp}(\phi) \cap \Delta = \emptyset. \\ \text{For a covering of } M \text{ by coordinate systems } F_j : U_j \longrightarrow U_j' \\ \text{and any elements } \psi_j, \psi_j' \in \mathcal{C}^\infty(M),\ \operatorname{supp}(\psi) \cup \operatorname{supp}(\psi_j') \subset U_j, \\ (A_\epsilon)_{\psi_j,\psi_j',F_j} \in \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n). \end{gathered} \tag{3.60}$$

This is just supposed to say that $A_\epsilon \in \Psi_{\mathrm{sl}}^{-\infty}(M)$ reduces to a semiclassical family on $\mathbb{R}^n$ in local coordinates. We do not really need quite as much as in the second part of the defintion, which involves *all* pairs of smooth functions ψ_j, ψ_j' with compact support in a covering by coordinate patches. There is an equivalent and more geometric characterizations of the kernels of these semiclassical families below.

For the moment we note the following more useful description of the local behaviour of these operators.

PROPOSITION 3.8. *On a compact manifold M,*

$$\{A \in \mathcal{C}^\infty([0,1]_\epsilon \times M^2; \pi_L^*\Omega); A \equiv 0 \text{ at } \{\epsilon = 0\}\} \subset \Psi_{\mathrm{sl}}^{-\infty}(M). \tag{3.61}$$

If $F : U \longrightarrow U' \in \mathbb{R}^n$ is a coordinate patch on M and $A \in \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n)$ has kernel with support in $[0,1]_\epsilon \times K \times K$, $K \subset U'$ compact then

$$\begin{gathered} A_F \in \Psi_{\mathrm{sl}}^{-\infty}(M) \text{ where} \\ (A_F)_\epsilon : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M),\ (A_F u) = F^*(A(F^{-1})^* u). \end{gathered} \tag{3.62}$$

Moreover any element of $\Psi_{\mathrm{sl}}^{-\infty}(M)$ is the sum of a family of the first type and a finite sum, over any covering by coordinate patches, of operators as in (3.62).

PROOF. For the moment, see the proof of the corresponding theorem for pseudodifferential operators, Lemmas 6.1 and 6.2. The present result is is a bit easier; I will move the proof here and change it a bit. □

We can capture the 'semiclassical symbol' by oscillatory testing.

LEMMA 3.5. *If $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M)$ then there exists a function $\sigma_{\mathrm{sl}}(A_\epsilon) \in \mathcal{S}(T^*M)$ such that whenever $f : M \longrightarrow \mathbb{R}$ and $\psi \in \mathcal{C}^\infty(M)$ are such that $df \neq 0$ on $\operatorname{supp}(\psi)$ then*

$$A_\epsilon e^{-if/\epsilon}\psi = e^{-if/\epsilon}b,\ \ b \in \mathcal{C}^\infty([0,1] \times M),\ \ b\big|_{\epsilon=0} = \sigma_{\mathrm{sl}}(A_\epsilon) \circ df. \tag{3.63}$$

I need to define $\mathcal{S}(T^*M)$ first!

PROOF. Do the local, Euclidean, and then patch. □

PROPOSITION 3.9. *The semiclassical symbol of an element of $\Psi^{-\infty}_{\mathrm{sl}}(M)$ is determined by* (3.63) *and gives a short exact, multiplicative, sequence*

$$0 \longrightarrow \epsilon\Psi^{-\infty}_{\mathrm{sl}}(M) \longrightarrow \Psi^{-\infty}_{\mathrm{sl}}(M) \longrightarrow \mathcal{S}(T^*M) \longrightarrow 0 \tag{3.64}$$

Later, after discussing pseudodifferential operators on manifolds, we will also discuss semiclassical families of pseudodifferential operators, generalizing the discussion here. However there is one case which is very elementary. Namely the identity operator can be considered as a semiclassical family, even though it is independent of the parameter ϵ. By *fiat* its semiclassical symbol is declared to be the constant function 1 on the cotangent bundle. This is consistent with the multiplicativity of the semiclassical symbol, since of course for any family $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M)$,

$$\sigma_{\mathrm{sl}}(A_\epsilon) = \sigma_{\mathrm{sl}}(\operatorname{Id} \circ A_\epsilon) = 1 \times \sigma_{\mathrm{sl}}(A_\epsilon). \tag{3.65}$$

We can also immediately allow the algebra $\Psi^{-\infty}_{\mathrm{sl}}(M)$ to be 'valued in matrices', just by taking matrices of operators; we will denote this algebra as $\Psi^{-\infty}_{\mathrm{sl}}(M; \mathbb{C}^N)$ since the act on N-vectors of smooth functions on M. The symbol is then also valued in matrices.

PROPOSITION 3.10. *If $a \in \mathcal{S}(T^*M; M(N, \mathbb{C}))$ is such that $\operatorname{Id}_{N\times N} - a$ is invertible at every point of T^*M then any semiclassical family $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M; \mathbb{C}^N)$ with $\sigma_{\mathrm{sl}}(A_\epsilon) = a$ is such that $\operatorname{Id} - A_\epsilon$ is invertible for small $\epsilon > 0$ with inverse of the form $\operatorname{Id} - B_\epsilon$ for some $B_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M; \mathbb{C}^N)$.*

3.13. Submanifolds and blow up

A brief description of blow up of a submanifold, enough to introduce the semiclassical resolution of $[0,1] \times M^2$ in the next section.

3.14. Resolution of semiclassical kernels

3.15. Quantization of projections

PROPOSITION 3.11. *If $a \in \mathcal{S}(T^*M; M(N, \mathbb{C}))$ is such that for a constant projection $\pi_0 \in M(N, \mathbb{C})$, i.e. such that $\pi_0^2 = \pi_0$, $\pi_0 + a$ is a smooth family of projections, $(\pi_0 + a)^2 = \pi_0 + a$ then there exists a semiclassical family $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M; \mathbb{C}^N)$ such that $\sigma_{\mathrm{sl}}(A_\epsilon) = a$ and such that*

$$(\pi_0 + A_\epsilon)^2 = \pi_0 + A_\epsilon \tag{3.66}$$

is a semiclassical family of projections.

PROOF. Just 'quantizing' a by choosing a semiclassical family $A'_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M; \mathbb{C}^n)$ with $\sigma_{\mathrm{sl}}(A'_\epsilon) = a$ ensures that

$$(\pi_0 + A'_\epsilon)^2 - (\pi_o + A'_\epsilon) = \epsilon E^{(1)}_\epsilon,\ \ E^{(1)}_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M; \mathbb{C}^N). \tag{3.67}$$

We proceed to show, inductively, that there is a series of 'correction terms' $A^{(j)} \in \Psi_{\mathrm{sl}}^{-\infty}(M;\mathbb{C}^N)$ such that for all l,

(3.68)
$$(\pi_0 + A'_\epsilon + \sum_{k=1}^{l} \epsilon^k A_\epsilon^{(k)})^2 - (\pi_o + A'_\epsilon \sum_{k=1}^{l} \epsilon^k A_\epsilon^{(k)}) = \epsilon^{l+1} E_\epsilon^{(l+1)},\ E_\epsilon^{(l)} \in \Psi_{\mathrm{sl}}^{-\infty}(M;\mathbb{C}^N).$$

Composing on the left and on the right with $\pi_o + A'_\epsilon \sum\limits_{k=1}^{l} \epsilon^k A_\epsilon^{(k)}$ and using the associativity of the product it follows that

$$\pi_0 \sigma_{\mathrm{sl}}(E_\epsilon^{(l+1)}) = \sigma_{\mathrm{sl}}(E_\epsilon^{(l+1)})\pi_0. \tag{3.69}$$

This in turn means that if $A_\epsilon^{(l+1)} \in \Psi_{\mathrm{sl}}^{-\infty}(M;\mathbb{C}^N)$ satisfies

$$\sigma_{\mathrm{sl}}(A_\epsilon^{(l+1)}) = (2\pi_0 - \mathrm{Id})\sigma_{\mathrm{sl}}(E_\epsilon^{(l+1)}) \tag{3.70}$$

then the next identity, (3.68), for $l+1$, holds.

Now, if A''_ϵ is an asymptotic sum of the series then

$$(\pi_0 + A''_\epsilon)^2 - \pi_0 + A''_\epsilon \in \{A \in \mathcal{C}^\infty([0,1];\Psi^{-\infty}(M;\mathbb{C}^N)); A \equiv 0 \text{ at } \{\epsilon = 0\}\}. \tag{3.71}$$

To correct this family of 'projections to infinite order' $P'_\epsilon = \pi_0 + A''_\epsilon$ to a true projection we may use the holomorphic calculus of smoothing operators. Thus, the family

$$Q(s) = s^{-1}(\mathrm{Id} - P') + (s-1)^{-1}P',\ s \in \mathbb{C} \setminus \{0,1\} \tag{3.72}$$

satisfies the 'resolvent identity' to infinite order in ϵ :

(3.73)
$$(s\,\mathrm{Id} - P')Q(s) = (s(\mathrm{Id} - P') - (1-s)P')\,(Q(s) =$$
$$(\mathrm{Id} - P')^2 + (P')^2 + s^{-1}(s-1)(\mathrm{Id} - P')P' + (s-1)^{-1}sP'(\mathrm{Id} - P') = \mathrm{Id} + R(s)$$

where $R_\epsilon(s)$ is a family of smoothing operators vanishing to infinite order at $\epsilon = 0$ and depending holomorphically on $s \in \mathbb{C} \setminus \{0,1\}$. Thus in any region $|s| \geq \delta$, $|1-s| \geq \delta$, that is away from $s=0$ and $s=1$, $R(s)$ has uniformly small norm as $\epsilon \to 0$. It follows that $(\mathrm{Id} + R(s))^{-1} = \mathrm{Id} + M(s)$ exists in this region, for $\epsilon > 0$ small, and $M(s)$ is a holomorphic family of smoothing operators vanishing to infinite order at $\epsilon = 0$.

Thus the resolvent exists in this region and

$$(s\,\mathrm{Id} - P')^{-1} = Q(s) + M'(s) \tag{3.74}$$

where $M'(s)$ is another holomorphic family of smoothing operators vanishing to infinite order at $\epsilon = 0$.

To 'correct' P' to a family of projections we simply define

$$P = \frac{1}{2\pi i} \oint_{|1-s| = \frac{1}{2}} (s - P'(s))^{-1} ds. \tag{3.75}$$

From the decomposition (3.74) and (3.72) we see immediately that

$$P = P' + M,\ M = \frac{1}{2\pi i} \oint_{|1-s| = \frac{1}{2}} M(s) ds \in \epsilon^\infty \Psi_{\mathrm{sl}}^{-\infty}(M). \tag{3.76}$$

Moreover it follows from (3.75) that P is a projection. First, using Cauchy's theorem, we can shift the contour away from $s = 1$ a little, to $|s-1| = \gamma$ for some $\gamma > 0$, small. Then

$$P^2 = \frac{1}{2\pi i}\oint_{|1-s|=\frac{1}{2}} \frac{1}{2\pi i}\oint_{|1-t|=\frac{1}{2}+\gamma} (t - P'(t))^{-1}(s - P'(s))^{-1} ds dt. \tag{3.77}$$

The resolvent identity

$$(t - P'(t))^{-1}(s - P'(s))^{-1} = (s-t)^{-1}\left((t - P'(t))^{-1} - (s - P'(s))^{-1}\right) \tag{3.78}$$

allows the integral to be split into two. In the first double integral there are no singularities in s within $|1-s| \le \frac{1}{2}$ since $|1-t| = \frac{1}{2}+\gamma$, so by Cauchy's theorem this evaluates to zero. In the remaining term the t integral can be evaluated by residues, with the only singular point being at $t = s$ so

$$\begin{aligned} P^2 &= -\frac{1}{2\pi i}\oint_{|1-s|=\frac{1}{2}} \frac{1}{2\pi i}\oint_{|1-t|=\frac{1}{2}+\gamma} (s-t)^{-1}(s - P'(s))^{-1} ds dt \\ &= \frac{1}{2\pi i}\oint_{|1-s|=\frac{1}{2}} (s - P'(s))^{-1} ds = P. \end{aligned} \tag{3.79}$$

Thus P is a semiclassical quantization of the projection-valued symbol to a family of projections. □

We will show below that this same argument works in other contexts.

CHAPTER 4

Isotropic calculus

The algebra of 'isotropic' pseudodifferential operators on $\mathbb{R}^n$ has global properties very similar to the algebra of pseudodifferential operators on a compact manifold discussed below. There are several reasons for the extensive discussion here. First it is pretty! Second it is useful in the sense that it embeds the harmonic oscillator in a broader context. Thirdly, many of the global constructions here carry over almost unchanged to the case of compact manifolds and it may help to see them in a somewhat simpler setting. Finally, it is useful in a geometric and topological sense as may become clearer below in the discussion of K-theory.

4.1. Isotropic operators

As noted in the discussion in Chapter 2, there are other sensible choices of the class of amplitudes which can be admitted in the definition of a space of pseudodifferential operators rather than the basic case of $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ discussed there. One of the smallest such choices is the class which is completely symmetric in the variables x and ξ and consists of the symbols on $\mathbb{R}^{2n}$. Thus, $a\in S^m(\mathbb{R}^{2n}_{x,\xi})$ satisfies the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi)|\le C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|} \tag{4.1}$$

for all multiindices α and β. Recall that there is a subspace of 'classical' or polyhomogeneous symbols

$$S^m_{\mathrm{ph}}(\mathbb{R}^{2n})\subset S^m(\mathbb{R}^{2n}) \tag{4.2}$$

defined by the condition that its elements are asymptotic sums of terms $a_j\in S^m(\mathbb{R}^{2n})$ with a_j positively homogeneous of degree $m-j$ in $|(x,\xi)|\ge 1$.

If $m\le 0$, it follows that $a\in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$; if $m>0$ this is not true, however,

Lemma 4.1. *For any p and n*

$$S^m(\mathbb{R}^{p+n})\subset\begin{cases}\bigcap\limits_{0\ge r\ge m}(1+|x|^2)^{r/2}S^{m-r}_\infty(\mathbb{R}^p_x;\mathbb{R}^n_\xi) & m\le 0\\ (1+|x|^2)^{m/2}S^m_\infty(\mathbb{R}^p_x;\mathbb{R}^n_\xi), & m>0.\end{cases} \tag{4.3}$$

Proof. This follows from (4.1) and the inequalities

$$1+|x|+|\xi|\le(1+|x|)(1+|\xi|),$$
$$1+|x|+|\xi|\ge(1+|x|)^t(1+|\xi|)^{1-t},\ 0\le t\le 1.$$

□

In view of these estimates the following definition makes sense.

Definition 4.1. *For any $m \in \mathbb{R}$ we define*

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n) \subset \langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n) \tag{4.4}$$

as the subspaces determined by

$$\begin{aligned} A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) &\Longleftrightarrow \sigma_L(A) \in S^m_{\mathrm{ph}}(\mathbb{R}^{2n}) \\ A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n) &\Longleftrightarrow \sigma_L(A) \in S^m(\mathbb{R}^{2n}). \end{aligned} \tag{4.5}$$

Note however that the notation has been switched here. The space with the absence of any subscript corresponds to classical symbols, whereas the '$\infty-\mathrm{iso}$' subscript refers to the symbols with 'bounds' as in (4.1).

As in the discussion in Chapter 2 the 'residual' algebra consists just of the intersection

$$\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \Psi^{-\infty}_{\infty-\mathrm{iso}}(\mathbb{R}^n) = \bigcap_m \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n). \tag{4.6}$$

From the discussion above, an element of either space on the left has left-reduced symbol in $S^{-\infty}(\mathbb{R}^{1n}) = \mathcal{S}(\mathbb{R}^{2n})$ so its kernel is also in $\mathcal{S}(\mathbb{R}^{2n})$ and conversely. This justifies the apparently different sense in which this notation is used in Chapter 3.

As in the discussion of the traditional algebra in Chapter 2 we show the $*$-invariance and composition properties of these spaces of operators by proving an appropriate 'reduction' theorem. However there is a small difficulty here. Namely it might be supposed that it is enough to analyse $I(a)$ for $a \in S^m(\mathbb{R}^{3n})$. This however is not the case. Indeed the definition above is in terms of left-reduced symbols. If $a \in S^m(\mathbb{R}^{2n})$ is regarded as a function on $\mathbb{R}^{3n}$ which is independent of one of the variables then it is in general *not* an element of $S^m(\mathbb{R}^{3n})$ (it is an element of $S^m_\infty(\mathbb{R}^n_y;\mathbb{R}^{2n})$ since it is constant in the first variables). For this reason we need to consider some more 'hybrid' estimates.

Consider a subdivision of $\mathbb{R}^{3n}$ into two closed regions:

$$\begin{aligned} R_1(\epsilon) &= \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \le \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\} \\ R_2(\epsilon) &= \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \ge \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\}. \end{aligned} \tag{4.7}$$

If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ consider the estimates

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma} \begin{cases} \langle (x,y,\xi)\rangle^{m-|\alpha|-|\beta|-|\gamma|} & \text{in } R_1(\frac18) \\ \langle (x,y)\rangle^{m_+}\langle \xi\rangle^{m-|\gamma|} & \text{in } R_2(\frac18). \end{cases} \tag{4.8}$$

The choice $\epsilon = \frac18$ here is rather arbitrary. However if ϵ is decreased, but kept positive the same estimates continue to hold for the new subdivision, since the estimates in R_1 are stronger than those in R_2 (which is increasing at the expense of R_1 as ϵ decreases). Notice too that these estimates do in fact imply that $a \in \langle x\rangle^{m_+}\langle y\rangle^{m_+}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and hence they do define operators in the weighted spaces – in principle $\langle x\rangle^{2m_+}\Psi^m_\infty(\mathbb{R}^n)$ although actually $\langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n)$ – that were analysed in Chapter 2.

Proposition 4.1. *If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ satisfies the estimates* (4.8) *then $A = I(a) \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ and* (2.58) *holds for $\sigma_L(A)$.*

Proof. We separate a into two pieces. Choose $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $0 \le \chi \le 1$, with support in $[-\frac18,\frac18]$ and with $\chi \equiv 1$ on $[-\frac19,\frac19]$. Then consider the cutoff function on $\mathbb{R}^{3n}$

$$\psi(x,y,\xi) = \chi\left(\frac{|x-y|}{\langle(x,y,\xi)\rangle}\right). \tag{4.9}$$

Clearly, ψ has support in $R_1(\frac{1}{8})$ and $\psi \in S^0_\infty(\mathbb{R}^{3n})$. It follows then that $a' = \psi a \in S^m_{\mathrm{iso}}(\mathbb{R}^{3n})$. On the other hand, $a'' = (1-\psi)a$ has support in $R_2(\frac{1}{9})$. In this region $|x-y|$, $\langle(x,y)\rangle$ and $\langle(x,y,\xi)\rangle$ are bounded by constant multiples of each other. Thus a'' satisfies the estimates

$$\begin{aligned} |D^\alpha_x D^\beta_y D^\gamma_\xi a''(x,y,\xi)| &\le C_{\alpha,\beta,\gamma}|x-y|^{m_+}\langle\xi\rangle^{m-|\gamma|} \\ &\le C'_{\alpha,\beta,\gamma}\langle(x,y,\xi)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|},\ \operatorname{supp}(a'') \subset R_2(\frac{1}{9}). \end{aligned} \tag{4.10}$$

First we check that $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$. On $R_2(\frac{1}{9})$ it is certainly the case that $|x-y| \ge \frac{1}{9}\langle(x,y)\rangle$ and by integration by parts

$$|x-y|^{2p} D^\alpha_x D^\beta_y I(a'') = I(|D_\xi|^{2p} D^\alpha_x D^\beta_y a'').$$

For all sufficiently large p it follows from (4.10) that this is the product of $\langle(x,y)\rangle^{m_+}$ and a bounded continuous function. Thus, $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$ is the kernel of an operator in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

So it remains only to show that $A' = I(a') \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Certainly this is an element of $\langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n)$. The left-reduced symbol of A' has an asymptotic expansion, as $\xi \to \infty$, given by the usual formula, namely (2.58). Each of the terms in this expansion

$$a_L(A') \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_y D^\alpha_\xi a(x,x,\xi)$$

is in the space $S^{m-2|\alpha|}(\mathbb{R}^{2n})$. Thus we can actually choose an asymptotic sum in the stronger sense that

$$b' \in S^m(\mathbb{R}^{2n}),\ b_N = b' - \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-2N}(\mathbb{R}^{2n})\ \forall\ N.$$

Consider the remainder term in (2.47), given by (2.44) and (2.45). Integrating by parts in ξ to remove the factors of $(x-y)^\alpha$ the remainder, R_N, can be written as a pseudodifferential operator with amplitude

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!} \int_0^1 dt(1-t)^N (D^\alpha_\xi D^\alpha_y a)((1-t)x+ty,\xi).$$

This satisfies the estimates (4.8) with m replaced by $m-2N$. Indeed from the symbol estimates on a' the integrand satisfies the bounds

$$\begin{aligned} &|D^\beta_x D^\gamma_y D^\delta_\xi D^\alpha_\xi D^\alpha_y a'((1-t)x+ty,\xi)| \\ &\qquad\le C(1+|(x+t(x-y)|+|\xi|)^{m-2N-|\beta|-|\gamma|-|\delta|}. \end{aligned}$$

In $R_1(\frac{1}{8})$, $|x-y| \le \frac{1}{8}\langle(x,y,\xi)\rangle$ so $|x+t(x-y)|+|\xi| \ge \frac{1}{2}\langle(x,y,\xi)\rangle$ and these estimates imply the full symbol estimates there. On R_2 we immediately get the weaker estimates in (4.7).

Thus, for large N, the remainder term gives an operator in $\langle x\rangle^{\frac{m}{2}-N}\Psi^{\frac{m}{2}-N}_\infty(\mathbb{R}^n)$. The difference between A' and the operator $B' \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, which is R_N plus an operator in $\Psi^{m-2N}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for any N is therefore in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$. $\square$

This is a perfectly adequate replacement in this context for our previous reduction theorem, so now we can show the basic result.

THEOREM 4.1. *The spaces* $\Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) of isotropic (resp. polyhomogeneous isotropic) pseudodifferential operators on* $\mathbb{R}^n$*, defined by* (4.5) *form an order-filtered* $*$*-algebra with residual space* $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$ *(resp. the same) as spaces of kernels.*

PROOF. The condition that a continuous linear operator A on $\mathcal{S}(\mathbb{R}^n)$ be an element of $\Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is that it be an element of $(1+|x|^2)^{m/2}\Psi^m_\infty(\mathbb{R}^n)$ if $m \geq 0$ or $\Psi^m_\infty(\mathbb{R}^n)$ if $m < 0$ with left-reduced symbol an element of $S^m(\mathbb{R}^{2n}_{x,\xi})$:

$$q_l : S^m_\infty(\mathbb{R}^{2n}) \longleftrightarrow \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n). \tag{4.11}$$

Thus A^* has right-reduced symbol in $S^m_\infty(\mathbb{R}^{2n})$. This satisfies the estimates (4.8) as a function of x, y and ξ. Thus Proposition 4.1 shows that $A^* \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, since its left-reduced symbol is in $S^m(\mathbb{R}^{2n})$, proving the $*$-invariance. Moreover it also follows that any $B \in \Psi^{m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ has right-reduced symbol in $S^{m'}(\mathbb{R}^{2n})$. Thus if $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ then using this result to right-reduce B we see that the composite operator has kernel $I(a_L(x,\xi)b_R(y,\xi))$ where $a_L \in S^m_\infty(\mathbb{R}^{2n})$ and $b_R \in S^{m'}_\infty(\mathbb{R}^{2n})$. Now it again follows that this product satisfies the estimates (4.8) of order $m + m'$. Hence, again applying Proposition 4.1, we conclude that $A \circ B \in \Psi^{m+m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. This proves the theorem for $\Psi^*_{\infty-\mathrm{iso}}(\mathbb{R}^n)$.

The proof for the polyhomogeneous space $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ follows immediately, since the symbol expansions all preserve polyhomogeneity. $\square$

One further property of the isotropic calculus that distinguishes it strongly from the traditional calculus is that it is invariant under Fourier transformation.

PROPOSITION 4.2. *If* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) then* $\hat{A} \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) where* $\widehat{\hat{A}u} = A\hat{u}$ *with* $\hat{u}$ *being the Fourier transform of* $u \in \mathcal{S}(\mathbb{R}^n)$*.*

The proof of this is outlined in Problem 2.20.

Also note that asymptotic completeness then carries over from the symbol spaces. If $B_j \in \Psi^{m-j}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ then there exists

$$B \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n),\ B \sim \sum_j B_j \text{ that is } B - \sum_{j=0}^{N-1} B_j \in \Psi^{m-N}_{\infty-\mathrm{iso}}(\mathbb{R}^n)\ \forall\ N. \tag{4.12}$$

4.2. Fredholm property

An element $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is said to be elliptic (of order m in the isotropic calculus) if its left-reduced symbol is elliptic in $S^m(\mathbb{R}^{2n})$.

THEOREM 4.2. *Each elliptic element* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *has a two-sided parametrix* $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *in the sense that*

$$A \circ B - \mathrm{Id},\ B \circ A - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.13}$$

which is unique up to an element of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and it follows that any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *satisfying* $Au \in \mathcal{S}(\mathbb{R}^n)$ *is an element of* $\mathcal{S}(\mathbb{R}^n)$*; if* $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ *is elliptic then its parametrix is in* $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$*.*

Proof. This is just the inductive argument used to prove Lemma 2.7. Nevertheless we repeat it here.

The ellipticity of $\sigma_m(A)$ means that it has a two-sided inverse $b \in S^{-m}(\mathbb{R}^{2n})$ modulo $S^{-\infty}(\mathbb{R}^{2n}) = \mathcal{S}(\mathbb{R}^{2n})$. This in turn means that the equation $\sigma_k(A)c = d$ always has a solution $c \in S^{-m+m'-[1]}(\mathbb{R}^{2n})$ for given $d \in S^{m'-[1]}(\mathbb{R}^{2n})$ namely $c = bd$. This in turn means that given $C_j \in \Psi^j_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ there always exists $B_j \in \Psi^{j-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ such that $AB_j - C_j \in \Psi^{j-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Choosing $B_0 \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ to have $\sigma_{-m}(B_0) = b$ we can define $C_1 = \mathrm{Id} - AB_0 \in \Psi^{-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then, proceeding inductively we may assume that B_j for $j < l$ have been chosen such that $A(B_0 + \cdots + B_{l-1}) - \mathrm{Id} = -C_l \in \Psi^{-l}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then using the solvability we may choose B_l so that $AB_l - C_l = -C_{l+1} \in \Psi^{-l-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ which completes the induction, since $A(B_0+\cdots+B_l)-\mathrm{Id} = AB_l - C_l = -C_{l+1}$. Finally by the asymptotic completeness we may choose $B \sim B_0 + B_1 + \dots$ which is a right parametrix.

The existence of a left parametrix follows from the ellipticity of A^* and the argument showing that a right parametrix is a two-sided parametrix is essentially the same as in Lemma 2.7. $\square$

Combining the earlier symbolic discussion and these analytic results we can see that elliptic operators are Fredholm as an operator

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ or } A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{4.14}$$

Proposition 4.3. *If $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is elliptic then it has a generalized inverse $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ satisfying*

$$AB - \mathrm{Id} = \Pi_1,\ BA - \mathrm{Id} = \Pi_0 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.15}$$

where Π_1 and Π_0 are the finite rank orthogonal (in $L^2(\mathbb{R}^n)$) projections onto the null spaces of A^* and A.

Proof. As discussed above, A has a parametrix $B' \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ modulo $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$\begin{aligned} AB' &= \mathrm{Id} - E_R,\ E_R \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\\ B'A &= \mathrm{Id} - E_L,\ E_L \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \end{aligned}$$

Using Proposition 3.2 it follows that the null space of A is contained in the null space of $B'A = \mathrm{Id} - E_L$, hence is finite dimensional. Similarly, the range of A contains the range of $AB' = \mathrm{Id} - E_R$ so is closed with a finite codimensional complement. Defining B as the linear map which vanishes on $\mathrm{Nul}(A^*)$, and inverts A on $\mathrm{Ran}(A)$ with values in $\mathrm{Ran}(A^*) = \mathrm{Nul}(A)^\perp$ gives (4.15). Furthermore these identities show that $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ since applying B' gives

$$\begin{gathered} B - E_L B = B'AB = B' - B'\Pi_1,\ B - BE_R = BAB' = B' - \Pi_0 B' \Longrightarrow \\ B = B' - B'\Pi_1 + E_L B' + E_L B E_R - E_L \Pi_0 B' \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n) \end{gathered} \tag{4.16}$$

where we use the fact that $EBE' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for any continuous linear operator B on $\mathcal{S}(\mathbb{R}^n)$ and elements $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. $\square$

Corollary 4.1. *If $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is elliptic then its generalized inverse lies in $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$.*

4.3. The harmonic oscillator

The harmonic oscillator is the differential operator on $\mathbb{R}^n$

$$H = \sum_{j=1}^{n} (D_j^2 + x_j^2) = \Delta + |x|^2.$$

This is an elliptic element of $\Psi^2_{\text{iso}}(\mathbb{R}^n)$. The main immediate interest is in the spectral decomposition of H. The ellipticity of $H - \lambda$, $\lambda \in \mathbb{C}$, shows that

$$(H - \lambda)u = 0,\ u \in \mathcal{S}'(\mathbb{R}^n) \Longrightarrow u \in \mathcal{S}(\mathbb{R}^n). \tag{4.17}$$

Since H is (formally) self-adjoint, i.e., $H^* = H$, there are no non-trivial tempered solutions of $(H - \lambda)u = 0$, $\lambda \notin \mathbb{R}$. Indeed if $(H - \lambda)u = 0$,

$$0 = \langle Hu, u\rangle - \langle u, Hu\rangle = (\lambda - \overline{\lambda})\langle u, u\rangle \Longrightarrow u = 0. \tag{4.18}$$

As we shall see below in more generality, the spectrum of H is a discrete subset of $\mathbb{R}$. In this case we can compute it explicitly.

The direct computation of eigenvalues and eigenfunctions is based on the properties of the creation and annihilation operators

$$C_j = D_j + ix_j,\ C_j^* = A_j = D_j - ix_j,\ j = 1, \dots, n. \tag{4.19}$$

These satisfy the elementary identities

$$[C_j, C_k] = [A_j, A_k] = 0,\ [A_j, C_k] = 2\delta_{jk},\ j, k = 1, \dots, n \tag{4.20}$$

$$H = \sum_{j=1}^{n} C_j A_j + n,\ [C_j, H] = -2C_j,\ [A_j, H] = 2A_j. \tag{4.21}$$

Now, if λ is an eigenvalue, $Hu = \lambda u$, then

$$\begin{aligned} H(C_j u) &= C_j(Hu + 2u) = (\lambda + 2)C_j u, \\ H(A_j u) &= A_j(Hu - 2u) = (\lambda - 2)A_j u. \end{aligned} \tag{4.22}$$

Proposition 4.4. *The eigenvalues of H are*

$$\sigma(H) = \{n, n + 2, n + 4, \dots\}. \tag{4.23}$$

Proof. We already know that eigenvalues must be real and from the decomposition of H in (4.21) it follows that, for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle Hu, u\rangle = \sum_j \|A_j u\|^2 + n\|u\|^2. \tag{4.24}$$

Thus if $\lambda \in \sigma(H)$ is an eigenvalue then $\lambda \geq n$.

By direct computation we see that n is an eigenvalue with a 1-dimensional eigenspace. Indeed, from (4.24), $Hu = nu$ iff $A_j u = 0$ for $j = 1, \dots, n$. In each variable separately

$$A_j u(x_j) = 0 \Leftrightarrow u(x_j) = c_1 \exp\left(-\frac{x_j^2}{2}\right).$$

Thus the only tempered solutions of $A_j u = 0$, $i = 1, \dots, n$ are the constant multiples of

$$u_0 = \exp\left(-\frac{|x|^2}{2}\right), \tag{4.25}$$

which is often called the *ground state*.

Now, if λ is an eigenvalue with eigenfunction $u \in \mathcal{S}(\mathbb{R}^n)$ it follows from (4.22) that $\lambda - 2$ is an eigenvalue with eigenfunction $A_j u$. Since all the $A_j u$ cannot vanish unless u is the ground state, it follows that the eigenvalues are contained in the set in (4.23). We can use the same argument to show that if u is an eigenfunction with eigenvalue λ then $C_j u$ is an eigenfunction with eigenvalue $\lambda + 2$. Moreover, $C_j u \equiv 0$ would imply $u \equiv 0$ since $C_j v = 0$ has no non-trivial tempered solutions, the solution in each variable being $\exp(x_j^2/2)$. $\square$

Using the creation operators we can parameterize the eigenspaces quite explicitly.

Proposition 4.5. *For each $k \in \mathbb{N}_0$ there is an isomorphism*

$$\{\textit{Polynomials, homogeneous of degree } k \textit{ on } \mathbb{R}^n\} \ni p \longmapsto p(C)\exp\left(-\frac{|x|^2}{2}\right) \in E_k \tag{4.26}$$

where E_k is the eigenspace of H with eigenvalue $n + 2k$.

Proof. Notice that the C_j, $j = 1, \dots, n$ are commuting operators, so $p(C)$ is well-defined. By iteration from (4.22),

$$HC^\alpha u_0 = C^\alpha(H + 2|\alpha|)u_0 = (n + 2|\alpha|)C^\alpha u_0. \tag{4.27}$$

Thus (4.26) is a linear map into the eigenspace as indicated.

To see that (4.26) is an isomorphism consider the action of the annihilation operators. Again from (4.22)

$$|\beta| = |\alpha| \Longrightarrow A^\beta C^\alpha u_0 = \begin{cases} 0 & \beta \neq \alpha \\ 2^{|\alpha|}\alpha! u_0 & \beta = \alpha. \end{cases} \tag{4.28}$$

This allows us to recover the coefficients of p from $p(C)u_0$, so (4.26) is injective. Conversely if $v \in E_k \subset \mathcal{S}(\mathbb{R}^n)$ is orthogonal to all the $C^\alpha u_0$ then

$$\langle A^\alpha v, u_0\rangle = \langle v, C^\alpha u_0\rangle = 0 \ \forall \ |\alpha| = k. \tag{4.29}$$

From (4.22), the $A^\alpha v$ are all eigenfunctions of H with eigenvalue n, so (4.29) implies that $A^\alpha v = 0$ for all $|\alpha| = k$. Proceeding inductively in k we see that $A^{\alpha'} A_j v = 0$ for all $|\alpha'| = k - 1$ and $A_j v \in E_{k-1}$ implies $A_j v = 0$, $j = 1, \dots, n$. Since $v \in E_k$, $k > 0$, this implies $v = 0$ so Proposition 4.5 is proved. $\square$

Thus H has eigenspaces as described in (4.26). The same argument shows that for any integer p, positive or negative, the eigenvalues of H^p are precisely $(n + 2k)^p$ with the same eigenspaces E_k. For $p < 0$, H^p is a compact operator on $L^2(\mathbb{R}^n)$; this is obvious for large negative p. For example, if $p \le -n - 1$ then

$$x_i^\beta D_j^\alpha H \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n),\ |\alpha| \le n + 1, |\beta| \le n + 1 \tag{4.30}$$

are all bounded on L^2. If $S \subset L^2(\mathbb{R}^n)$ is bounded this implies that $H^{-n-1}(S)$ is bounded in $\langle x\rangle^{n+1}C^1_\infty(\mathbb{R}^n)$, so compact in $\langle x\rangle^n C^0_\infty(\mathbb{R}^n)$ and hence in $L^2(\mathbb{R}^n)$. It is a general fact that for compact self-adjoint operators, such as H^{-n-2}, the eigenfunctions span $L^2(\mathbb{R}^n)$. We give a brief proof of this for the sake of 'completeness'.

Lemma 4.2. *The eigenfunction of H, $u_\alpha = \pi^{-\frac{n}{4}}(2^{|\alpha|}\alpha!)^{-1/2}C^\alpha u_0$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.*

PROOF. Let $V \subset L^2(\mathbb{R}^n)$ be the closed subspace consisting of the orthocomplements of all the u_α's. Certainly H^{-n-2} acts on it as a compact self-adjoint operator. Since we have found all the eigenvalues of H, and hence of H^{-n-1}, it has no eigenvalue in V. We wish to conclude that $V = \{0\}$. Set

$$\tau = \|H^{-n-1}\|_V = \sup\{\|H^{-n-1}\varphi\|; \varphi \in V, \|\varphi\| = 1\}.$$

Then there is a weakly convergent sequence $\varphi_j \rightharpoonup \varphi$, $\|\varphi_j\| = 1$, so $\|\varphi\| \leq 1$, with $\|H^{-n-1}\varphi_j\| \to \tau$. The compactness of H^{-n-2} allows a subsequence to be chosen such that $H^{-n-1}\varphi_j \to \psi$ in $L^2(\mathbb{R}^n)$. So, by the continuity of H^{-n-1}, $H^{-n-1}\varphi = \psi$ and $\|H^{-n-1}\varphi\| = \tau$, $\|\varphi\| = 1$. If $\varphi' \in V$, $\varphi' \perp \varphi$, $\|\varphi'\| = 1$ then

$$\begin{aligned} \tau^2 \geq \|H^{-n-2}\left(\frac{\varphi + t\varphi'}{\sqrt{1+t^2}}\right)\|^2 &= \tau^2 + 2t\langle H^{-2n-2}\varphi, \varphi'\rangle + 0(t^2) \\ &\Longrightarrow \langle H^{-2n-2}\varphi, \varphi'\rangle = 0 \Longrightarrow H^{-2n-2}\varphi = \tau^2\varphi. \end{aligned}$$

This contradicts the fact that H^{-2n-2} has no eigenvalues in V, so $V = \{0\}$ and the eigenbasis is complete. □

Thus, if $u \in L^2(\mathbb{R}^n)$

$$u = \sum_\alpha c_\alpha u_\alpha,\ c_\alpha = \langle u, u_\alpha\rangle \tag{4.31}$$

with convergence in L^2.

LEMMA 4.3. *If $u \in \mathcal{S}(\mathbb{R}^n)$ the convergence in (4.31) is rapid, i.e., $|c_\alpha| \leq C_N(1+|\alpha|)^{-N}$ for all N and the series converges in $\mathcal{S}(\mathbb{R}^n)$.*

PROOF. Since $u \in \mathcal{S}(\mathbb{R}^n)$ implies $H^N u \in L^2(\mathbb{R}^n)$ we see that

$$C_N \geq |\langle H^N u, u_\alpha\rangle| = |\langle u, H^N u_\alpha\rangle| = (n + 2|\alpha|)^N |c_\alpha|\ \forall\ \alpha.$$

Furthermore, $2ix_j = C_j - A_j$ and $2D_j = C_j + A_j$ so the polynomial derivatives of the u_α can be estimated (using the Sobolev embedding theorem) by polynomials in α; this implies that the series converges in $\mathcal{S}(\mathbb{R}^n)$. □

COROLLARY 4.2. *Finite rank elements are dense in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^{2n})$.*

PROOF. Consider the approximation (4.31) to the kernel A of an element of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ as an element of $\mathcal{S}(\mathbb{R}^{2n})$. In this case the ground state is

$$U_0 = \exp\left(-\frac{|x|^2}{2} - \frac{|y|^2}{2}\right) = \exp\left(-\frac{|x|^2}{2}\right)\exp\left(-\frac{|y|^2}{2}\right)$$

and so has rank one as an operator. The higher eigenfunctions

$$C^\alpha U_0 = Q_\alpha(x, y)U_0$$

are products of U_0 and a polynomial, so are also of finite rank. □

4.4. L^2 boundedness and compactness

The results above have obvious extension to the case of $N\times N$ matrices of operators, which we denote by $\Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ since they act on $\mathbb{C}^N$ valued functions. Recall that $\Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n)\subset\Psi^0_{\infty}(\mathbb{R}^n)$ so, by Proposition 2.6, these operators are bounded on $L^2(\mathbb{R}^n)$. Using the same argument the bound on the L^2 norm can be related to the norm of the principal symbol as an $N\times N$ matrix.

PROPOSITION 4.6. *If $A\in\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ has homogeneous principal symbol*

$$a=\sigma_L(A)\big|_{\mathbb{S}^{2n-1}}\in\mathcal{C}^\infty(\mathbb{S}^{2n-1};M(N,\mathbb{C}))$$

then

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)}\|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N))}=\sup_{p\in\mathbb{S}^{2n-1}}\|a(p)\|. \tag{4.32}$$

A similar result is true without the assumption that the principal symbol is homogeneous. It is simply necessary to replace the supremum on the right by

$$\lim_{R\to\infty}\sup_{|(x,\xi)|\ge R}\|\sigma_L(A)(x,\xi)\| \tag{4.33}$$

where the norm on the symbol is the Euclidean norm on $N\times N$ matrices.

PROOF. It suffices to prove (4.32) for all single operators $A\in\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$. Indeed if $j_v(z)=zv$ is the linear map from $\mathbb{C}$ to $\mathbb{C}^N$ defined by $v\in\mathbb{C}^N$ then

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N))}=\sup_{\{v,w\in\mathbb{C}^N;\|v\|=\|w\|=1\}}\|j_w^*Aj_v\|_{\mathcal{B}(L^2(\mathbb{R}))}. \tag{4.34}$$

Since the symbol of $j_w^*Aj_v$ is just $j_w^*\sigma(A)j_v$, (4.32) follows from the corresponding equality for a single operator:

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)}\|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n)}\le\sup_{p\in\mathbb{S}^{2n-1}}|a(p)|,\ a=\sigma_L(A)\big|_{\mathbb{S}^{2n-1}}. \tag{4.35}$$

The construction of the approximate square-root of $C-A^*A$ in Proposition 2.7 only depends on the existence of a positive smooth square-root for $C-|a|^2$, so can be carried out for any

$$C>\sup_{p\in\mathbb{S}^{2n-1}}|a(p)|^2. \tag{4.36}$$

Thus we conclude that with such a value of C

$$\|Au\|^2\le C\|u\|^2+\|\langle Gu,u\rangle|\ \forall\ u\in L^2(\mathbb{R}^n),$$

where $G\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Since G is an isotropic smoothing operator, for any $\delta>0$ there is a finite dimensional subspace $W\subset\mathcal{S}(\mathbb{R}^n)$ such that

$$\|\langle Gu,u\rangle\|\le\delta\|u\|^2\ \forall\ u\in W^\perp. \tag{4.37}$$

Thus if we replace A by $A(\mathrm{Id}-\Pi_W)=A+E$ where E is a (finite rank) smoothing operator we see that

$$\|(A+E)u\|^2\le(C+\delta)\|Gu\|^2\ \forall\ u\in L^2(\mathbb{R}^n)\Longrightarrow\|(A+E)\|\le(C+\delta)^{\frac12}.$$

This proves half of the desired estimate (4.34), namely

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)}\|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n)}\le\sup_{p\in\mathbb{S}^{2n-1}}|a(p)|. \tag{4.38}$$

To prove the opposite inequality, leading to (4.32), it is enough to arrive at a contradiction by supposing to the contrary that there is some $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ satisfying the strict inequality

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R}^n))} < \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|.$$

From this it follows that we may choose $c > 0$ such that $c = |a(p)|^2$ for some $p \in \mathbb{S}^{2n-1}$ and yet $A' = A^*A-c$ has a bounded inverse, B. By making an arbitrariy small perturbation of the full symbol of A' we may assume that it vanishes identically near p. By (4.38) we may choose $G \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with arbitrariy small L^2 such that $\tilde{A} = A' + B$ has left symbol rapidly vanishing near p. When the norm of the perturbation is small enough, $\tilde{A}$ will still be invertible, with inverse $\tilde{B} \in \mathcal{B}(L^2(\mathbb{R}^n))$. Now choose an element $G \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with left symbol supported sufficiently near p, so that $G \circ \tilde{A} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ but yet the principal symbol of G should not vanish at p. Thus

$$G = G \circ \tilde{A} \circ \tilde{B} : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),$$
$$G* = G = \tilde{B}^* \circ \tilde{A}^* \circ G^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

It follows that $G^*G : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is an isotropic smoothing operator. This is the expected contradiction, since G, and hence G^*G, may be chosen to have non-vanishing principal symbol at p. Thus we have proved (4.38) and hence the Proposition. □

It is then easy to characterize the compact operators amongst the polyhomogeneous isotropic operators as those of negative.

Lemma 4.4. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then, as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$, A is compact if and only if it has negative order.*

Proof. The necessity of the vanishing of the principal symbol for compactness follows from Proposition 4.6 and the sufficiency follows from the density of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ in $\Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ in the topology of $\Psi^{-\frac{1}{2}}_{\infty-\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ and hence in the topology of bounded operators. Thus, such an operator is the norm limit of compact operators so itself is compact. □

Also as a consequence of Proposition 4.6 we can see the necessity of the assumption of ellipticity in Proposition 4.3.

Corollary 4.3. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then A is Fredholm as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ if and only if it is elliptic.*

4.5. Sobolev spaces

The space of square-integrable functions plays a basic rôle in the theory of distributions; one reason for this is that it is associated with the embedding of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$. We know that pseudodifferential operators of order 0 are bounded on $L^2(\mathbb{R}^n)$. There is also a natural collection of Sobolev spaces associated to the isotropic calculus. The isotropic Sobolev space of order m may be defined as the collection of distributions mapped into $L^2(\mathbb{R}^n)$ by any one elliptic operator of order $-m$.

Note that a differential operator $P(x, D_x)$ on $\mathbb{R}^n$ is an isotropic pseudodifferential operator if and only if its coefficients are polynomials. The fundamental symmetry between coefficients and differentiation suggest that the isotropic Sobolev spaces of non-negative integral order be defined by

$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n) \text{ if } |\alpha| + |\beta| \le k\},\ k \in \mathbb{N}. \tag{4.39}$$

The norms

$$\|u\|^2_{k,\mathrm{iso}} = \sum_{|\alpha|+|\beta|\le k} \int_{\mathbb{R}^n} |x^\alpha D^\beta_x u|^2 dx \tag{4.40}$$

turn these into Hilbert spaces. For negative integral orders we identify the isotropic Sobolev spaces with the duals of these spaces

$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = (H^{-k}_{\mathrm{iso}}(\mathbb{R}^n))' \hookrightarrow \mathcal{S}'(\mathbb{R}^n),\ k \in -\mathbb{N}. \tag{4.41}$$

The (continuous) injection into tempered distributions here arises from the density of the image of the inclusion $\mathcal{S}(\mathbb{R}^n) \longrightarrow H^k_{\mathrm{iso}}(\mathbb{R}^n)$.

Lemma 4.5. *For any $k \in \mathbb{Z}$,*

$$\begin{aligned} H^k_{\mathrm{iso}}(\mathbb{R}^n) &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-k}_{\mathrm{iso}}\right\} \\ &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); \exists\ A \in \Psi^{-k}_{\mathrm{iso}} \textit{ elliptic and such that } Au \in L^2(\mathbb{R}^n)\right\} \end{aligned} \tag{4.42}$$

and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\mathrm{iso}}(\mathbb{R}^n)$ is dense for each $k \in \mathbb{Z}$.

Proof. [1] For $k \in \mathbb{N}$, the functions $x^\alpha \xi^\beta$ for $|\alpha| + |\beta| = k$ are 'collectively elliptic' in the sense that

$$q_k(x, \xi) = \sum_{|\alpha|+|\beta|=k} (x^\alpha \xi^\beta)^2 \ge c(|x|^2 + |\xi|^2)^k,\ c > 0. \tag{4.43}$$

Thus $Q_k = \sum\limits_{|\alpha|+|\beta|\le k} (D^\beta x^\alpha x^\alpha D^\beta) \in \Psi^{2k}_{\mathrm{iso}}(\mathbb{R}^n)$, which has principal reduced symbol q_k, has a left parameterix $A_k \in \Psi^{-2k}_{\mathrm{iso}}(\mathbb{R}^n)$. This gives the identity

$$\sum_{|\alpha|+|\beta|\le k} R_{\alpha,\beta} x^\alpha D^\beta = A_k Q_k = \mathrm{Id} + E, \text{ where} \tag{4.44}$$

$$R_{\alpha,\beta} = A_k D^\beta x^\alpha \in \Psi^{-2k+|\alpha|+|\beta|}_{\mathrm{iso}}(\mathbb{R}^n),\ E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Thus if $A \in \Psi^k_{\mathrm{iso}}(\mathbb{R}^n)$

$$Au = -AEu + \sum_{|\alpha|+|\beta|\le k} AR_{\alpha,\beta} x^\alpha D^\beta u.$$

If $u \in H^k_{\mathrm{iso}}(\mathbb{R}^n)$ then by definition $x^\alpha D^\beta u \in L^2(\mathbb{R}^n)$. By the boundedness of operators of order 0 on L^2, all terms on the right are in $L^2(\mathbb{R}^n)$ and we have shown the inclusion of $H^k_{\mathrm{iso}}(\mathbb{R}^n)$ in the first space space on the right in (4.42). The converse is immediate, so this proves the first equality in (4.42) for $k > 0$. Certainly the third space in (4.42) contains in the second. The existence of an elliptic parametrix B for the ellipic operator A proves the converse since any isotropic pseudodifferential operator of order A' of order k can be effectively factorized as

$$A' = A'(BA + E) = B'A + E',\ B' \in \Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n),\ E' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Thus, $Au \in L^2(\mathbb{R}^n)$ implies that $A'u \in L^2(\mathbb{R}^n)$.

[1] This is an essentially microlocal proof.

It also follows from second identification that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k_{\text{loc}}(\mathbb{R}^n)$. Thus, if $Au \in L^2(\mathbb{R}^n)$ and we choose $f_n \in \mathcal{S}(\mathbb{R}^n)$ with $f_n \to Au$ in $L^2(\mathbb{R}^n)$ then, with B a parametrix for A, $u_n' = Bf_n \to BAu = u + Eu$. Thus $u_n = u_n' - Eu \in \mathcal{S}(\mathbb{R}^n) \to u$ in $L^2(\mathbb{R}^n)$ and $Au_n \to u$ in $L^2(\mathbb{R}^n)$ proving the density.

The Riesz representation theorem shows that $v \in \mathcal{S}'(\mathbb{R}^n)$ is in the dual space, $H^{-k}_{\text{iso}}(\mathbb{R}^n)$, if and only if there exists $v' \in H^k_{\text{iso}}(\mathbb{R}^n)$ such that

$$(4.45) \quad v(u) = \langle u, v' \rangle_{k,\text{iso}} = \langle u, Q_{2k} v' \rangle_{L^2}, \ \forall \ u \in \mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$$
$$\text{with } Q_{2k} = \sum_{|\alpha|+|\beta| \le k} D^\beta x^{2\alpha} D^\beta.$$

This shows that Q_{2k} is an isomorphism of $H^k_{\text{iso}}(\mathbb{R}^n)$ onto $H^{-k}_{\text{iso}}(\mathbb{R}^n)$ as subspaces of $\mathcal{S}'(\mathbb{R}^n)$. Notice that $Q_{2k} \in \Psi^{2k}_{\text{iso}}(\mathbb{R}^n)$ is elliptic, self-adjoint and invertible, since it is strictly positive. This now gives the same identification (4.42) for $k < 0$.

The case $k = 0$ follows directly from the L^2 boundedness of operators of order 0 so the proof is complete. □

In view of this identification we define the isotropic Sobolev spaces or any real order the same way

$$(4.46) \qquad H^s_{\text{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n) \ \forall \ A \in \Psi^{-s}_{\text{iso}}\right\}, \ s \in \mathbb{R}.$$

These are Hilbertable spaces, with the Hilbert norm being given by $\|Au\|_{L^2(\mathbb{R}^n}$ for any $A \in \Psi^s_{\text{iso}}(\mathbb{R}^n)$ which is elliptic and invertible.

Proposition 4.7. *Any element $A \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$, $m \in \mathbb{R}$, defines a bounded linear operator*

$$(4.47) \qquad A : H^s_{\text{iso}}(\mathbb{R}^n) \longrightarrow H^{s-m}_{\text{iso}}(\mathbb{R}^n), \ \forall \ s \in \mathbb{R}.$$

This operator is Fredholm if and only if A is elliptic. For any $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^s_{\text{iso}}(\mathbb{R}^n)$ is dense and $H^{-s}_{\text{iso}}(\mathbb{R}^n)$ may be identified as the dual of $H^s_{\text{iso}}(\mathbb{R}^n)$ with respect to the continuous extension of the L^2 pairing.

Proof. A straightforward application of the calculus, with the exception of the necessity of ellipticity for an isotropic pseudodifferential operator to be Fredholm. This is discussed in the problems beginning at Problem 4.10. □

4.6. Representations

In §1.9 the compactification of Euclidean space to a ball, or half-sphere, is described. We make the following definition, recalling that $\rho \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$ is a boundary defining function.

Definition 4.2. *The space of of 'Laurent functions' on the half-sphere is*

$$(4.48) \quad \mathcal{L}(\mathbb{S}^{n,+}) = \bigcup_{k \in \mathbb{N}_0} \rho^{-k} \mathcal{C}^\infty(\mathbb{S}^{n,+}),$$
$$\rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}) = \{u \in \mathcal{C}^\infty(\text{int}(\mathbb{S}^{n,+})); \rho^k u \in \mathcal{C}^\infty(\mathbb{S}^{n,+}).$$

More generally if $m \in \mathbb{R}$ we denote by $\rho^m \mathcal{C}^\infty(\mathbb{S}^{n,+})$ the space of functions which can be written as products $u = \rho^m v$, with $v \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$; again it can be identified with a subspace of the space of $\mathcal{C}^\infty$ functions on the open half-sphere.

PROPOSITION 4.8. *The compactification map* (1.96) *extends from* (1.98) *to give, for each* $m \in \mathbb{R}$, *an identification of* $\rho^{-m}\mathcal{C}^\infty(\mathbb{S}^{n,+})$ *and* $S^m_{\text{cl}}(\mathbb{R}^n)$.

Thus, the fact that the $\Psi^{\mathbb{Z}}_{\text{iso}}(\mathbb{R}^n)$ form an order-filtered $*$-algebra means that $\rho^{\mathbb{Z}}\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ has a non-commutative product defined on it, with $\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ a sub-algebra, using the left symbol isomorphism, followed by compactification.

4.7. Symplectic invariance of the isotropic product

The composition law for the isotropic calculus, and in particular for it smoothing part, is derived from its identification as a subalgebra of the (weighted) spaces of pseudodifferential operator on $\mathbb{R}^n$. There is a much more invariant formulation of the product which puts into evidence more of the invariance properties.

Let W be a real symplectic vector space. Thus, W is a vector space equipped with a real, antisymmetic and non-degenerate bilinear form

$$\begin{aligned}(4.49)\quad \omega : W \times W \longrightarrow \mathbb{R},\ \omega(w_1, w_2) + \omega(w_2, w_1) = 0\ \forall\ w_1, w_2 \in W,\\ \omega(w_1, w) = 0\ \forall\ w \in W \Longrightarrow w_1 = 0.\end{aligned}$$

A Lagrangian subspace of W is a vector space $V \subset W$ such that ω vanishes when restricted to V and such that $2\dim V = \dim W$.

LEMMA 4.6. *Every symplectic vector space has a Lagrangian subspace and for any choice of Lagrangian subspace* U_1 *there is a second Lagrangian subspace* U_2 *such that* $W = U_1 \oplus U_2$ *is a* Lagrangian decomposition.

PROOF. First we show that there is a Lagrangian subspace. If $\dim W > 0$ then the antisymmetry of ω shows that any 1-dimensional vector subspace is *isotropic*, that is ω vanishes when restricted to it. Let V be a maximal isotropic subspace, that is an isotropic subspace of maximal dimension amongst isotropic subspaces. Let U be a complement to V in W. Then

$$(4.50)\qquad \omega : V \times U \longrightarrow \mathbb{R}$$

is a non-degenerate pairing. Indeed $u \in U$ and $\omega(v, u) = 0$ for all $v \in V$ then $V + \mathbb{R}\{u\}$ is also isotropic, so $u = 0$ by the assumed maximality. Similarly if $v \in V$ and $\omega(v, u) = 0$ for all $u \in U$ then, recalling that ω vanishes on V, $\omega(v, w) = 0$ for all $w \in W$ so $v = 0$. The pairing (4.50) therefore identifies U with V', the dual of V. In particular $\dim w = 2\dim V$.

Now, choose any Lagrangian subspace U_1. We proceed to show that there is a complementary Lagrangian subspace. Certainly there is a 1-dimensional subspace which does not meet U_1. Let V be an isotropic subspace which does not meet U_1 and is of maximal dimension amongst such subspaces. Suppose that $\dim V < \dim U_1$. Choose $w \in W$ with $w \notin V \oplus U_1$. Then $V \ni v \longrightarrow \omega(w, v)$ is a linear functional on U_1. Since U_1 can be completed to a complement, any such linear functional can be written $\omega(u_1, v)$ for some $u_1 \in U_1$. It follows that $\omega(w - u_1, v) = 0$ for all $v \in V$. Thus $V \oplus \mathbb{R}\{w - u_1\}$ a non-trivial isotropic extension of V, contradicting the assumed maximality. Thus $V = U_2$ is a complement of U_1. □

Given such a Lagrangian decomposition of the symplectic vector space W, let $X_1, \dots X_n$ be a basis for the dual of U_1, and let $\Xi_1, \dots, \Xi_n$ be the dual basis, of U_1 itself. The pairing (4.50) with $U = U_1$ and $V = U_2$ identifies $U_2 = U_1'$ so the Ξ_i

can also be regarded as a basis of the dual of U_2. Thus $X_1 \dots X_n, \Xi_1, \dots, \Xi_n$ gives a basis of $W' = U_1' \oplus U_2'$. The symplectic form can then be written

$$\omega(w_1, w_2) = \sum_{i=1}^{n} (\Xi_i(w_1) X_i(w_2) - \Xi_i(w_2) X_i(w_1)). \tag{4.51}$$

This is the *Darboux* form of ω. If the X_i, Ξ_i are thought of as linear functions x_i, ξ_i on W now considered as a manifold then these are *Darboux coordinates* in which(4.51) becomes

$$\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{4.52}$$

The symplectic form ω defines a volume form on W, namely the n-fold wedge product ω^n. In Darboux coordinates this is just, up to sign, the Lebesgue form $d\xi dx$.

Proposition 4.9. *On any symplectic vector space, W, the bilinear map on $\mathcal{S}(W)$,*

$$a\#b(w) = (2\pi)^{-2n} \int_{W^2} e^{i\omega(w_1,w_2)} a(w + w_1) b(w + w_2) \omega^n(w_1)\omega^2(w_2), \ \dim W = 2n \tag{4.53}$$

defines an associative product isomorphic to the composition of $\Psi^{-\infty}_{\mathrm{iso}}(U_1)$ for any Lagrangian decomposition $W = U_1 \oplus U_2$.

Corollary 4.4. *Extended by continuity in the symbol space* (4.53) *defines a filtered product on $S^\infty(W)$ which is isomorphic to the isotropic algebra on $\mathbb{R}^{2n}$ and is invariant under symplectic linear transformation of W.*

Proof. Written in the form (4.53) the symplectic invariance is immediate. That is, if F is a linear transformation of W which preserves the symplectic form, $\omega(Fw_1, Fw_2) = \omega(w_1, w_2)$ then

$$F^*(a\#b) = (F^*a)\#(F^*b) \ \forall \ a, b \in \mathcal{S}(W). \tag{4.54}$$

The same result holds for general symbols once the continuity is established.

Let us start from the Weyl quatization of the isotropic algebra. As usual for computations we may assume that the amplitudes are of order $-\infty$. Thus, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ may be written

$$Au(x) = \int A(x, y)u(y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(\frac{1}{2}(x + y), \xi)u(y)dyd\xi. \tag{4.55}$$

Both the kernel $A(x, y)$ and the amplitude $a(x, \xi)$ are elements of $\mathcal{S}(\mathbb{R}^{2n})$. The relationship (4.55) and its inverse may be written

$$\begin{aligned} A(s + \frac{t}{2}, s - \frac{t}{2}) &= (2\pi)^{-n} \int e^{it\cdot\xi} a(s, \xi)d\xi, \\ a(x, \xi) &= \int e^{-it\cdot\xi} A(x + \frac{t}{2}, x - \frac{t}{2})dt. \end{aligned} \tag{4.56}$$

If A has Weyl symbol a and B has Weyl symbol b let c be the Weyl symbol of the composite $A \circ B$. Using (4.56) and (4.55)

$$c(s,\zeta) = \int e^{-it\cdot\zeta} A(s+\frac{t}{2}, z)B(z, s-\frac{t}{2})dt$$

$$= (2\pi)^{-2n} \int\int\int dtdzd\xi d\eta e^{i\Phi} a(\frac{s}{2}+\frac{t}{4}+\frac{z}{2},\xi)a(\frac{z}{2}+\frac{s}{2}-\frac{t}{4},\eta)$$

$$\text{where } \Phi = -t\cdot\zeta + (s+\frac{t}{2}-z)\cdot\xi + (z-s+\frac{t}{2})\cdot\eta.$$

Changing variables of integration to $X = \frac{z}{2}+\frac{t}{4}-\frac{s}{2}$, $Y = \frac{z}{2}-\frac{t}{4}-\frac{s}{2}$, $\Xi = \xi - \zeta$ and $H = \eta - \zeta$ this becomes

$$c(s,\zeta) = (2\pi)^{-2n}4^n \int\int\int dY dX d\Xi dH \\ e^{2i(X\cdot H - Y\cdot\Xi)} a(X+s, \Xi+\zeta)a(Y+s, H+\zeta).$$

This reduces to (4.53), written out in Darboux coordinates, after the change of variable $H' = 2H$, $\Xi' = 2\Xi$ and $\zeta' = 2\zeta$. Thus the precise isomorphism with the product in Weyl form is given by

$$A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\omega(\frac{1}{2}(x+y), 2\xi)u(y)dyd\xi \tag{4.57}$$

so that composition of kernels reduces to (4.53). □

4.8. Metaplectic group

The discussion of the metaplectic group in this section might, or might not, be relevant for later material. For the moment you can freely ignore it, but it is amusing enough. The operators constructed here are 'Fourier integral operators' in the isotropic sense – but by no means all such Fourier integral operator. In particular they correspond to linear symplectic transformations of the underlying space, rather than more general homogeneous symplectic diffeomorphisms.

As we shall see below the discussion of the metaplectic group reduces to the computation of some constants, these are bound up with the standard formula for the Fourier transform of 'Gaussians'. Namely, if $z \in \mathbb{C}$ has positive real part then

$$\mathcal{F}(\exp(-zx^2)) = \frac{\sqrt{\pi}}{\sqrt{z}} \exp(-\frac{1}{4z}\xi^2) \tag{4.58}$$

where the square-root is the standard branch, having positive real part for z in this half-plane. One can carry out the integrals directly. In fact both sides are holomorphic in $\Re z > 0$ so it suffices to check the formula on the positive real axis in z where it is easy.

Now, recall that the symplectic group on $\mathbb{R}^{2n}$, denoted $\mathrm{Sp}(2n)$, is the group of linear transformations preserving a given non-degenerate antisymmetric bilinear form. We will take the standard (well, standard up to sign and maybe constants) Darboux form

$$\omega_D((x,\xi),(x',\xi')) = \xi'\cdot x - \xi\cdot x'. \tag{4.59}$$

Recall that this is not a restriction in the sense that

LEMMA 4.7 (Linear Darboux Theorem). *If ω is a non-degenerate antisymetric real bilinear form on a (necessarily even-dimensional) real vector space then there is a linear isomorphism to $\mathbb{R}^{2n}$ reducing ω to the Darboux form* (4.59).

BRIEF PROOF. Construct a basis by induction. First choose a non-zero element e_1 and then a second element e_2 such that $\omega(e_1, e_2) = 1$, which is possible by the assumed non-degeneracy. Then look at the subspace spanned by those vector satisfying $\omega(e_1, f) = \omega(e_2, f) = 0$. This is complementary to the span of e_1, e_2 and ω is the direct sum of ω_D for $n = 1$ on the span of e_1, e_2 and ω on this complement. After a finite number of steps one arrives at (4.59) with the x's corresponding to the odd basis elements and the ξ's to the even ones. □

We will need properties of the symplectic group below, but I will just work them out as the need arises.

Let me define a group of operators on $L^2(\mathbb{R}^n)$ which also map $\mathcal{S}(\mathbb{R}^n)$ to itself, by the crude method of taking products of some obvious invertible operators. The basic list is:-

(F.1) Multiplication by constants.
(F.2) Multiplication by functions $e^{iq(x)}$ where q is a real quadratic form,
(F.3) The Fourier transforms in each variable

(4.60)
$$\mathcal{F}_j u(x', \tau, x'') = \int e^{-t\tau} u(x', t, x'') dt,\ x' = (x_1, \dots, x_j),\ x'' = (x_{j+1}, \dots, x_n).$$

(F.4) Pull-back under any linear isomorphism

$$T^* u(x) = u(Tx),\ T \in \mathrm{GL}(n, \mathbb{R}). \tag{4.61}$$

Obviously the multiples of the identity in (F.1) commute with the other operators. Moreover

$$e^{iq(x)} T^* = T^* e^{iq'(x)},\ q'(x) = q((T^t)^{-1} x) \tag{4.62}$$

so (F.2) and (F.4) may be interchanged.

In fact it is convenient to reorganize the products of these elements. Observe that conjugation by the Fourier transform (in all variables) of an operator (F.2)

$$\mathcal{F}^{-1} e^{iq(\cdot)} \mathcal{F} u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} e^{iq(\xi)} u(u) dy d\xi. \tag{4.63}$$

gives a convolution operator which we can, and will, denote $e^{iq(D)}$. Then the operators in this list which are 'close to the identity' are

(S.1) Multiplication by constants near 1
(S.2) Multiplication by functions $e^{iq(x)}$ where q is a small real quadratic form,
(S.3) Application of $e^{iq(D)}$ where q is a small real quadratic form and
(S.4) Pull-back under any linear isomorphism close to the identity.

For definiteness sake:-

DEFINITION 4.3. *Let $\mathcal{M}(2n)$ denote the space of operators on $\mathcal{S}(\mathbb{R}^n)$ which are finite products of elements of the form*

$$M = c e^{iq_2} L_2^* \mathcal{F}_I e^{iq_1} \mathcal{F}_I L_1^* \tag{4.64}$$

where $\mathcal{F}_I$ denotes the product of the Fourier transforms in the variables corresponding to $i \in I$ for some subset $I \subset \{1, \dots, n\}$.

As we shall see, these operators form a Lie group; it contains but is not equal to the metaplectic group. The products of elements in (S.1) – (S.4) give a neighbourhood of the identity in this group.

First we need to see how these operators are related to the symplectic group.

LEMMA 4.8. *If M is of the form* (4.64) *then*

$$\begin{aligned} M(x_j u) &= (\sum_i A_{kj} x_k + \sum_k B_{kj} D_k) M u, \\ M(D_j u) &= (\sum_i i C_{kj} x_k + \sum_k D_{kj} D_k) M u \end{aligned} \tag{4.65}$$

where A, B, C and D are real $n \times n$ matrices and

$$S(M) = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathrm{Sp}(2n). \tag{4.66}$$

Furthermore all symplectic matrices arise this way and all symplectic matrices close to the identity arise from products operators in (S.1) – (S.4) (one of each of type).

PROOF. To prove (4.66) we will check that it holds for each of the factors in (4.64). Then from (4.65)

$$S(M_1 M_2) = S(M_1) S(M_2), \tag{4.67}$$

i.e. this will be a group homomorphism.

For (F.1), (4.65) and (4.66) are obvious, with the matrix being the identity. For M as in (F.2), $A = \mathrm{Id}$, $B = 0$, $D = \mathrm{Id}$ and $Cx = -q'(x)$ is given by the derivative of q and

$$S(M) = \begin{pmatrix} \mathrm{Id} & C \\ 0 & \mathrm{Id} \end{pmatrix} \in \mathrm{Sp}(2n) \text{ for any symmetric } C. \tag{4.68}$$

The matrix for $\mathcal{F}_l$ is the identity outside the 2×2 block corresponding to x_l and D_l where it is just

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \tag{4.69}$$

which is certainly symplectic. Finally the matrix for L^* is just

$$\begin{pmatrix} L & 0 \\ 0 & (L^{-1})^t \end{pmatrix}. \tag{4.70}$$

So this gives a group homomorphism, we proceed to check the surjectivity of this map to $\mathrm{Sp}(2n)$. By the conjugation result (4.62) an operator of the form (4.64) remains so under conguation by some L^*. This in particular conjugates the upper left block A in S to $L^{-1}AL$. The rank of the matrix is a complete invariant under conjugation by $\mathrm{GL}(n, \mathbb{R})$ so we may arrange that

$$A = \pi_k \text{ projection onto the first } k \text{ components} \tag{4.71}$$

without affecting the overall problem. The symplectic condition then implies that

$$S = \begin{pmatrix} \pi_k & C' \\ B' & D' \end{pmatrix}, \ \pi_k B'(\mathrm{Id} - \pi_k) = 0 \tag{4.72}$$

and where $(\operatorname{Id}-\pi_k)B'(\operatorname{Id}-\pi_k)$ must be an isomorphism on the range of $\operatorname{Id}-\pi_k$ to preserve invertibility. Thus a further choice of L_1, not affecting the special form of A allows us to arrange that

$$S = \begin{pmatrix} \pi_k & C' \\ -(\operatorname{Id}-\pi_k) + B'' & D' \end{pmatrix},\ B''\pi_k = B''. \tag{4.73}$$

Again from the symplectic condition it follows that B'' is symmetric. Now choosing I to be the first k elements arranges that $A = \operatorname{Id}$. For the new matrix, B must be symmetric. $\square$

To proceed further consider the operators of the form (4.64) for which $S(M) = \operatorname{Id}$.

Proposition 4.10. *The space of operators $\mathcal{M}$ defined by* (4.64) *is a group with a multiplicative short exact sequence*

$$\mathbb{C}^* \longrightarrow \mathcal{M} \longrightarrow \operatorname{Sp}(2n). \tag{4.74}$$

Proof. Consider the elements of $\mathcal{M}$ such that $S(M) = \operatorname{Id}$. By definition of $S(M)$ these have the property that they commute with x_k and D_j for all j. Recalling the proof of the invertibility of the Fourier transform, this shows that the kernel of M satisfies the differential equations

$$(x_j - y_j)M(x,y) = 0,\ (D_{x_j} + D_{y_j})M(x,y) = 0 \Longrightarrow M(x,y) = c\delta(x,y) \tag{4.75}$$

for some constant c. Thus

$$\ker(S : \mathcal{M} \longrightarrow \operatorname{Sp}(2n)) = \mathbb{C}^* \operatorname{Id}. \tag{4.76}$$

Now Lemma 4.8 combined with this argument shows that $\mathcal{M}$ actually consists of the operators of the form (4.64), without having to take further products. Indeed, given a finite product $M_1M_2 \dots M_p$, of elements of $\mathcal{M}$ we can use Lemma 4.8 to find a single element $M \in \mathcal{M}$ such that $S(M) = S(M_1)\dots S(M_p)$. Composing on the right with M gives a product $M^{-1}M_1 \dots M_p$ which commutes with x_j and D_j as above, so is a multiple of the identity, which proves that $M_1 \dots M_p$ is of the form (4.64). The inverse of an element of $\mathcal{M}(2n)$, as an operator is not quite of the same form directly, but the same argument applies. $\square$

Thus $\mathcal{M}$ is two real dimensions larger than $\operatorname{Sp}(2n)$. Notice that all the elements in the products (4.64) are unitary up to positive constant multiples – and all multiples occur. So we can kill one dimension by looking at the unitary elements

$$\mathbb{S} \longrightarrow (\mathcal{M}(2n) \cap \operatorname{U}(L^2(\mathbb{R}^n)) \longrightarrow \operatorname{Sp}(2n) \text{ is exact.} \tag{4.77}$$

In fact we can do more than this, namely we can define in a reasonably natural way a lift of a neighbourhood of the identity in $\operatorname{Sp}(2n)$ into $\mathcal{M} \cap \operatorname{U}(L^2(\mathbb{R}^n))$. If $S \in \operatorname{Sp}(2n)$ is close to the identity then it has a 'generating function'. Namely if we write

$$S(x,\xi) = (x',\xi') \text{ then } \frac{\partial \xi'(x,\xi)}{\partial \xi} \text{ is invertible} \tag{4.78}$$

since it is close to the identity. So, the corresponding linear map is invertible, and x and ξ' may be introduced as linear coordinates on the graph

$$\xi = \Xi(x,\xi'),\ x' = X'(x,\xi') \text{ on the graph of } S. \tag{4.79}$$

Now the symplectic condition can be rewritten as

$$-d\xi' \wedge dx' + d\xi \wedge dx = d(\Xi \cdot dx + X' \cdot d\xi') = 0 \Longrightarrow \Xi \cdot dx + X' \cdot d\xi = d\Phi(x, \xi') \tag{4.80}$$

where Φ is a quadratic form (since there is no 1-dimensional cohomolgy in $\mathbb{R}^{2n}$ such a smooth function exists but by homogeneity we may replace it by its quadratic part at the origin) such that

$$\Xi(x, \xi') = \frac{\partial \Phi(x, \xi'}{\partial x}, \ X'(x, \xi') = \frac{\partial \Phi(x, \xi'}{\partial \xi'} \text{ defines } S'. \tag{4.81}$$

So now we 'lift' S to the element

$$M(S)u(x) = c(S) \int e^{i\Phi(x,\eta)} \hat{u}(\eta) d\eta \tag{4.82}$$

defined by the construction of the generating function above.

Proposition 4.11. *For S in a small neighbourhood of the identity in* $\mathrm{Sp}(2n)$ *here is a unique choice of $c(S) > 0$ in* (4.82) *such that $M \in \mathcal{M} \cap \mathrm{U}(L^2(\mathbb{R}^n)$ and this choice is smooth in S, the subgroup of* $\mathrm{Mp}(2n) \subset \mathcal{M}(2n)$ *generated by the finite products of these elements is a Lie group giving a 2-fold cover*

$$\mathbb{Z}_2 \longrightarrow \mathrm{Mp}(2n) \longrightarrow \mathrm{Sp}(2n). \tag{4.83}$$

This is either the metaplectic group or else is a faithful representation of it, depending on your attitude; I will call it the metaplectic group!

Proof. For S close to the identity the discussion above shows that Φ is close to $x \cdot \eta$ as a quadratic form, meaning that

$$\Phi_S(x, \eta) = q_2(x) + Lx \cdot \eta + q_1(\eta), \ L \in \mathrm{GL}(n, \mathbb{R}). \tag{4.84}$$

In fact L is close to the identity. The definition of $M(S)$ in (4.82) can therefore be rewritten

$$M(S) = c(S) e^{iq_2} L^* \mathcal{F}^{-1} e^{iq_1} \mathcal{F}. \tag{4.85}$$

The desired unitarity then fixes $c(S) > 0$ and in fact

$$c(S) = \sqrt{|\det T|} \tag{4.86}$$

and it follows that $M(S)$ depends smoothly on $S \in \mathrm{Sp}(2n)$, near Id.

The next important thing to check is that this lift is multiplicative near the identity, i.e. gives a local Lie group. From the discussion above we know that

$$M(S_1)M(S_2) = e^{i\theta} M(S_1 S_2), \ S_1, \ S_2 \in \mathrm{Sp}(2n) \text{ near } \mathrm{Id} \tag{4.87}$$

up to the possibility of factor of absolute value 1 – we proceed to show that there is no such factor locally, although as we shall see there is one globally.

Lemma 4.9. *If q_i, $i = 1, 2$, are real quadratic forms which are sufficiently small and $L \in \mathrm{GL}(n, \mathbb{R})$ is sufficiently close to the identity then there exist unique small quadratic forms q_i', $i = 1, 2$ and $L' \in \mathrm{GL}(n, \mathbb{R})$ close to the identity such that*

$$e^{iq_1(x)} L e^{iq_2(D)} = \delta' e^{iq_2'(D)} e^{iq_1'(x)} L', \ \delta' > 0. \tag{4.88}$$

PROOF. We can move the linear transformations to the left, so it suffices to show the existence of q_i' and L' such that

$$M = e^{iq_1(x)}e^{iq_2(D)} = \delta e^{iq_2'(D)}e^{iq_1'(x)}L',\ \delta > 0 \tag{4.89}$$

under the same hypotheses. By making an overal orthogonal transformation, we may suppose that q_2 is a non-degenerate quadratic form in the duals of the first variables in a splitting $x = (x', x'')$ and is trivial in the second variables. Since e^{iq_2} is a product of terms in each of the variables, it suffices (by renumbering the coordinates) to consider the case that $q_2 = \xi_1^2$. Then, after another orthogonal transformation close to the identity, we may suppose that $q_1 = ax^2 + bxy$ where a, b and c are all small and we are reduced to two variables which we denote x and y with $q = c\xi^2$. Now we will show directly that

$$e^{ibxy+cb^2y^2}e^{icD_x^2} = e^{icD_x^2}e^{ibxy}T^*,\ T^*x = x - 2cby,\ T^*y = y \tag{4.90}$$

where, up to a constant of absolute value one, this comes from the computation of the corresponding symplectic transformations. To see (4.90) insert the Fourier transform on the right and change variables

$$\begin{aligned}
&(e^{icD_x^2}e^{ibxy}T^*)u(x,y)\\
&= (2\pi)^{-1}\int e^{ic\xi^2+i(x-x')\xi}e^{ibx'y}u(x'-2cby,y)dx'd\xi\\
&= e^{ibxy}(2\pi)^{-1}\int e^{ic\xi^2+i(x-x''-2cby)\xi+ib(x''+2cby)y)}u(x'',y)dx''d\xi\\
&= e^{ibxy+cb^2y^2}(2\pi)^{-1}\int e^{ic(\xi-by)^2+i(x-x'')(\xi-by)}u(x'',y)dx''d\xi\\
&= e^{ibxy+cb^2y^2}(2\pi)^{-1}\int e^{ic(\xi')^2+i(x-x'')\xi'}u(x'',y)dx''d\xi'\\
&= e^{ibxy+cb^2y}e^{icD_x^2}u.
\end{aligned} \tag{4.91}$$

where $x'' = x' - 2cby$ and $\xi' = \xi - by$. Whilst these are really oscillatory integrals, the formal manipulation is easily justified by regularization, as usual.

Since (4.90) can be rewritten

$$e^{ibxy+cb^2y^2}e^{icD_x^2} = e^{ibxy}e^{icD_x^2}e^{cb^2y^2} = e^{icD_x^2}e^{ibxy}T^* \Longrightarrow e^{ibxy}e^{icD_x^2} = e^{icD_x^2}e^{ibxy-icb^2y^2}T^* \tag{4.92}$$

we are reduced to the case $q_1 = ax^2$, $q_2 = c\xi^2$ which is purely one-dimensional. By a similar computation it can be checked that

$$\begin{gathered}
e^{iax^2}e^{icD_x^2} = DT^*e^{ic'D^2}e^{ia'x^2}\\
\text{if } a' = \frac{a}{1-4ac},\ c' = c(1-4ac),\ T^*x = (1-4ac)x,\ D = 1-4ac
\end{gathered} \tag{4.93}$$

where again the basic formula comes from comparing the symplectic transformations, namely under the operator on the left

$$x \longmapsto (1-4ac)x + 2xD_x,\ D_x \longmapsto D_x - 2ax \tag{4.94}$$

and on the right, before the application of T^*,

$$x \longmapsto x + 2cD_x,\ D_x \longmapsto (1-4a'c')D_x - 2a'x. \tag{4.95}$$

Comparing these leads to (4.93). Thus we know that the sides are equal up to a multiplicative constant and this can be computed by applying the operators to one non-trivial function. For example applying the operators to e^{-x^2}, and using (4.58), the right side gives

$$
\begin{aligned}
(4.96)\qquad \frac{D}{\sqrt{A}\sqrt{1-ia'}}&\exp(-(\frac{1}{4A}-ia)(1-4ac)^2x^2)\\
&=\frac{2D}{\sqrt{1-4ac}\sqrt{1-4ic}}\exp(-Bx^2),\\
&\qquad\text{since } A=\frac{1}{4(1-ia')}-ic'=(1-4ac)\frac{1-4ic}{4(1-ia')}
\end{aligned}
$$

and on the left

$$
\frac{2}{\sqrt{1-4ic}}\exp(-Bx^2),\ B=\frac{1-4ac-ia}{1-4ic} \tag{4.97}
$$

which gives the formula for D and shows most significantly that it is positive. $\square$

Returning to the proof of the Propostion, we have now checked that the lift is well-defined near the identity and defines a local group. In fact it follows from this discussion that all the operators of the form

$$
M=e^{iq_1(x)}Le^{iq_2(D)},\ \det L>0, \tag{4.98}
$$

where we no longer assume that the quadratic forms are small, are products of elements from a neighbourhood of the identity, and hence are in $\mathrm{Mp}(2n)$ and have a unique representation (4.98). Indeed, we can certainly connect such an element to the identity by connecting L to the identity by a curve $L_t\in\mathrm{GL}(n,\mathbb{R})$ and replacing q_1 and q_2 by tq_1 and tq_2. The corresponding element $M_t\in\mathrm{Mp}(2n)$ for small t and by continuity it follows that it is in $\mathrm{Mp}(2n)$ for all $t\in[0,1]$. Indeed, let T be the supremum of those t for which it remains in the group, and is therefore a finite product of elements in a fixed small neighbourhood of the identity for each $t<T$. Consider the image curve $S(M_t)$ in $\mathrm{Sp}(2n)$. For $0\le s<\epsilon$ for some $\epsilon>0$, $S(M_t)=R_sS(M_{t-\epsilon+s})$ where $[0,\epsilon]\ni s\longmapsto R_s$ is a curve starting at the identity in $\mathrm{Sp}(2n)$. Thus, from the discussion above, R_s has a unique lift N_s as in Lemma 4.9. The uniqueness of the representation shows that $M_{t-\epsilon+s}=N_sM_{t-\epsilon}$ for $s<\epsilon$ and since $M_{t-\epsilon}$ has a finite product representation, so does M_t for $t\le\epsilon$ and so this is true of M_T. Thus, $M_t\in\mathrm{Mp}(2n)$ and the unqueness follows from the earlier discussion.

In fact we can now check that the metaplectic group, defined by iterated composition of the elements near the identity just discussed, consists precisely of the unitary operators of the form

$$
\pm D\exp\left(i\frac{\pi}{2}(1-\operatorname{sgn}\det(T)-|I|)\right)e^{iq_1(x)}e^{iq_2(D)}T^*\mathcal{F}_I,\ D>0. \tag{4.99}
$$

Notice that if $I=0$, so no explicit partial Fourier transforms are present, then the complex factor is $\pm i$ if $\det T<0$ and ±1 if $\det T>0$ which shows that $\mathcal{M}$ is a double cover of $\mathrm{Sp}(2n)$. $\square$

THEOREM 4.3. *The metaplectic group of operators on $\mathcal{S}(\mathbb{R}^n)$ acts by conjugation on $\Psi^k_{\mathrm{iso}}(\mathbb{R}^n)$ and gives an action of $\mathrm{Sp}(2n)$ as a group of outer automorphisms of the algebra.*

4.9. Complex order

The identification of polyhomogeneous symbols of order zero on $\mathbb{R}^{2n}$ with the smooth functions on the radial compactification allows us to define the isotropic operators of a given complex order $z \in \mathbb{C}$. Namely, we use the left quantization map to identify

$$\Psi^z_{\mathrm{iso}}(\mathbb{R}^n) = \rho^{-z}\mathcal{C}^\infty(\mathbb{S}^{2n,1}) \subset \Psi^{\Re z}_{\infty-\mathrm{iso}}(\mathbb{R}^n). \tag{4.100}$$

Here, $\rho \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$ is a boundary defining function. Any other boundary defining function is of the form $a\rho$ with $0 < a \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$. It follows that the definition is independent of the choice of ρ since $a^z \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$ for any $z \in \mathbb{Z}$.

In fact it is even more useful to consider holomorphic families. Thus if $\Omega \subset \mathbb{C}$ is an open set and $h : \Omega \longrightarrow \mathbb{C}$ is holomorphic then we may consider holomorphic families of order h as elements of

$$\Psi^{h(z)}_{\mathrm{iso}}(\mathbb{R}^{2n}) = \big\{A : \Omega \longrightarrow \Psi^\infty_{\infty-\mathrm{iso}}(\mathbb{R}^{2n}); \\ \Omega \ni z \longmapsto \rho^{h(z)}A(z) \in \mathcal{C}^\infty(\mathbb{S}^{2n,1}) \text{ is holomorphic.}\big\} \tag{4.101}$$

Note that a map from $\Omega \subset \mathbb{C}$ into $\mathcal{C}^\infty(\mathbb{S}^{2n1,})$ is said to be holomorphic it is defines an element of $\mathcal{C}^\infty(\Omega \times \mathbb{S}^{2n,1})$ which satisies the Cauchy-Riemann equation in the first variable.

Proposition 4.12. *If h and g are holomorphic functions on an open set $\Omega \subset \mathbb{C}$ and $A(z)$, $B(z)$ are holomorphic familes of isotropic operators of orders $h(z)$ and $g(z)$ then the composite family $A(z) \circ B(z)$ is holomorphic of order $h(z) + g(z)$.*

Proof. It suffices to consider an arbitrary open subset $\Omega' \subset \Omega$ with compact closure inside Ω. Then h and g have bounded real parts, so $A(z)$, $B(z) \in \Psi^M_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$ for $z \in \Omega'$ for some fixed M. It follows that the composite $A(z) \circ B(z) \in \Psi^{2M}_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$. The symbol is given by the usual formula. Furthermore □

4.10. Resolvent and spectrum

One direct application of analytic Fredholm theory is to the resolvent of an elliptic operator of positive order. For simplicity we assume that $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ with $m \in \mathbb{N}$, although the case of non-integral positive order is only slightly more complicated.

Proposition 4.13. *If $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$, $m \in \mathbb{N}$, and there exists one point $\lambda' \in \mathbb{C}$ such that $A - \lambda'$ and $A^* - \overline{\lambda'}$ both have trivial null space, then*

$$(A - \lambda)^{-1} \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \tag{4.102}$$

is a meromorphic family with all residues finite rank smoothing operators; the span of the ranges of the residues at any $\tilde{\lambda}$ is the linear space of generalized eigenvalues, the solutions of

$$(A - \tilde{\lambda})^p u = 0 \textit{ for some } p \in \mathbb{N}. \tag{4.103}$$

Proof. Since A is elliptic and of positive integral order, m, $A - \lambda \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is and entire elliptic family. By hypothesis, its inverse exists for some $\lambda' \in \mathbb{C}$. Thus, by Proposition **??** $(A - \lambda)^{-1} \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ is a meromorphic family in the complex plane, with all residues finite rank smoothing operators.

Let $\tilde{\lambda}$ be a pole of $A-\lambda$. Since we can replace A by $A-\tilde{\lambda}$ we may suppose without loss of generality that $\tilde{\lambda}=0$. Thus, for some k the product $\lambda^k(A-\lambda)^{-1}$ is holomorphic near $\lambda=0$. Differentiating the identities

$$(A-\lambda)[\lambda^k(A-\lambda)^{-1}]=\lambda^k\operatorname{Id}=[\lambda^k(A-\lambda)^{-1}](A-\lambda)$$

up to k times gives the relations

$$\begin{gathered}A\circ R_{k-j}=R_{k-j}\circ A=R_{k-j+1},\ j=0,\cdots,k-1,\\ A\circ R_0=R_0\circ A=\operatorname{Id}+R_1,\ \text{where}\\ (A-\lambda)^{-1}=R_k\lambda^{-k}+R_{k-1}\lambda^{-k+1}+\cdots+R_0+\cdots,\ R_{k+1}=0.\end{gathered}\tag{4.104}$$

Thus $A^p\circ R_{k-p+1}=0=R_{k-p+1}\circ A^p$ for $0<p\le k$, which shows that all the residues, R_j, $1\le j\le k$, have ranges in the generalized eigenfunctions. $\square$

Notice also from (4.104) that the range of R_{k-j+1} is contained in the range of R_{k-j} for each $j=0,\dots,k-1$, and conversely for the null spaces

$$\begin{gathered}\operatorname{Ran}(R_k)\subset\operatorname{Ran}(R_{k-1})\subset\cdots\subset\operatorname{Ran}(R_1)\\ \operatorname{Nul}(R_k)\supset\operatorname{Nul}(R_{k-1})\supset\cdots\supset\operatorname{Nul}(R_1).\end{gathered}$$

Thus,

$$u\in\operatorname{Ran}(R_p),\ p\ge1\Longleftrightarrow\exists\ u_1\in\operatorname{Ran}(R_1)\text{ s.t. }A^{p-1}u_1=u.\tag{4.105}$$

4.11. Residue trace

We have shown, in Proposition 3.4, the existence of a unique trace functional on the residual algebra $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. We now follow ideas originating with Seeley, [**13**], and developed by Guillemin [**7**], [**8**] and Wodzicki [**16**], [**15**] to investigate the traces on the full algebra $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ of polyhomogeneous operators of integral order. We will prove the existence of a trace but defer until later the proof of its uniqueness.

Observe that for $A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the kernel can be written

$$A(x,y)=(2\pi)^{-n}\int e^{i(x-y)\xi}a_L(x,\xi)d\xi$$

and hence the trace, from (3.24), becomes

$$\operatorname{Tr}(A)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}a_L(x,\xi)dxd\xi,\tag{4.106}$$

just the integral of the left-reduced symbol. In fact this is true for *any* amplitude (of order $-\infty$) representing A :

$$A=(2\pi)^{-n}\int e^{i(x-y)}a(x,y,\xi)d\xi\Longrightarrow\operatorname{Tr}(A)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}a(x,x,\xi)dxd\xi.\tag{4.107}$$

The integral in (4.106) extends by continuity to $a_L\in S^m_\infty(\mathbb{R}^{2n})$ provided $m<-2n$. Thus, as a functional,

$$\operatorname{Tr}:\Psi^{-2n-\epsilon}_{\infty,\mathrm{iso}}(\mathbb{R}^n)\to\mathbb{C},\ \text{for any }\epsilon>0.\tag{4.108}$$

To extend it further we need somehow to *regularize* the resultant divergent integral in (4.106) (and to pay the price in terms of properties). One elegant way to do this is to use a holomorphic family as discussed in Section 4.9. Notice that we are passing from the algebra-with-bounds in (4.108) to polyhomogeneous operators.

LEMMA 4.10. *If $A(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family then $f(z) = \mathrm{Tr}(A(z))$, defined by (4.107) when $\Re(z) < -2n$, extends to a meromorphic function of z with at most simple poles on the divisor*

$$\{-2n, -2n+1, \dots, -1, 0, 1, \dots\} \subset \mathbb{C}.$$

PROOF. We know that $A(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family if and only if its left-reduced symbol is of the form

$$\sigma_L(A(z)) = (1 + |x|^2 + |z|^2)^{z/2} a(z; x, \xi)$$

where $a(z; x, y)$ is an entire function with values in $S^0_{\mathrm{phg}}(\mathbb{R}^n)$. For $\Re z < -2n$ the trace of $A(z)$ is

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (1 + |x|^2 + |\xi|^2)^{z/2} a(z; x, \xi) dx d\xi.$$

Consider the part of this integral on the ball

$$f_1(z) = (2\pi)^{-n} \int_{|x|^2 + |\xi|^2 \le 1} (1 + |x|^2 + |\xi|^2)^{z/2} a(z, x, y) dx d\xi.$$

This is clearly an entire function of z, since the integrand is entire and the domain compact.

To analyze the remaining part $f_2(z) = f(z) - f_1(z)$ let us introduce polar coordinates

$$r = (|x|^2 + |\xi|^2)^{1/2}, \ \theta = \frac{(x, \xi)}{r} \in \mathbb{S}^{2n-1}.$$

The integral, convergent in $\Re z < -2n$, becomes

$$f_2(z) = (2\pi)^{-n} \int_1^\infty \int_{\mathbb{S}^{2n-1}} (1 + r^2)^{z/2} \tilde{a}(z; r, \theta) d\theta r^{2n-1} dr.$$

Let us now pass to the radical compactification of $\mathbb{R}^{2n}$ or more prosaically, introduce $t = 1/r \in [0, 1]$ as variable of integration, so

$$f_2(z) = (2\pi)^{-n} \int_0^1 \int_{\mathbb{S}^{2n-1}} t^{-z} (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta) d\theta t^{-2n} \frac{dt}{t}.$$

Now the definition of $S^0_{\mathrm{phg}}(\mathbb{R}^{2n})$ reduces to the statement that

$$b(z; t, \theta) = (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta) \in \mathcal{C}^\infty(\mathbb{C} \times [0, 1] \times \mathbb{S}^{2n-1}) \tag{4.109}$$

is holomorphic in z.

If we replace b by its Taylor series at $t = 0$ to high order,

$$b(z; t, \theta) = \sum_{j=0}^k \frac{t^j}{j!} b_j(z; \theta) + t^{k+1} b_{(k)}(z; t, \theta), \tag{4.110}$$

where $b_{(k)}(z; t, \theta)$ has the same regularity (4.109), then $f_2(z)$ is decomposed as

$$f_2(z) = (2\pi)^{-n} \sum_{j=0}^k \int_0^1 \int_{\mathbb{S}^{2n-1}} \frac{t^{-z+j}}{j!} b_j(z; \theta) t^{-2n} \frac{dt}{t} + f_2^{(k)}(z). \tag{4.111}$$

The presence of this factor t^k in the remainder in (4.110) shows that $f_2^{(k)}(z)$ is holomorphic in $\Re z < -2n+k$. On the other hand the individual terms in the sum in (4.111) can be computed (for $\Re z < -2n$) as

$$(2\pi)^{-n}\left[\frac{t^{-z+j-2n}}{(-z+j-2n)}\right]_0^1 \int_{\mathbb{S}^{2n-1}} b_j(z,\theta)\frac{d\theta}{j!} \\ = (2\pi)^{-n}\frac{1}{(z-j+2n)}\int_{\mathbb{S}^{2n-1}} b_j(z,\theta)\frac{d\theta}{j!}.$$

Each of these terms extends to be meromorphic in the entire complex plane, with a simple pole (at most) at $z=-2n+j$. This shows that $f(z)$ has a meromorphic continuation as claimed. □

By this argument we have actually computed the residues of the analytic continuation of $\mathrm{Tr}(A(z))$ as

$$\lim_{z\to -2n+j}(z-j+2n)\,\mathrm{Tr}(A(z)) = (2\pi)^{-n}\int_{\mathbb{S}^{2n-1}} a_j(\theta)d\theta \tag{4.112}$$

when $a_j(\theta)\in\mathcal{C}^\infty(S^{2n-1})$ is the function occurring in the asymptotic expansion of the left symbol of $A(z)$:

$$\sigma_L(A(z)) \sim \sum_{j=0}^{\infty}(|x|^2+|\xi|^2)^{z/2-j}\tilde{a}_j(z,\theta) \\ |x|^2+|\xi|^2\to\infty,\ \theta=\frac{(x,\xi)}{(|x|^2+|\xi|^2)^{1/2}},\ a_j(\theta)=\tilde{a}_j(-2n+j,\theta). \tag{4.113}$$

More generally, if $m\in\mathbb{Z}$ and $A(z)\in\Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family then

$$\mathrm{Tr}(A(z)) \text{ is meromorphic with at most} \\ \text{simple poles at } -2n-m+\mathbb{N}_0.$$

Indeed this just follows by considering the family $A(z-m)$.

We are especially interested in the behavior at $z=0$. Since the residue there is an integral of the term of order $-2n$, we know that

$$A(z)\in\Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ holomorphic with } A(0)=0 \\ \Longrightarrow \mathrm{Tr}(A(z)) \text{ is regular at } z=0. \tag{4.114}$$

This allows us to make the following definition:

$$\mathrm{Tr}_{\mathrm{Res}}(A) = \lim_{z\to 0} z\,\mathrm{Tr}(A(z)) \text{ if} \\ A(z)\in\Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ is holomorphic with } A(0)=A.$$

We know that such a holomorphic family exists, since we showed in Section 4.9 the existence of a holomorphic family $F(z)\in\Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ with $F(0)=\mathrm{Id}$; $A(z)=AF(z)$ is therefore an example. Similarly we know that $\mathrm{Tr}_{\mathrm{Res}}(A)$ is independent of the choice of holomorphic family $A(z)$ because of (4.114) applied to the difference, which vanishes at zero.

Lemma 4.11. *The residue functional* $\mathrm{Tr}_{\mathrm{Res}}(A)$, $A\in\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$, *is a trace:*

$$\mathrm{Tr}_{\mathrm{Res}}([A,B]) = 0\ \forall\ A,B\in\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.115}$$

which vanishes on $\Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n)$ *and is given explicitly by*

$$\operatorname{Tr}_{\mathrm{Res}}(A) = (2\pi)^{-n} \int_{\mathcal{S}^{2n-1}} a_{-2n}(\theta) d\theta \tag{4.116}$$

where $a_{-2n}(\theta)$ *is the term of order* $-2n$ *in the expansion of the left (or right) symbol of* a.

PROOF. We have already shown that $\operatorname{Tr}_{\mathrm{Res}}(A)$ is well-defined and (4.116) follows from (4.112) with $a_{-2n}(\theta)$ the term of order $-2n$ in the left-reduced symbol of $A = A(0)$. On the other hand, the same argument applies for the right-reduced symbol.

To see (4.115) just note that if $A(z)$ and $B(z)$ are holomorphic families with $A(0) = A$, and $B(0) = B$ then $C(z) = [A(z), B(z)]$ is a holomorphic family with $C(0) = [A, B]$. On the other hand, $\operatorname{Tr}(C(z)) = 0$ when $\Re z \gg 0$, so the analytic continuation of $\operatorname{Tr}(C(z))$ vanishes identically and (4.115) follows. □

As we shall see below, $\operatorname{Tr}_{\mathrm{Res}}$ is the unique trace (up to a multiple of course) on $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$.

4.12. Exterior derivation

Let $A(z) \in \Psi^{z}_{\mathrm{iso}}(\mathbb{R}^n)$ be a holomorphic family with $A(0) = \operatorname{Id}$. Then

$$G(z) = A(z) \cdot A(-z) \in \Psi^{0}_{\mathrm{iso}}(\mathbb{R}^n)$$

is a holomorphic family of fixed order with $G(0) = \operatorname{Id}$. By analytic Fredholm theory

$$G^{-1}(z) \in \Psi^{0}_{\mathrm{iso}}(\mathbb{R}^n) \text{ is a meromorphic family with finite rank poles.} \tag{4.117}$$

It follows that $A^{-1}(z) = A(-z)G^{-1}(z)$ is a meromorphic family of order $-z$ with at most finite rank poles and regular near 0. Set

$$\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n) \ni B \mapsto A(z)BA^{-1}(z) = B(z). \tag{4.118}$$

Thus $B(z)$ is a meromorphic family of order m with $B(0) = B$. The derivative gives a linear map.

$$\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n) \ni B \mapsto D_A B = \frac{d}{dz} A(z)BA^{-1}(z)|_{z=0} \in \Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n). \tag{4.119}$$

PROPOSITION 4.14. *For any holomorphic family of order* z, *with* $A(0) = \operatorname{Id}$, *the map* (4.119), *defined through* (4.118), *is a derivation and for two choices of* $A(z)$ *the derivations differ by an inner derivation.*

PROOF. Since

$$A(z)B_1B_2A^{-1}(z) = A(z)B_1A^{-1}(z)A(z)B_2A^{-1}(z)$$

it follows that

$$\frac{d}{dz} A(z)B_1B_2A^{-1}(z)|_{z=0} = (D_AB_1) \circ B_2 + B_1 \circ (D_AB_2).$$

If $A_1(z)$ and $A_2(z)$ are two holomorphic families of order z with $A_1(0) = A_2(0) = \operatorname{Id}$ then

$$A_2(z) = A_1(z)G(z)$$

when $G(z) \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a meromorphic family, with finite rank poles. Thus

$$\begin{aligned} A_2(z)BA_2^{-1}(z) &= A_1(z)G(z)BG^{-1}(z)A_1^{-1}(z) \\ &= A_1(z)BA^{-1}(z) + zA_1(z)H(z)A_1^{-1}(z). \end{aligned}$$

Here $H(z) = (G(z)BG^{-1}(z) - B)/z$ is a holomorphic family of degree m with $H(0) = G'(0)B - BG'(0)$. Thus

$$\frac{d}{dz}A_2(z)BA_2^{-1}(z)|_{z=0} = \frac{d}{dz}A_1(z)BA^{-1}(z)|_{z=0} + [G'(0), B]$$

which shows that the two derivations differ by an inner derivation, which is to say commutation with an element of $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$. □

Note that in fact

$$D_A : \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \to \Psi^{m-1}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\ m$$

since the symbol of $A(z)BA^{-1}(z)$ is equal to the principal symbol of B for all z.

For the specific choice of $A(z) = H(z)$ given by

$$\sigma_L(H(z)) = (1 + |x|^2 + |\xi|^2)^{z/2}$$

we shall set

$$D_A B = D_H B.$$

Observe that $\frac{1}{2}\log(1 + |x|^2 + |\xi|^2) \in S^{\epsilon}_{\infty}(\mathbb{R}^{2n})\ \forall\ \epsilon > 0$. Thus $\log(1 + |x|^2 + |\xi|^2)$, defined by Weyl quantization, is an element of $\Psi^{-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for all $\epsilon > 0$. By differentiation the symbols satisfy

$$D_H B = [\frac{1}{2}\log(1 + |x|^2 + |D|^2), B] + [G, B]$$

where $G \in \Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus D_H is *not* itself an interior derivation. It is therefore an *exterior* derivation.

4.13. Regularized trace

In Section 4.11 we defined the residue trace of B as the residue at $z = 0$ of the analytic continuation of $\mathrm{Tr}(BA(z))$, where $A(z)$ is a holomorphic family of order z with $A(0) = \mathrm{Id}$. Next we consider the functional

$$\overline{\mathrm{Tr}}_{\mathrm{A}}(B) = \lim_{z=0}(\mathrm{Tr}(BA(z)) - \frac{1}{z}\mathrm{Tr}_{\mathrm{Res}}(B)). \tag{4.120}$$

In contrast to the residue trace, $\overline{\mathrm{Tr}}_{\mathrm{A}}(z)$ *does* depend on the choice of analytic family $A(z)$.

Lemma 4.12. *If $A_i(z)$, $i = 1, 2$, are two holomorphic families of order z with $A_i(0) = \mathrm{Id}$ and $G'(0) = \frac{d}{dz}A_2(z)A_1^{-2}(z)|_{z=0}$ then*

$$\overline{\mathrm{Tr}}_{\mathrm{A}_2}(B) - \overline{\mathrm{Tr}}_{\mathrm{A}_1}(B) = \mathrm{Tr}_{\mathrm{Res}}(BG'(0)). \tag{4.121}$$

Proof. Writing $G(z) = A_2(z)A_1^{-1}(z)$, which is a meromorphic family of order 0 with $G(0) = \mathrm{Id}$,

$$\begin{aligned}\mathrm{Tr}(BA_2(z)) &= \mathrm{Tr}(BG(z)A_1(z))\\ &= \mathrm{Tr}(BA_1(z)) + z\,\mathrm{Tr}(BG'(0)A_1(z)) + z^2\,\mathrm{Tr}(H(z)A_1(z))\end{aligned}$$

where $H(z) = \frac{B}{z^2}(G(z) - \mathrm{Id} - zG'(0))$ is then meromorphic with only finite rank poles and is regular near $z = 0$. Thus the analytic continuation of $z^2\,\mathrm{Tr}(H(z)A(z))$ vanishes at zero from which (4.121) follows. □

This regularized trace $\overline{\mathrm{Tr}}_{\mathrm{A}}(B)$ therefore only depends on the first order, in z, term in $A(z)$ at $z = 0$. It is important to note that it is *not* itself a trace.

Lemma 4.13. *If* $B_1,\ B_2 \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ *then*

$$\overline{\mathrm{Tr}}_{\mathrm{A}}([B_1, B_2]) = \mathrm{Tr}_{\mathrm{Res}}(B_2 D_A B_1). \tag{4.122}$$

Proof. Since $\overline{\mathrm{Tr}}_{\mathrm{A}}([B_1, B_2])$ is the regularized value at 0 of the analytic continuation of the trace of

$$\begin{aligned} B_1 B_2 A(z) - B_2 B_1 A(z) &= B_2[A(z), B_1] + [B_1, B_2 A(z)] \\ &= B_2([A(z), B]A^{-1}(z))A_1(z) + [B_1 B_2 A(z)]. \end{aligned} \tag{4.123}$$

The second term on the right in (4.123) has zero trace before analytic continuation. Thus $\overline{\mathrm{Tr}}_A([B_1, B_2])$ is the regularized value of the analytic continuation of the trace of $Q(z)A(z)$ where

$$Q(z) = B_2[A(z), B_1]A^{-1}(z) = zD_A B_1 + z^2 L(z)$$

with $L(z)$ meromorphic of fixed order and regular at $z = 0$. Thus (4.122) follows. □

Note that

$$\mathrm{Tr}_{\mathrm{Res}}(D_A B) = 0\ \forall\ B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.124}$$

and any family A. Indeed the residue trace is the residue of $z = 0$ of the analytic continuation of $\mathrm{Tr}(H(z)A(z))$ when $A(z)$ is any meromorphic family of fixed order with $H(0) = D_A B$. In particular we can take

$$H(z) = \tfrac{1}{z}(A(z)BA^{-1}(z) - B).$$

Then $H(z)A(z) = \frac{1}{z}[A(z), B]$ so the trace vanishes before analytic continuation.

4.14. Projections

4.15. Complex powers

4.16. Index and invertibility

We have already seen that the elliptic elements

$$E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \subset \Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \hookrightarrow \mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N)) \tag{4.125}$$

define Fredholm operators. The *index* of such an operator

$$\mathrm{Ind}(A) = \dim \mathrm{Nul}(A) - \dim \mathrm{Nul}(A^*) \tag{4.126}$$

is a measure of its non-invertibility. Set

$$E^0_{\mathrm{iso},k}(\mathbb{R}^n; \mathbb{C}^N) = \left\{A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N); \mathrm{Ind}(A) = k\right\},\ k \in \mathbb{Z}. \tag{4.127}$$

Proposition 4.15. *If* $A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ *and* $\mathrm{Ind}(A) = 0$ *then there exists* $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ *such that* $A + E$ *is invertible in* $\mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N))$ *and the inverse then lies in* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$.

Proof. Let B be the generalized inverse of A, assumed to be elliptic. The assumption that $\mathrm{Ind}(A) = 0$ means that $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$ have the same dimension. Let $e_1, \cdots, e_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ and $f_1, \cdots, f_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ be bases of $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$. Then consider

$$E = \sum_{j=1}^{p} f_j(x)\overline{e_j(y)} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N). \tag{4.128}$$

By construction E is an isomorphism (in fact an arbitrary one) between $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$. Thus $A+E$ is continuous, injective and surjective, hence has an inverse in $\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N))$. Indeed this inverse is $B+E^{-1}$ where E^{-1} is the inverse of E as a map from $\mathrm{Nul}(A)$ to $\mathrm{Nul}(A^*)$. This shows that A can be perturbed by a smoothing operator to be invertible. $\square$

Let

$$G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \subset E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N) \subset E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \subset \Psi 0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.129}$$

denote the group of the invertible elements (invertibility being either in $\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N)$ or in $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$) in the ring of elliptic elements of index 0.

Corollary 4.5. *The first inclusion in* (4.129) *is dense in the topology of* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

Proof. This follows from the proof of Proposition 4.15, since $A+sE$ is invertible for all $s\neq 0$. $\square$

We next derive some simple formulæ for the index of an element of $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. First observe that the trace of a finite dimensional projection is its rank, the dimension of its range. Thus

$$\mathrm{Ind}(A) = \mathrm{Tr}(\Pi_{\mathrm{Nul}(A)}) - \mathrm{Tr}(\Pi_{\mathrm{Nul}(A^*)}) \tag{4.130}$$

where the trace may be reinterpreted as the trace on smoothing operators. The identities, (4.15), satisfied by the generalized inverse of A shows that this can be rewritten

$$\mathrm{Ind}(A) = -\,\mathrm{Tr}(BA-\mathrm{Id}) + \mathrm{Tr}(AB-\mathrm{Id}) = \mathrm{Tr}([A,B]). \tag{4.131}$$

Here $[A,B] = \Pi_{\mathrm{Nul}(A)} - \Pi_{\mathrm{Nul}(A^*)}$ is a smoothing operator, even though both A and B are elliptic of order 0.

Lemma 4.14. *If $A\in E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ the identity* (4.131)*, which may be rewritten*

$$\mathrm{Ind}(A) = \mathrm{Tr}([A,B]), \tag{4.132}$$

holds for any parametrix B.

Proof. If B' is a parametrix and B is the generalized inverse then $B'-B = E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. Thus

$$[A,B'] = [A,B] + [A,E].$$

Since $\mathrm{Tr}([A,E] = 0$, one of the arguments being a smoothing operator, (4.132) follows in general from the particular case (4.131). $\square$

Note that it follows from (4.132) that $\mathrm{Ind}(A) = \mathrm{Ind}(A+E)$ if E is smoothing. In fact the index is even more stable than this as we shall see, since it is locally constant on $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. In any case this shows that

$$\begin{gathered}\mathrm{Ind}:\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow\mathbb{Z},\ \mathrm{Ind}(a)=\mathrm{Ind}(A)\text{ if }a=[A],\\ \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) = E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\\ \subset\mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) = \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\end{gathered} \tag{4.133}$$

is well-defined.

The argument of the trace functional in (4.132) is a smoothing operator, but we may still rewrite the formula in terms of the regularized trace, with respect to the standard regularizer $H(z)$ with left symbol $(1+|x|^2+|\xi|^2)^{\frac{z}{2}}$. The advantage of doing so is that we can then use the trace defect formula (4.122). Thus for any elliptic isotropic operator of order 0

$$\operatorname{Ind}(A) = \operatorname{Tr}_{\mathrm{Res}}(BD_H A). \tag{4.134}$$

Here B is a parametrix for A. The residue trace is actually a functional

$$\operatorname{Tr}_{\mathrm{Res}} : \mathcal{A}^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$

so if we write a^{-1} for the inverse of a in the ring $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then

$$\operatorname{Ind}(a) = \operatorname{Tr}_{\mathrm{Res}}(a^{-1}D_H a),\ D_H : \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.135}$$

being the induced derivation (since D_H clearly preserves the ideal $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

From this simple formula we can easily deduce two elementary properties of elliptic operators. These actually hold in general for Fredholm operators, although the proofs here are not valid in that generality. Namely

$$\operatorname{Ind} : \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \text{ is locally constant and} \tag{4.136}$$

$$\operatorname{Ind}(a_1a_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2)\ \forall\ a_1,\ a_2 \in \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N). \tag{4.137}$$

The first of these follows the continuity of the formula (4.135) since under deformation of a in $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ the inverse a^{-1} varies continuously, so Ind is continuous and integer-valued, hence locally constant. Similarly the second, logarithmic additivity, property follows from the fact that D_H is a derivation, so

$$D_H(a_1a_2) = (D_Ha_1)a_2 + a_1D_Ha_2$$

and the the trace property of $\operatorname{Tr}_{\mathrm{Res}}$ which shows that

$$\begin{aligned}\operatorname{Ind}(a_1a_2) = \operatorname{Tr}_{\mathrm{Res}}((a_1a_2)^{-1}D_H(a_1a_2) &= \operatorname{Tr}(a_2^{-1}a_1^{-1}\left((D_Ha_1)a_2 + a_1D_Ha_2\right)\\ &= \operatorname{Tr}(a_2^{-1}a_1^{-1}(D_Ha_1)a_2) + \operatorname{Tr}(a_2^{-1}D_Ha_2) = \operatorname{Ind}(a_1)+\operatorname{Ind}(a_2).\end{aligned} \tag{4.138}$$

4.17. Variation 1-form

In the previous section we have seen that the index

$$\operatorname{Ind} : E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \tag{4.139}$$

is a multiplicative map which is the obstruction to perturbative invertibility. In the next two sections we will derive a closely related obstruction to the perturbative invertibility of a family of elliptic operators. Thus, suppose

$$Y \ni y \longmapsto A_y \in E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.140}$$

is a family of elliptic operators depending smoothly on a parameter in the compact manifold Y. We are interested in the *families perturbative invertibility question.* That is, does there exist a smooth family

$$Y \ni y \longmapsto E_y \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \text{ such that } (A_y+E_y) \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\ \forall\ y. \tag{4.141}$$

We have assumed that the operators have index zero since this is necessary (and sufficient) for E_y to exist for any one $y \in Y$. Thus the issue is the smoothness (really just the continuity) of the perturbation E_y.

We shall start by essentially writing down such a putative obstruction directly and then subsequently we shall investigate its topological origins.

PROPOSITION 4.16. *If a smooth family* (4.140)*, parameterized by a compact manifold* Y*, is perturbatively invariant in the sense that there is a smooth family as in* (4.141)*, then the closed* 2*-form on* Y

$$\beta = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y \cdot a_y^{-1}D_Ha_y) \in \mathcal{C}^\infty(Y;\Lambda^2), \qquad a_y = [A_y] \in \mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N), \tag{4.142}$$

is exact.

PROOF. Note first that β is indeed a smooth form, since the full symbolic inverse depends smoothly on parameters. Next we show that β is always closed. The 1-forms $a_y^{-1}d_ya_ya_y^{-1}$ and da_y are exact so differentiating directly gives

$$\begin{aligned} d\beta &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y \wedge d(a_y^{-1}D_Ha_y)) \\ &= -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y^{-1}D_Ha_y)) \\ &\qquad +\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y \wedge a_y^{-1}D_H(da_y)) \\ &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y \wedge D_H(a_y^{-1}da_y)). \end{aligned} \tag{4.143}$$

Using the trace property and the commutativity of a 2-form with other forms the last expression can be written

$$\frac{1}{6}\operatorname{Tr}_{\mathrm{Res}}(D_H(a_y^{-1}d_ya_y \wedge a_y^{-1}d_ya_y \wedge a_y^{-1}da_y)) = 0 \tag{4.144}$$

by property (4.124) of the residue trace.

Now, suppose that a smooth perturbation as in (4.141) does exist. We can replace A_y by $A_y + E_y$ without affecting β, since the residue trace vanishes on the ideal of smoothing operators. Thus we can assume that A_y itself is invertible. Then consider the 1-form defined using the regularized trace

$$\overline{\alpha} = \overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}d_yA_y). \tag{4.145}$$

This is an extension of the 1-form $d\log\det_F$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. The extension is not in general closed, because the regularized trace does not satisfy the trace condition. Using the stanadard formula for the variation of the inverse, $dA_y^{-1} = -A_y^{-1}dA_yA_y^{-1}$, the exterior derivative is the 2-form

$$d\overline{\alpha} = -\overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}(d_yA)A_y^{-1}d_yA_y). \tag{4.146}$$

The 2-form argument is a commutator. Indeed, in terms of local coordinates we can write

$$\begin{aligned} A_y^{-1}(d_yA)A_y^{-1}d_yA_y &= \sum_{j,k=1}^{p} A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k})dy_j \wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}\left(A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k}) - A_y^{-1}(\frac{\partial A}{\partial y_k})A_y^{-1}(\frac{\partial A}{\partial y_j})\right)dy_j \wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}[A_y^{-1}(\frac{\partial A}{\partial y_j}), A_y^{-1}(\frac{\partial A}{\partial y_k})]dy_j \wedge dy_k \end{aligned}$$

Applying the trace defect formula (4.122) shows that

$$d\overline{\alpha} = -\frac{1}{2}\operatorname{Tr}_{\text{Res}}\left(A_y^{-1}d_yA_y \wedge D_H(A_y^{-1}d_yA_y)\right), \tag{4.147}$$

locally and hence globally.

Expanding the action of the derivation D_H gives

$$\begin{aligned} d\overline{\alpha} = \beta - \frac{1}{2}\operatorname{Tr}_{\text{Res}}\left(A_y^{-1}d_yA_y \wedge A_y^{-1}d_y(D_HA_y)\right) = \beta - d\gamma, \text{ where} \\ \gamma = \frac{1}{2}\operatorname{Tr}_{\text{Res}}\left(A_y^{-1}d_yA_y \wedge .A_y^{-1}D_HA_y\right). \end{aligned} \tag{4.148}$$

We conclude that if A_y has an invertible lift then β is exact. $\square$

Note that the form γ in (4.148) is well-defined as a form on $\mathcal{E}^0_{\text{iso},0}(\mathbb{R}^n;\mathbb{C}^N)$, and is independent of the perturbation. Thus the cohomology class which we have constructed as the obstruction to perturbative invertibility can be written

$$[\beta] = [\beta - d\gamma] \in H^2(\mathcal{E}^0_{\text{iso},0}(\mathbb{R}^n;\mathbb{C}^N)). \tag{4.149}$$

4.18. Determinant bundle

To better explain the topological origin of the cohomology class (4.149) we construct the determinant bundle. This was originally introduced for families of Dirac operators by Quillen [**12**]. Recall that the Fredholm determinant is a character

$$\begin{aligned} &\det{}_{\text{Fr}} : \operatorname{Id} + \Psi^{-2n-1}_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C}, \\ &\qquad \det{}_{\text{Fr}}(AB) = \det{}_{\text{Fr}}(A)\det{}_{\text{Fr}}(B) \forall\ A,\ B \in \operatorname{Id} + \Psi^{-2n-1}_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N). \end{aligned} \tag{4.150}$$

As we shall see, it is not possible to extend the Fredholm determinant as a multiplicative function to $G^0_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N)$, essentially because of the non-extendibility of the trace.

However in trying to extend the determinant we can consider the possible values it would take on a point $A \in G^0_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ as the set of pairs (A,z), $z \in \mathbb{C}$. Thus we simple consider the product

$$D^0 = G^0 \times \mathbb{C}, \tag{4.151}$$

where from now on we simplify the notation and write $G^0 = G_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ etc. Although it is not reasonable to expect full multiplicative of the determinant, it is more reasonable to expect the determinant of $A(\operatorname{Id} + B)$, $B \in \Psi^{-2n-1}$ to be related to the product of determinants. Thus it is natural to identify pairs in D^0,

$$\begin{aligned} (A,z) \sim_p (A',z') \text{ if} \\ A, A' \in G^0,\ A' = A(\operatorname{Id} + B),\ z' = \det{}_{\text{Fr}}(\operatorname{Id} + B)z,\ B \in \Psi^p,\ p < -2n. \end{aligned} \tag{4.152}$$

The equivalence relations here are slightly different, depending on p. In all cases the action of the determinant is linear, so the quotient is a line bundle.

LEMMA 4.15. *For any integer $p < -2n$, and also $p = -\infty$, the quotient*

$$\mathcal{D}^0_p = D^0/\sim_p \tag{4.153}$$

is a smooth line bundle over $\mathcal{G}^0_p = G^0/G^p$.

Proof. The projection is just the quotient in the first factor and this clearly defines a commutative square

$$
\begin{tikzcd}
D^0 \arrow[r, "{[\sim_p]}"] \arrow[d, "\pi"] & \mathcal{D}^0_p \arrow[d, "\pi"] \\
G^0 \arrow[r, "/G^p"] & \mathcal{G}^0_p.
\end{tikzcd}
\tag{4.154}
$$

□

4.19. Index bundle

4.20. Index formulæ

4.21. Isotropic essential support

4.22. Isotropic wavefront set

4.23. Isotropic FBI transform

4.24. Problems

Problem 4.1. Define the *isotropic* Sobolev spaces of integral order by

$$
H^k_{\mathrm{iso}}(\mathbb{R}^n) = \begin{cases} \left\{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| + |\beta| \le k\right\} & k \in \mathbb{N} \\ \left\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum\limits_{|\alpha|+|\beta|\le -k} x^\alpha D^\beta_x u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n)\right\} & k \in -\mathbb{N}. \end{cases}
\tag{4.155}
$$

Show that if $A \in \Psi^p_{\mathrm{iso}}(\mathbb{R}^n)$ with p an integer, then $A : H^k_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{k-p}_{\mathrm{iso}}(\mathbb{R}^n)$ for any integral k. Deduce (using the properties of elliptic isotropic operators) that the general definition

$$
H^m_{\mathrm{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n),\ \forall\ A \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)\right\},\ m \in \mathbb{R}
\tag{4.156}
$$

is consistent with (4.155) and has the properties

$$
A \in \Psi^M_{\mathrm{iso}}(\mathbb{R}^n) \Longrightarrow A : H^m_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{m-M}_{\mathrm{iso}}(\mathbb{R}^n),
\tag{4.157}
$$

$$
\bigcap_m H^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n),\ \bigcup_m H^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)
\tag{4.158}
$$

$$
A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),\ u \in \mathcal{S}'(\mathbb{R}^n),\ Au \in H^{m'}(\mathbb{R}^n) \Longrightarrow u \in H^{m'-m}(\mathbb{R}^n),
\tag{4.159}
$$

Problem 4.2. Show that if $\epsilon > 0$ then

$$
H^\epsilon_{\mathrm{iso}}(\mathbb{R}^n) \subsetneq (1 + |x|)^{-\epsilon} L^2(\mathbb{R}^n) \cap H^\epsilon(\mathbb{R}^n)
$$

Deduce that $H^\epsilon_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is a compact inclusion.

Problem 4.3. Using Problem 4.2, or otherwise, show that each element of $\Psi^{-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$, $\epsilon > 0$, defines a compact operator on $L^2(\mathbb{R}^n)$.

Problem 4.4. Show that if $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then there exists $F \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that

$$
(\mathrm{Id} + E)(\mathrm{Id} + F) = \mathrm{Id}_G \text{ with } G \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ of finite rank,}
$$

that is, $G \cdot \mathcal{S}(\mathbb{R}^n)$ is finite dimensional.

Problem 4.5. Using Problem 4.4 show that an elliptic element $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ has a parametrix $B \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ *up to finite rank error*; that is, such that $A \circ B - \mathrm{Id}$ and $B \circ A - \mathrm{Id}$ are finite rank elements of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Deduce that such an elliptic A defines a Fredholm operator

$$A : H^M_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{M-m}_{\mathrm{iso}}(\mathbb{R}^n)$$

for any M. [The requirements for an operator A between Hilbert spaces to be Fredholm are that it be bounded, have finite-dimensional null space and closed range with a finite-dimensional complement.]

Problem 4.6. [The harmonic oscillator] Show that the 'harmonic oscillator'

$$H = |D|^2 + |x|^2,\ Hu = \sum_{j=1}^{n} D_j^2 u + |x|^2 u,$$

is an elliptic element of $\Psi^2_{\mathrm{iso}}(\mathbb{R}^n)$. Consider the 'creation' and 'annihilation' operators

$$C_j = D_j + ix_j,\ A_j = D_j - ix_j = C_j^*, \tag{4.160}$$

and show that

$$H = \sum_{j=1}^{n} C_j A_j + n = \sum_{j=1}^{n} A_j C_j - n, \tag{4.161}$$

$$[A_j, H] = 2A_j,\ [C_j, H] = -2C_j,\ [C_l, C_j] = 0,\ [A_l, A_j] = 0,\ [A_l, C_j] = 2\delta_{lk}\,\mathrm{Id},$$

where $[A, B] = A \circ B - B \circ A$ is the commutator bracket and δ_{lk} is the Kronecker symbol. Knowing that $(H - \lambda)u = 0$, for $\lambda \in \mathbb{C}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{S}(\mathbb{R}^n)$ (why?) show that

$$E_\lambda = \{u \in \mathcal{S}'(\mathbb{R}^n); (H - \lambda)u = 0\} \neq \{0\} \Longleftrightarrow \lambda \in n + 2\mathbb{N}_0 \tag{4.162}$$

$$\text{and } E_{-n+2k} = \left\{ \sum_{|\alpha|=k} c_\alpha C^\alpha \exp(-|x|^2/2),\ c_\alpha \in \mathbb{C} \right\},\ k \in \mathbb{N}_0. \tag{4.163}$$

Problem 4.7. [Definition of determinant of matrices.]

Problem 4.8. [Proof that $d\alpha = 0$ in (3.32).] To prove that the 1-form is closed it suffices to show that it is closed when restricted to any 2-dimensional submanifold. Thus we may suppose that $A = A(s,t)$ depends on 2 parameters. In terms of these parameters

$$\alpha = \mathrm{Tr}(A(s,t)^{-1}\frac{dA(s,t)}{ds})ds + \mathrm{Tr}(A(s,t)^{-1}\frac{dA(s,t)}{dt})dt. \tag{4.164}$$

Show that the exterior derivative can be written

$$d\alpha = \mathrm{Tr}([A(s,t)^{-1}\frac{dA(s,t)}{dt}, A(s,t)^{-1}\frac{dA(s,t)}{ds}])ds \wedge dt \tag{4.165}$$

and hence that it vanishes.

Problem 4.9. If E and F are vector spaces, show that the space of operators $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n; E, F)$ from $\mathcal{S}'(\mathbb{R}^n; E)$ to $\mathcal{S}'(\mathbb{R}^n; F)$ is well-defined as the matrices with entries in $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ for any choice of bases of E and F.

Problem 4.10. Necessity of ellipticity for a psuedodifferential operator to be Fredholm on the isotropic Sobolev spaces.

(1) Reduce to the case of operators of order 0.
(2) Construct a sequence in L^2 such that $\|u_n\| = 1$, $u_n \to 0$ weakly and $Au_n \to 0$ strongly in L^2.

PROBLEM 4.11. [Koszul complex] Consider the form bundles over $\mathbb{R}^n$. That is $\Lambda^k\mathbb{R}^n$ is the vector space of dimension $\binom{n}{k}$ consisting of the totally antisymmetric k-linear forms on $\mathbb{R}^n$. If $e_1, e_2, \dots, e_n$ is the standard basis for $\mathbb{R}^n$ then for a k-tuple α e^α defined on basis elements by

$$e^\alpha(e_{i_1}, \dots, e_{i_k}) = \prod_{j=1}^{k} \delta_{1_j \alpha_j}$$

extends uniquely to a k-linear map. Elements $dx^\alpha \in \Lambda^k\mathbb{R}^n$ are defined by the total antisymmetrization of the e^α. Explicitly,

$$dx^\alpha(v_1, \dots, v_k) = \sum_{\pi} \operatorname{sgn} \pi e^\alpha(v_{\pi_1}, \dots, v_{\pi_n})$$

where the sum is over permutations π of $\{1, \dots, n\}$ and $\operatorname{sgn} \pi$ is the parity of π. The dx^α for strictly increasing k-tuples α of elements of $\{1, \dots, n\}$ give a basis for $\Lambda^k\mathbb{R}^n$. The wedge product is defined by $dx^\alpha \wedge dx^\beta = dx^{\alpha,\beta}$.

Now let $\mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ be the tensor product, that is $u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ is a finite sum

$$u = \sum_{\alpha} u_\alpha dx^\alpha. \tag{4.166}$$

The annihilation operators in (4.160) define an operator, for each k,

$$D : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k+1}),\ Du = \sum_{j=1}^{n} A_j u_\alpha dx^j \wedge dx^\alpha.$$

Show that $D^2 = 0$. Define inner products on the $\Lambda^k\mathbb{R}^n$ by declaring the basis introduced above to be orthonormal. Show that the adjoint of D, defined with respect to these inner products and the L^2 pairing is

$$D^* : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ D^*u = \sum_{j=1}^{n} C_j u_\alpha \iota_j dx^\alpha.$$

Here, ι_j is 'contraction with e_j;' it is the adjoint of $dx^j \wedge$. Show that $D + D^*$ is an elliptic element of $\Psi^1_{\mathrm{iso}}(\mathbb{R}^n; \Lambda^*)$. Maybe using Problem 4.6 show that the null space of $D + D^*$ on $\mathcal{S}'(\mathbb{R}^n; \Lambda^*\mathbb{R}^n)$ is 1-dimensional. Deduce that

$$\begin{gathered}\{u \in \mathcal{S}'(\mathbb{R}^n); Du = 0\} = \mathbb{C} \exp(-|x|^2/2),\\ \{u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k); Du = 0\} = (\mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ k \geq 1.\end{gathered} \tag{4.167}$$

Observe that, as an operator from $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\mathrm{odd}})$ to $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\mathrm{even}})$, $D + D^*$ is an elliptic element of $\Psi^1_{\mathrm{iso}}(\mathbb{R}^n; \Lambda^{\mathrm{odd}}, \Lambda^{\mathrm{even}})$ and has index 1.

PROBLEM 4.12. [Isotropic essential support] For an element of $S^m(\mathbb{R}^n)$ define (isotropic) essential support, or operator wavefront set, of $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ by

$$\mathrm{WF}_{\mathrm{iso}}(A) = \operatorname{cone\,supp}(\sigma_L(A)) \subset \mathbb{R}^{2n} \setminus \{0\}. \tag{4.168}$$

Show that $\mathrm{WF}_{\mathrm{iso}}(A) = \operatorname{cone}\operatorname{supp}(\sigma_L(A))$ and check the following

$$\mathrm{WF}'_{\mathrm{iso}}(A+B) \cup \mathrm{WF}'_{\mathrm{iso}}(A \circ B) \subset \mathrm{WF}'_{\mathrm{iso}}(A) \cap \mathrm{WF}'_{\mathrm{iso}}(B), \tag{4.169}$$

$$\mathrm{WF}'_{\mathrm{iso}}(A) = \emptyset \Longleftrightarrow A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \tag{4.170}$$

PROBLEM 4.13. [Isotropic partition of unity] Show that if $U_i \subset \mathbb{S}^{n-1}$ is an open cover of the unit sphere and $\tilde{U}_i = \{Z \in \mathbb{R}^{2n} \setminus \{0\}; \frac{Z}{|Z|} \in U_i\}$ is the corresponding *conic* open cover of $\mathbb{R}^{2n} \setminus \{0\}$ then there exist (finitely many) operators $A_i \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with $\mathrm{WF}'_{\mathrm{iso}}(A_i) \subset \tilde{U}_i$, such that

$$\mathrm{Id} - \sum_i A_i \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \tag{4.171}$$

PROBLEM 4.14. Suppose $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$, is elliptic and has index zero as an operator on $\mathcal{S}'(\mathbb{R}^n)$. Show that there exists $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that $A + E$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$.

PROBLEM 4.15. [Isotropic wave front set] For $u \in \mathcal{S}'(\mathbb{R}^n)$ define

$$\mathrm{WF}_{\mathrm{iso}}(u) = \bigcap \left\{ \mathrm{WF}'_{\mathrm{iso}}(A); A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n),\ Au \in \mathcal{S}(\mathbb{R}^n) \right\}. \tag{4.172}$$

CHAPTER 5

Microlocalization

5.1. Calculus of supports

Recall that we have already defined the support of a tempered distribution in the slightly round-about way:

$$\text{if } u \in \mathcal{S}'(\mathbb{R}^n),\ \operatorname{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^{\complement}. \tag{5.1}$$

Now if $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is any continuous linear operator we can consider the support of the kernel:

$$\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}. \tag{5.2}$$

We write out the space as a product here to point to the fact that *any* subset of the product defines (is) a *relation* i.e. a map on subsets:

$$\begin{gathered} G \subset \mathbb{R}^n \times \mathbb{R}^n, \quad S \subset \mathbb{R}^n \Longrightarrow \\ G \circ S = \big\{x \in \mathbb{R}^n;\ \exists\ y \in S \text{ s.t. } (x, y) \in G\big\}. \end{gathered} \tag{5.3}$$

One can write this much more geometrically in terms of the two projection maps

$$\begin{tikzcd} & \mathbb{R}^{2n} \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ \mathbb{R}^n & & \mathbb{R}^n. \end{tikzcd} \tag{5.4}$$

Thus $\pi_R(x, y) = y$, $\pi_L(x, y) = x$. Then (5.3) can be written in terms of the action of maps on sets as

$$G \circ S = \pi_L\left(\pi_R^{-1}(S) \cap G\right). \tag{5.5}$$

From this it follows that if S is compact and G is closed, then $G \circ S$ is closed, since its intersection with any compact set is the image of a compact set under a continuous map, hence compact. Now, by the *calculus of supports* we mean the 'trivial' result.

PROPOSITION 5.1. *If $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear map then*

$$\operatorname{supp}(A\phi) \subset \operatorname{supp}(A) \circ \operatorname{supp}(\phi)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.6}$$

PROOF. Since we want to *bound* $\operatorname{supp}(A\phi)$ we can use (5.1) directly, i.e. show that

$$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \Longrightarrow x \notin \operatorname{supp}(A\phi). \tag{5.7}$$

Since we know $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ to be closed, the assumption that x is outside this set means that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with

$$\psi(x) \neq 0 \text{ and } \operatorname{supp}(\psi) \cap \operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \emptyset.$$

From (5.3) or (5.5) this means

$$\operatorname{supp}(A)\cap(\operatorname{supp}(\psi)\times\operatorname{supp}(\phi))=\emptyset \text{ in } \mathbb{R}^{2n}. \tag{5.8}$$

But this certainly implies that

$$\begin{gathered} K_A(x,y)\psi(x)\phi(y)=0 \\ \Longrightarrow \psi A(\phi)=\int K_A(x,y)\psi(x)\phi(y)dy=0. \end{gathered} \tag{5.9}$$

Thus we have proved (5.6) and the lemma. □

Diff ops.

5.2. Singular supports

As well as the support of a tempered distribution we can consider the singular support:

$$\operatorname{sing\,supp}(u)=\left\{x\in\mathbb{R}^n;\exists\ \phi\in\mathcal{S}(\mathbb{R}^n),\phi(x)\neq 0,\phi u\in\mathcal{S}(\mathbb{R}^n)\right\}^{\complement}. \tag{5.10}$$

Again this is a closed set since $x\notin\operatorname{sing\,supp}(u)\Longrightarrow\exists\ \phi\in\mathcal{S}(\mathbb{R}^n)$ with $\phi u\in\mathcal{S}(\mathbb{R}^n)$ and $\phi(x)\neq 0$ so $\phi(x')\neq 0$ for $|x-x'|<\epsilon$, some $\epsilon>0$ and hence $x'\notin\operatorname{sing\,supp}(u)$ i.e. the complement of $\operatorname{sing\,supp}(u)$ is open.

Directly from the definition we have

$$\operatorname{sing\,supp}(u)\subset\operatorname{supp}(u)\ \forall\ u\in\mathcal{S}'(\mathbb{R}^n) \text{ and} \tag{5.11}$$

$$\operatorname{sing\,supp}(u)=\emptyset\Longleftrightarrow u\in\mathcal{C}^\infty(\mathbb{R}^n). \tag{5.12}$$

Examples

5.3. Pseudolocality

We would like to have a result like (5.6) for singular support, and indeed we can get one for pseudodifferential operators. First let us work out the singular support of the kernels of pseudodifferential operators.

Proposition 5.2. *If* $A\in\Psi^m_\infty(\mathbb{R}^n)$ *then*

$$\operatorname{sing\,supp}(A)=\operatorname{sing\,supp}(K_A)\subset\left\{(x,y)\in\mathbb{R}^{2n};x=y\right\}. \tag{5.13}$$

Proof. The kernel is defined by an oscillatory integral

$$I(a)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi. \tag{5.14}$$

If the order m is $<-n$ we can show by integration by parts that

$$(x-y)^\alpha I(a)=I\left((-D_\xi)^\alpha a\right), \tag{5.15}$$

and then this must hold by continuity for all orders. If a is of order m and $|\alpha|>m+n$ then $(-D_\xi)^\alpha a$ is of order less than $-n$, so

$$(x-y)^\alpha I(a)\in\mathcal{C}^0_\infty(\mathbb{R}^n),|\alpha|>m+n. \tag{5.16}$$

In fact we can also differentiate under the integral sign:

$$D^\beta_x D^\gamma_y(x-y)^\alpha I(a)=I\left(D^\beta_x D^\gamma_y(-D_\xi)^\alpha a\right) \tag{5.17}$$

so generalizing (5.16) to

$$(x-y)^\alpha I(a)\in\mathcal{C}^k_\infty(\mathbb{R}^n) \text{ if } |\alpha|>m+n+k. \tag{5.18}$$

This implies that $I(A)$ is $\mathcal{C}^\infty$ on the complement of the diagonal, $\{x=y\}$. This proves (5.13). □

An operator is said to be *pseudolocal* if it satisfies the condition

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{5.19}$$

PROPOSITION 5.3. *Pseudodifferential operators are pseudolocal.*

PROOF. Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\overline{x} \notin \operatorname{sing\,supp}(u)$. Then we can choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi \equiv 1$ near $\overline{x}$ and $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (by definition). Thus

$$u = u_1 + u_2,\ u_1 = (1-\phi)u, \quad u_2 \in \mathcal{S}(\mathbb{R}^n). \tag{5.20}$$

Since $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, $Au_2 \in \mathcal{S}(\mathbb{R}^n)$ so

$$\operatorname{sing\,supp}(Au) = \operatorname{sing\,supp}(Au_1) \text{ and } \overline{x} \notin \operatorname{supp}(u_1). \tag{5.21}$$

Choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $\psi(\overline{x}) = 1$ and

$$\operatorname{supp}(\psi) \cap \operatorname{supp}(1-\phi) = \emptyset. \tag{5.22}$$

Thus

$$\psi A u_1 = \psi A(1-\phi)u = \tilde{A}u \tag{5.23}$$

where

$$K_{\tilde{A}}(x,y) = \psi(x) K_A(x,y)(1-\phi(y)). \tag{5.24}$$

Combining (5.22) and (5.13) shows that $K_{\tilde{A}} \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ so, by Lemma 2.8, $\tilde{A}u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \operatorname{sing\,supp}(Au)$ by (5.13)(?). This proves the proposition. □

5.4. Coordinate invariance

If $\Omega \subset \mathbb{R}^n$ is an open set, put

$$\begin{aligned} \mathcal{C}_c^\infty(\Omega) &= \left\{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\right\} \\ \mathcal{C}_c^{-\infty}(\Omega) &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\right\} \end{aligned} \tag{5.25}$$

respectively the space of $\mathcal{C}^\infty$ functions of compact support in Ω and of distributions of compact support in Ω. Here $K \Subset \Omega$ indicates that K is a compact subset of Ω. Notice that if $u \in \mathcal{C}_c^{-\infty}(\Omega)$ then u defines a continuous linear functional

$$\mathcal{C}^\infty(\Omega) \ni \phi \longmapsto u(\phi) = u(\psi\phi) \in \mathbb{C} \tag{5.26}$$

where if $\psi \in \mathcal{C}_c^\infty(\Omega)$ is chosen to be identically one near $\operatorname{supp}(u)$ then (5.26) is independent of ψ. [Think about what continuity means here!]

Now suppose

$$F : \Omega \longrightarrow \Omega' \tag{5.27}$$

is a diffeomorphism between open sets of $\mathbb{R}^n$. The pull-back operation is

$$F^* :\ \mathcal{C}_c^\infty(\Omega') \longleftrightarrow \mathcal{C}_c^\infty(\Omega),\ F^*\phi = \phi \circ F. \tag{5.28}$$

LEMMA 5.1. *If F is a diffeomorphism,* (5.27)*, between open sets of $\mathbb{R}^n$ then there is an extension by continuity of* (5.28) *to*

$$F^* : \mathcal{C}_c^{-\infty}(\Omega') \longleftrightarrow \mathcal{C}_c^{-\infty}(\Omega). \tag{5.29}$$

PROOF. The density of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{C}_c^{-\infty}(\Omega)$, in the weak topology given by the seminorms from (5.26), can be proved in the same way as the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ (see Problem 5.5). Thus, we only need to show continuity of (5.29) in this sense. Suppose $u \in \mathcal{C}_c^\infty(\Omega')$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ then

$$
\begin{aligned}
(F^*u)(\phi) &= \int u(F(x))\phi(x)dx \\
&= \int u(y)\phi(G(y))|J_G(y)|dy
\end{aligned} \tag{5.30}
$$

where $J_G(y) = \left(\frac{\partial G(y)}{\partial y}\right)$ is the Jacobian of G, the inverse of F. Thus (5.28) can be written

$$F^*u(\phi) = (|J_G|u)\,(G^*\phi) \tag{5.31}$$

and since $G^* : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega')$ is continuous (!) we conclude that F^* is continuous as desired. $\square$

Now suppose that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

has

$$\operatorname{supp}(A) \Subset \Omega \times \Omega \subset \mathbb{R}^{2n}. \tag{5.32}$$

Then

$$A : \mathcal{C}_c^\infty(\Omega) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega) \tag{5.33}$$

by Proposition 5.1. Applying a diffeomorphism, F, as in (5.27) set

$$A_F : \mathcal{C}_c^\infty(\Omega') \longrightarrow \mathcal{C}_c^{-\infty}(\Omega'),\ A_F = G^* \circ A \circ F^*. \tag{5.34}$$

LEMMA 5.2. *If A satisfies (5.32) and F is a diffeomorphism (5.27) then*

$$K_{A_F}(x,y) = (G \times G)^* K \cdot |J_G(y)| \text{ on } \Omega' \times \Omega' \tag{5.35}$$

has compact support in $\Omega' \times \Omega'$.

PROOF. Essentially the same as that of (5.30). $\square$

PROPOSITION 5.4. *Suppose $A \in \Psi_\infty^m(\mathbb{R}^n)$ has kernel satisfying (5.32) and F is a diffeomorphism as in (5.27) then A_F, defined by (5.34), is an element of $\Psi_\infty^m(\mathbb{R}^n)$.*

PROOF. See Proposition 2.11. $\square$

5.5. Problems

PROBLEM 5.1. Show that *Weyl quantization*

$$S_\infty^\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto q_W(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)d\xi \tag{5.36}$$

is well-defined by continuity from $S_\infty^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$ and induces an isomorphism

$$S_\infty^m(\mathbb{R}^n;\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi_\infty^m(\mathbb{R}^n)\ \forall\ m \in \mathbb{R}. \tag{5.37}$$

Find an asymptotic formula relating $q_W(A)$ to $q_L(A)$ for any $A \in \Psi_\infty^m(\mathbb{R}^n)$.

PROBLEM 5.2. Show that if $A \in \Psi_\infty^m(\mathbb{R}^n)$ then $A^* = A$ if and only if $\sigma_W(A)$ is real-valued.

Problem 5.3. Is it true that every $E \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ defines a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$?

Problem 5.4. Show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ by proving that if $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ has compact support and is identically equal to 1 near the origin then

$$u_n(x) = (2\pi)^{-n}\phi(\frac{x}{n})\int e^{ix\cdot\xi}\phi(\xi/n)\hat{u}(\xi)d\xi \in \mathcal{S}(\mathbb{R}^n) \text{ if } u \in L^2(\mathbb{R}^n) \tag{5.38}$$

and $u_n \to u$ in $L^2(\mathbb{R}^n)$. Can you see any relation to pseudodifferential operators here?

Problem 5.5. Check carefully that with the definition

$$H^k(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\alpha|\le -k} D^\alpha u_\alpha,\ u_\alpha \in L^2(\mathbb{R}^n) \right\} \tag{5.39}$$

for $-k \in \mathbb{N}$ one does have

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n) \tag{5.40}$$

as claimed in the text.

Problem 5.6. Suppose that $a(x) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$ and that $a(x) \ge 0$. Show that the operator

$$A = \sum_{j=1}^n D^2_{x_j} + a(x) \tag{5.41}$$

can have no solution which is in $L^2(\mathbb{R}^n)$.

Problem 5.7. Show that for any open set $\Omega \subset \mathbb{R}^n$, $\mathcal{C}^\infty_c(\Omega)$ is dense in $\mathcal{C}^{-\infty}_c(\Omega)$ in the weak topology.

Problem 5.8. Use formula (2.204) to find the principal symbol of A_F; more precisely show that if $F^* : T^*\Omega' \longrightarrow T^*\omega$ is the (co)-differential of F then

$$\sigma_m(A_F) = \sigma_m(A) \circ F^*.$$

We have now studied special distributions, the Schwartz kernels of pseudodifferential operators. We shall now apply this knowledge to the study of general distributions. In particular we shall examine the wavefront set, a refinement of singular support, of general distributions. This notion is fundamental to the general idea of 'microlocalization.'

5.6. Characteristic variety

If $A \in \Psi^m_\infty(\mathbb{R}^n)$, the left-reduced symbol is elliptic at $(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ if there exists $\epsilon > 0$ such that

$$\begin{gathered} \big|\sigma_L(A)(x,\xi)\big| \ge \epsilon|\xi|^m \quad \text{in} \\ \big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,;|x-\overline{x}| \le \epsilon,\ \big|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\big| \le \epsilon,\ |\xi| \ge \frac{1}{\epsilon}\big\}. \end{gathered} \tag{5.42}$$

Directly from the definition, ellipticity at $(\overline{x}, \overline{\xi})$ is actually a property of the principal symbol, $\sigma_m(A)$ and if A is elliptic at $(\overline{x}, \overline{\xi})$ then it is elliptic at $(\overline{x}, t\overline{\xi})$ for any $t > 0$. Clearly

$$\big\{(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\});\ A \text{ is elliptic (of order } m) \text{ at } (\overline{x}, \overline{\xi})\big\}$$

is an *open* cone in $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$. The complement

$$\Sigma_m(A) = \big\{(\overline{x},\overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,;\ A \text{ is } \textit{not} \text{ elliptic of order } m \text{ at } (\overline{x},\overline{\xi})\big\} \tag{5.43}$$

is therefore a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$; it is the characteristic set (or variety) of A. Since the product of two symbols is only elliptic at $(\overline{x},\overline{\xi})$ if they are both elliptic there, if follows from the composition properties of pseudodifferential operators that

$$\Sigma_{m+m'}(A \circ B) = \Sigma_m(A) \cup \Sigma_{m'}(B). \tag{5.44}$$

5.7. Wavefront set

We adopt the following bald definition:

$$\begin{gathered}\text{If } u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n);\ \operatorname{supp}(u) \Subset \mathbb{R}^n\big\} \text{ then}\\ \operatorname{WF}(u) = \bigcap\big\{\Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n)\big\}.\end{gathered} \tag{5.45}$$

Thus $\operatorname{WF}(u) \subset \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ is always a closed conic set, being the intersection of such sets. The first thing we wish to show is that $\operatorname{WF}(u)$ is a refinement of $\operatorname{sing\,supp}(u)$. Let

$$\pi :\ \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}) \ni (x,\xi) \longmapsto x \in \mathbb{R}^n \tag{5.46}$$

be projection onto the first factor.

Proposition 5.5. *If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ then*

$$\pi(\operatorname{WF}(u)) = \operatorname{sing\,supp}(u). \tag{5.47}$$

Proof. The inclusion $\pi(\operatorname{WF}(u)) \subset \operatorname{sing\,supp}(w)$ is straightforward. Indeed, if $\overline{x} \notin \operatorname{sing\,supp}(u)$ then there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that $\phi u \in \mathcal{C}^\infty(\mathbb{R}^n)$. Of course as a multiplication operator, $\phi \in \Psi^0_\infty(\mathbb{R}^n)$ and $\Sigma_0(\phi) \not\ni (\overline{x},\overline{\xi})$ for any $\overline{\xi} \neq 0$. Thus the definition (5.45) shows that $(\overline{x},\overline{\xi}) \notin \operatorname{WF}(u)$ for all $\overline{\xi} \in \mathbb{R}^n\diagdown 0$ proving the inclusion.

Using the calculus of pseudodifferential operators, the opposite inclusion,

$$\pi(\operatorname{WF}(u)) \supset \operatorname{sing\,supp}(u) \tag{5.48}$$

is only a little more complicated. Thus we have to show that if $(\overline{x},\overline{\xi}) \notin \operatorname{WF}(u)$ for all $\overline{\xi} \in \mathbb{R}^n\diagdown 0$ then $\overline{x} \notin \operatorname{sing\,supp}(u)$. The hypothesis is that for each $(\overline{x},\overline{\xi}), \overline{\xi} \in \mathbb{R}^n\diagdown 0$, there exists $A \in \Psi^0_\infty(\mathbb{R}^n)$ such that A is elliptic at $(\overline{x},\overline{\xi})$ and $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. The set of elliptic points is open so there exists $\epsilon = \epsilon(\overline{\xi}) > 0$ such that A is elliptic on

$$\big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\diagdown 0); |x-\overline{x}| < \epsilon, \Big|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\Big| < \epsilon\big\}. \tag{5.49}$$

Let B_j, $j = 1,\dots,N$ be a finite set of such operators associated to $\overline{\xi}_j$ and such that the corresponding sets in (5.49) cover $\{\overline{x}\} \times (\mathbb{R}^n\diagdown 0)$; the finiteness follows from the compactness of the sphere. Then consider

$$B = \sum_{j=1}^N B_j^* B_j \Longrightarrow Bu \in \mathcal{C}^\infty(\mathbb{R}^n).$$

This operator B is elliptic at $(\overline{x}, \xi)$, for all $\xi \neq 0$. Thus if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $0 \leq \phi(x) \leq 1$, has support sufficiently close to $\overline{x}$, $\phi(x) = 1$ in $|x - \overline{x}| < \epsilon/2$ then, since B has non-negative principal symbol

$$B + (1 - \phi) \in \Psi^0_\infty(\mathbb{R}^n) \tag{5.50}$$

is globally elliptic. Thus, by Lemma 2.7, there exists $G \in \Psi^0_\infty(\mathbb{R}^n)$ which is a parametrix for $B + (1 - \phi)$:

$$\mathrm{Id} \equiv G \circ B + G(1 - \phi) \mod \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{5.51}$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}(\psi) \subset \{\phi = 1\}$ and $\psi(\overline{x}) \neq 0$. Then, from the reduction formula

$$\psi \circ G \circ (1 - \phi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Thus from (5.51) we find

$$\psi u = \psi G \circ B u + \psi G(1 - \phi) u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Thus $\overline{x} \notin \operatorname{sing\,supp}(u)$ and the proposition is proved. $\square$

We extend the definition to general tempered distributions by setting

$$\mathrm{WF}(u) = \bigcup_{\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\phi u),\ u \in \mathcal{S}'(\mathbb{R}^n). \tag{5.52}$$

Then (5.47) holds for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

5.8. Essential support

Next we shall consider the notion of the essential support of a pseudodifferential operator. If $a \in S^m_\infty(\mathbb{R}^N; \mathbb{R}^n)$ we define the cone support of a by

$$\begin{aligned} \operatorname{cone\,supp}(a) = \big\{ &(\overline{x}, \overline{\xi}) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus 0); \exists\, \epsilon > 0 \text{ and } \forall\, M \in \mathbb{R}, \exists\, C_M \text{ s.t.} \\ &|a(x, \xi)| \leq C_M \langle \xi \rangle^{-M} \text{ if } |x - \overline{x}| \leq \epsilon, \Big| \frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|} \Big| \leq \epsilon \big\}^{\complement}. \end{aligned} \tag{5.53}$$

This is clearly a closed conic set in $\mathbb{R}^N \times (\mathbb{R}^n \setminus 0)$. By definition the symbol decays rapidly outside this cone, in fact even more is true.

Lemma 5.3. *If $a \in S^\infty_\infty(\mathbb{R}^N; \mathbb{R}^n)$ then*

$$\begin{aligned} &(\overline{x}, \overline{\eta}) \notin \operatorname{cone\,supp}(a) \Longrightarrow \\ &\exists\, \epsilon > 0 \textit{ s.t. } \forall\, M, \alpha, \beta\ \exists\, C_M \textit{ with} \\ &\big| D_x^\alpha D_\xi^\beta a(x, \eta) \big| \leq C_M \langle \eta \rangle^{-M} \textit{ if } |x - \overline{x}| < \epsilon, \Big| \frac{\eta}{|\eta|} - \frac{\overline{\eta}}{|\overline{\eta}|} \Big| < \epsilon. \end{aligned} \tag{5.54}$$

Proof. To prove (5.54) it suffices to show it to be valid for $D_{x_j} a$, $D_{\xi_k} a$ and then use an inductive argument, i.e. to show that

$$\operatorname{cone\,supp}(D_{x_j} a), \operatorname{cone\,supp}(D_{\xi_k} a) \subset \operatorname{cone\,supp}(a). \tag{5.55}$$

Arguing by contradiction suppose that $D_{x_\ell} a$ does not decay to order M in any cone around $(\overline{x}, \overline{\xi}) \notin \operatorname{cone\,supp}$. Then there exists a sequence (x_j, ξ_j) with

$$\begin{cases} x_j \longrightarrow \overline{x},\ \big| \frac{\xi_j}{|\xi_j|} - \frac{\overline{\xi}}{|\overline{\xi}|} \big| \longrightarrow 0,\ |\xi_j| \longrightarrow \infty \\ \text{and } \big| D_{x_\ell} a(x_j, \xi_j) \big| > j \langle \xi_j \rangle^M. \end{cases} \tag{5.56}$$

We can assume that $M < m$, since $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^N\right)$. Applying Taylor's formula with remainder, and using the symbol bounds on $D^2_{x_j}a$, gives

$$a(x_j + te_\ell, \xi_j) = a(x_j, \xi_j) + it(D_{x_j}a)(x_j, \xi_j) + O\left(t^2\langle\xi_j\rangle^m\right),\ (e_\ell)_j = \delta_{\ell j} \tag{5.57}$$

providing $|t| < 1$. Taking $t = \langle\xi_j\rangle^{M-m} \longrightarrow 0$ as $j \longrightarrow \infty$, the first and third terms on the right in (5.57) are small compared to the second, so

$$\left|a\left(x_j + \langle\xi_j\rangle^{\frac{M-m}{2}}, \xi_j\right)\right| > \langle\xi_j\rangle^{2M-m}, \tag{5.58}$$

contradicting the assumption that $(\overline{x}, \overline{\xi}) \notin \operatorname{cone\,supp}(a)$. A similar argument applies to $D_{\xi_\ell}a$ so (5.54), and hence the lemma, is proved. $\square$

For a pseudodifferential operator we define the *essential support* by

$$\mathrm{WF}'(A) = \operatorname{cone\,supp}\left(\sigma_L(A)\right) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)\,. \tag{5.59}$$

Lemma 5.4. *For every* $A \in \Psi^m_\infty(\mathbb{R}^n)$

$$\mathrm{WF}'(A) = \operatorname{cone\,supp}(\sigma_R(A)). \tag{5.60}$$

Proof. Using (5.54) and the formula relating $\sigma_R(A)$ to $\sigma_L(A)$ we conclude that

$$\operatorname{cone\,supp}(\sigma_L(A)) = \operatorname{cone\,supp}(\sigma_R(A)), \tag{5.61}$$

from which (5.60) follows. $\square$

A similar argument shows that

$$\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B). \tag{5.62}$$

Indeed the asymptotic formula for $\sigma_L(A \circ B)$ in terms of $\sigma_L(A)$ and $\sigma_L(B)$ shows that

$$\operatorname{cone\,supp}(\sigma_L(A \circ B)) \subset \operatorname{cone\,supp}\left(\sigma_L(A)\right) \cap \operatorname{cone\,supp}\left(\sigma_L(B)\right) \tag{5.63}$$

which is the same thing.

5.9. Microlocal parametrices

The concept of essential support allows us to refine the notion of a parametrix for an elliptic operator to that of a *microlocal parametrix.*

Lemma 5.5. *If* $A \in \Psi^m_\infty(\mathbb{R}^n)$ *and* $z \notin \Sigma_m(A)$ *then there exists a microlocal parametrix at* z, $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$ *such that*

$$z \notin \mathrm{WF}'(\mathrm{Id} - AB) \text{ and } z \notin \mathrm{WF}'(\mathrm{Id} - BA). \tag{5.64}$$

Proof. If $z = (\overline{x}, \overline{\xi})$, $\overline{\xi} \neq 0$, consider the symbol

$$\gamma_\epsilon(x, \xi) = \phi\left(\frac{x - \overline{x}}{\epsilon}\right)(1 - \phi)(\epsilon\xi)\phi\left(\left(\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\right)/\epsilon\right) \tag{5.65}$$

where as usual $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(\zeta) = 1$ in $|\zeta| \leq \frac{1}{2}$, $\phi(\zeta) = 0$ in $|\zeta| \geq 1$. Thus $\gamma_\epsilon \in S^0_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)$ has support in

$$|x - \overline{x}| \leq \epsilon,\ |\xi| \geq \frac{1}{2\epsilon},\ \left|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\right| \leq \epsilon \tag{5.66}$$

and is identically equal to one, and hence elliptic, on a similar smaller set

$$|x-\overline{x}|<\frac{\epsilon}{2}, |\xi|\geq\frac{1}{\epsilon},\ \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\leq\frac{\epsilon}{2}. \tag{5.67}$$

Define $L_\epsilon\in\Psi^0_\infty(\mathbb{R}^n)$ by $\sigma_L(L_\epsilon)=\gamma_\epsilon$. Thus, for any $\epsilon>0$,

$$z\notin\mathrm{WF}'(\mathrm{Id}-L_\epsilon),\ \mathrm{WF}'(L_\epsilon)\subset\left\{(x,\xi);|x-\overline{x}|\leq\epsilon\text{ and }\left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\leq\epsilon\right\}. \tag{5.68}$$

Let $G_{2m}\in\Psi^{2m}_\infty(\mathbb{R}^n)$ be a globally elliptic operator with positive principal symbol. For example take $\sigma_L(G_{2m})=(1+|\xi|^2)^m$, so $G_s\circ G_t=G_{s+t}$ for any s, $t\in\mathbb{R}$. Now consider the operator

$$J=(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A^*A\in\Psi^{2m}_\infty(\mathbb{R}^n). \tag{5.69}$$

The principal symbol of J is $(1-\gamma_\epsilon)(1+|\xi|^2)^m+|\sigma_m(A)|^2$ which is globally elliptic if $\epsilon>0$ is small enough (so that $\sigma_m(A)$ is elliptic on the set (5.66)). According to Lemma 2.75, J has a global parametrix $H\in\Psi^{-2m}_\infty(\mathbb{R}^n)$. Then

$$B=H\circ A^*\in\Psi^{-m}_\infty(\mathbb{R}^n) \tag{5.70}$$

is a microlocal right parametrix for A in the sense that $B\circ A-\mathrm{Id}=R_R$ with $z\notin\mathrm{WF}'(R_R)$ since

$$\begin{aligned}R_R=B\circ A-\mathrm{Id}&=H\circ A^*\circ A-\mathrm{Id}\\&=(H\circ J-\mathrm{Id})+H\circ(\mathrm{Id}-L_\epsilon)G_{2m}\circ A\end{aligned} \tag{5.71}$$

and the first term on the right is in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ whilst z is not in the operator wavefront set of $(\mathrm{Id}-L_\epsilon)$ and hence not in the operator wavefront set of the second term.

By a completely analogous construction we can find a left microlocal parametrix. Namely $(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A\circ A^*$ is also globally elliptic with parametrix H' and then $B'=A^*\circ H'$ satisfies

$$B'\circ A-\mathrm{Id}=R_L,\ z\notin\mathrm{WF}'(R_L). \tag{5.72}$$

Then, as usual,

$$B=(B'\circ A-R_L)\,B=B'\,(A\circ B)-R_LB=B'+B'R_R-R_LB \tag{5.73}$$

so $z\notin\mathrm{WF}'(B-B')$, which implies that B is both a left and right microlocal parametrix. □

In fact this argument shows that such a left parametrix is essentially unique. See Problem 5.29.

5.10. Microlocality

Now we can consider the relationship between these two notions of wavefront set.

Proposition 5.6. *Pseudodifferential operators are* microlocal *in the sense that*

$$\mathrm{WF}(Au)\subset\mathrm{WF}'(A)\cap\mathrm{WF}(u)\quad\forall\ A\in\Psi^\infty_\infty(\mathbb{R}^n),\ u\in\mathcal{C}^{-\infty}_c(\mathbb{R}^n). \tag{5.74}$$

Proof. We need to show that

$$\operatorname{WF}(Au) \subset \operatorname{WF}'(A) \text{ and } \operatorname{WF}(Au) \subset \operatorname{WF}(u). \tag{5.75}$$

the second being the usual definition of microlocality. The first inclusion is easy. Suppose $(\overline{x}, \overline{\xi}) \notin \operatorname{cone\,supp} \sigma_L(A)$. If we choose $B \in \Psi^0_\infty(\mathbb{R}^n)$ with $\sigma_L(B)$ supported in a small cone around $(\overline{x}, \overline{\xi})$ then we can arrange

$$(\overline{x}, \overline{\xi}) \notin \Sigma_0(B),\ \operatorname{WF}'(B) \cap \operatorname{WF}'(A) = \emptyset. \tag{5.76}$$

Then from (5.62), $\operatorname{WF}'(BA) = \emptyset$ so $BA \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ and $BAu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(Au)$.

Similarly suppose $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(u)$. Then there exists $G \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\overline{x}, \overline{\xi})$ with $Gu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Let B be a microlocal parametrix for G at $(\overline{x}, \overline{\xi})$ as in Lemma 5.5. Thus

$$u = BGu + Su,\ (\overline{x}, \overline{\xi}) \notin \operatorname{WF}'(S). \tag{5.77}$$

Now apply A to this identity. Since, by assumption, $Gu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ the first term on the right in

$$Au = ABGu + ASu \tag{5.78}$$

is smooth. Since, by (5.62), $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}'(AS)$ it follows from the first part of the argument above that $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(ASu)$ and hence $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(Au)$. □

We can deduce from the existence of microlocal parametrices at elliptic points a partial converse of (8.24).

Proposition 5.7. *For any $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and any $A \in \Psi^m_\infty(\mathbb{R}^n)$*

$$\operatorname{WF}(u) \subset \operatorname{WF}(Au) \cup \Sigma_m(A). \tag{5.79}$$

Proof. If $(\overline{x}, \overline{\xi}) \notin \Sigma_m(A)$ then, by definition, A is elliptic at $(\overline{x}, \overline{\xi})$. Thus, by Lemma 5.5, A has a microlocal parametrix B, so

$$u = BAu + Su,\ (\overline{x}, \overline{\xi}) \notin \operatorname{WF}'(S). \tag{5.80}$$

It follows that $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(Au)$ implies that $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(u)$ proving the Proposition. □

5.11. Explicit formulations

From this discussion of $\operatorname{WF}'(A)$ we can easily find a 'local coordinate' formulations of $\operatorname{WF}(u)$ in general.

Lemma 5.6. *If $(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ then $(\overline{x}, \overline{\xi}) \notin \operatorname{WF}(u)$ if and only if there exists $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that for some $\epsilon > 0$, and for all M there exists C_M with*

$$\big|\widehat{\phi u}(\xi)\big| \le C_M \langle\xi\rangle^M \ \text{in}\ \big|\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|}\big| < \epsilon. \tag{5.81}$$

Proof. If $\zeta \in \mathcal{C}^\infty(\mathbb{R})$, $\zeta(\xi) \equiv 1$ in $|\xi| < \frac{\epsilon}{2}$ and $\operatorname{supp}(\zeta) \subset \left[\frac{-3\epsilon}{4}, \frac{3\epsilon}{4}\right]$ then

$$\gamma(\xi) = (1-\zeta)(\xi) \cdot \zeta\big(\frac{\xi}{|\xi|} - \frac{\overline{x}}{|\overline{x}|}\big) \in S^0_\infty(\mathbb{R}^n) \tag{5.82}$$

is elliptic at $\bar{\xi}$ and from (5.81)

$$\gamma(\xi) \cdot \widehat{\phi u}(\xi) \in \mathcal{S}(\mathbb{R}^n). \tag{5.83}$$

Thus if $\sigma_R(A) = \phi_1(x)\gamma(\xi)$ then $A(\phi_2 u) \in \mathcal{C}^\infty$ where $\phi_1\phi_2 = \phi$, $\phi_1(\overline{x}), \phi_2(\overline{x}) \neq 0$, $\phi_1, \phi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Thus $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$. Conversely if $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ and A is chosen as above then $A(\phi_1 u) \in \mathcal{S}(\mathbb{R}^n)$ and Lemma 5.6 holds. □

5.12. Wavefront set of K_A

At this stage, a natural thing to look at is the wavefront set of the kernel of a pseudodifferential operator, since these kernels are certainly an interesting class of distributions.

PROPOSITION 5.8. *If $A \in \Psi_\infty^m(\mathbb{R}^n)$ then*

$$\begin{aligned}\mathrm{WF}(K_A) = \Big\{ &(x, y, \xi, \eta) \in \mathbb{R}^{2n} \times \left(\mathbb{R}^{2n} \setminus 0\right); \\ &x = y,\ \xi + \eta = 0 \text{ and } (x, \xi) \in \mathrm{WF}'(A)\Big\}.\end{aligned} \tag{5.84}$$

In particular this shows that $\mathrm{WF}'(A)$ determines $\mathrm{WF}(K_A)$ and conversely.

PROOF. Using Proposition 5.5 we know that $\pi\left(\mathrm{WF}(K_A)\right) \subset \big\{(x, x)\big\}$ so

$$\mathrm{WF}(K_A) \subset \big\{(x, x; \xi, \eta)\big\}.$$

To find the wave front set more precisely consider the kernel

$$K_A(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x, \xi) d\xi$$

where we can assume $|x - y| < 1$ on $\operatorname{supp}(K_A)$. Thus is $\phi \in \mathcal{C}_c^\infty(X)$ then

$$g(x, y) = K_A(x, y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$$

and

$$\begin{aligned}\hat{g}(\zeta, \eta) &= (2\pi)^{-n} \int e^{-i\zeta x - i\eta y} e^{i(x-y)\cdot\zeta} (\phi b)(x, \xi) d\zeta dx dy \\ &= \int e^{-i(\zeta+\eta)\cdot x} (\phi b)(x, -\eta) dx \\ &= \widehat{\phi b}(\zeta + \eta, -\eta).\end{aligned}$$

The fact that ϕb is a symbol of compact support in x means that for every M

$$\left|\widehat{\phi b}(\zeta + \eta, -\eta)\right| \leq C_M \left(\langle \zeta + \eta \rangle\right)^{-M} \langle \eta \rangle^m.$$

This is rapidly decreasing if $\zeta \neq -\eta$, so

$$\mathrm{WF}(K_A) \subset \big\{(x, x, \eta, -\eta)\big\} \text{ as claimed.}$$

Moreover if $(\overline{x}, \overline{\eta}) \notin \mathrm{WF}'(A)$ then choosing ϕ to have small support near $\overline{x}$ makes $\widehat{\phi b}$ rapidly decreasing near $-\overline{\eta}$ for all ζ. This proves Proposition 5.8. □

5.13. Hypersurfaces and Hamilton vector fields

In the Hamiltonian formulation of classical mechanics the dynamical behaviour of a 'particle' is fixed by the choice of an energy function ('the Hamiltonian') $h(x, \xi)$ depending on the position and momentum vectors (both in $\mathbb{R}^3$ you might think, but maybe in $\mathbb{R}^{3N}$ because there are really N particles). In fact one can think of a system confined to a surface in which case the variables are in the cotangent bundle of a manifold. However, in the local coordinate description the motion of the particle is given by Hamilton's equations:-

$$\frac{dx_i}{dt} = \frac{\partial h}{\partial \xi_i}(x,\xi),\ \frac{d\xi_i}{dt} = -\frac{\partial h}{\partial x_i}(x,\xi). \tag{5.85}$$

This means that the trajectory $(x(t),\xi(t))$ of a particle is an integral curve of the vector field

$$H_h(x,\xi) = \sum_i \left(\frac{\partial h}{\partial \xi_i}(x,\xi)\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}(x,\xi)\frac{\partial}{\partial \xi_i}(x,\xi) \right). \tag{5.86}$$

This, of course, is called the Hamilton vector field of h. The most important basic fact is that h itself is constant along integral curves of H_h, namely

$$H_h h = \sum_i \left(\frac{\partial}{\partial h \xi_i}(x,\xi)\frac{\partial h}{\partial x_i}(x,\xi) - \frac{\partial h}{\partial x_i}(x,\xi)\frac{\partial h}{\partial \xi_i}(x,\xi)h(x,\xi) \right) = 0. \tag{5.87}$$

More generally the action of H_h on any other function defines the Poisson bracket between h and g and

$$H_h g = \{h,g\} = -\{g,h\} = -H_g h \tag{5.88}$$

from which (5.87) again follows. See Problem 5.18.

More invariantly the Hamilton vector can be constructed using the symplectic form

$$\omega = \sum_i d\xi_i \wedge dx_i = d\alpha,\ \alpha = \sum_i \xi_i dx_i. \tag{5.89}$$

Here α is the 'tautological' 1-form. If we think of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi = (x,\xi)'$ as the pull back under $\pi : (x,\xi) \longmapsto x$ of β as a 1-covector on $\mathbb{R}^n$. In this sense the tautological form α is well defined on the cotangent bundle of any manifold and has the property that if one introduces local coordinates in the manifold x and the canonically dual coordinates in the cotangent bundle (by identifying a 1-covector as $\xi \cdot dx$) then it takes the form of α in (5.89). Thus the symplectic form, as $d\alpha$, is well-defined on T^*X for any manifold X.

Returning to the local discussion it follows directly from (5.86) that

$$\omega(\cdot, H_h) = dh(\cdot) \tag{5.90}$$

and conversely this determines H_h. See Problem 5.19.

Now, we wish to apply this discussion of 'Hamiltonian mechanics' to the case that $h = p(x,\xi)$ is the principal symbol of some pseudodifferential operator. We shall in fact take p to be homogeneous of degree m (later normalized to 1) in $|\xi| > 1$. That is,

$$p(x,s\xi) = s^m p(x,\xi)\ \forall\ x \in \mathbb{R}^n,\ |\xi| \geq 1, s|\xi| \geq 1,\ s > 0. \tag{5.91}$$

The effect of this is to ensure that

$$H_p \text{ is homogeneous of degree } m-1 \text{ under } (x,\xi) \longmapsto (x,s\xi) \tag{5.92}$$

in the same region. One consequence of this is that

$$H_p : S^M_{\mathrm{c}}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow S^{M-1}_{\mathrm{c}}(\mathbb{R}^n;\mathbb{R}^n). \tag{5.93}$$

(where the subscript 'c' just means supports are compact in the first variable). To see this it is convenient to again rewrite the definition of symbol spaces. Since supports are compact in x we are just requiring uniform smoothness in those variables. Thus, we are first requiring that symbols be smooth. Now, consider any point $\bar{\xi} \neq 0$. Thus $\bar{\xi}_j \neq 0$ for some j and we can consider a conic region around $\bar{\xi}$ of the form

$$\xi_j/\bar{\xi}_j \in (0, \infty),\ |\xi_k/\xi_j - \bar{\xi}_j/\bar{\xi}_j| < \epsilon \tag{5.94}$$

where $\epsilon > 0$ is small. Then the symbolic conditions on $a \in S^M_{\mathrm{c}}(\mathbb{R}^n; \mathbb{R}^n)$ imply

$$\begin{gathered} b(x, t, r) = a(x, rt_1, \dots, rt_{j-1}, r \operatorname{sgn} \bar{\xi}_j, rt_j, \dots, rt_{n-1}) \\ \text{satisfies} |D^\alpha_x D^\gamma_t D^k_r b(x, t, r)| \leq C_{\alpha,\gamma,k} r^{M-k} \text{ in } r \geq 1. \end{gathered} \tag{5.95}$$

See Problem 5.20.

For the case of a homogeneous function (away from $\xi = 0$) such as p the surface $\Sigma_m(P) = \{p = 0\}$ has already been called the 'characteristic variety' above. Correspondingly the integral curves of H_p on $\Sigma_m(p)$ (so the ones on which p vanishes) are called null bicharacteristics, or sometimes just bicharacteristics. Note that $\Sigma_m(P)$ may well have singularities, since dp may vanish somewhere. However this is not a problem with the general discussion, since H_p vanishes at such points – and it is only singular in this sense of vanishing. The integral curves through such a point are necessarily constant.

Now we are in a position to state at least a local form of the propagation theorem for operators of 'real principal type'. This means $dp \neq 0$, and in fact even more, that dp and α are linearly independent. The theorems below in fact apply in general when p is real even if there are points where dp is a multiple of α – they just give no information in those cases.

Theorem 5.1 (Hörmander's propagation theorem, local version). *Suppose $P \in \Psi^m_\infty(M)$ has real principal symbol homogeneous of degree m, that $c : (a, b) \longrightarrow \Sigma_m(P)$ is an interval of a null bicharacteristic curve (meaning $c_*(\frac{d}{dt}) = H_p$) and that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$$c(a, b) \cap \mathrm{WF}(Pu) = \emptyset \tag{5.96}$$

then

$$\begin{cases} \textit{either} & c(a, b) \cap \mathrm{WF}(u) = \emptyset \\ \textit{or} & c(a, b) \subset \mathrm{WF}(u). \end{cases} \tag{5.97}$$

5.14. Relative wavefront set

Although we could proceed directly by induction over the (Sobolev) order of regularity to prove a result such as Theorem 5.1 it is probably better to divide up the proof a little. To do this we can introduce a refinement of the notion of wavefront set, which is actually the wavefront set relative to a Sobolev space. So, fixing $s \in \mathbb{R}$ we can simply define by direct analogy with (5.45)

$$\mathrm{WF}_s(u) = \bigcap \left\{ \Sigma_0(A); A \in \Psi^0_\infty(\mathbb{R}^n); Au \in H^s(\mathbb{R}^n) \right\},\ u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n). \tag{5.98}$$

Notice that this would not be a very good definition if extended directly to $u \in \mathcal{S}'(\mathbb{R}^n)$ if we want to think of it as only involving local regularity (because growth of u might stop Au from being in $H^s(\mathbb{R}^n)$ even if it is smooth). So we will just localize the definition in general

$$\mathrm{WF}_s(u) = \bigcap \left\{\Sigma_0(A); A \in \Psi^0_\infty(\mathbb{R}^n); A(\psi u) \in H^s(\mathbb{R}^n)\ \forall\ \psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\right\}, \quad u \in \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{5.99}$$

In this sense the regularity is with respect to $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ – is purely local.

Lemma 5.7. *If $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ then $\mathrm{WF}_s(u) = \emptyset$ if and only if $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. The same proof as in the case of the original wavefront set works, only now we need to use Sobolev boundedness as well. Certainly if $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ then $\psi u \in H^s(\mathbb{R}^n)$ for each $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and hence $A(\psi u) \in H^s(\mathbb{R}^n)$ for every $A \in \Psi^0_\infty(\mathbb{R}^n)$. Thus $\mathrm{WF}_s(u) = \emptyset$.

Conversely if $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and $\mathrm{WF}_s(u) = \emptyset$ then for each point (x, ξ) with $x \in \mathrm{supp}(u)$ and $|\xi| = 1$ there exists $A_{x,\xi} \in \Psi^0_\infty(\mathbb{R}^n)$ such that $Au \in H^s(\mathbb{R}^n)$ with $(x, \xi) \notin \Sigma_0(A_{x,\xi})$. That is $A_{(x,\xi)}$ is elliptic at (x, ξ). By compactness (given the conic property of the elliptic set) a finite collection $A_i = A_{(x_i,\xi_i)}$ have the property that the union of their elliptic sets cover some set $K \times (\mathbb{R}^n \setminus 0)$ where K is compact and $\mathrm{supp}(u)$ is contained in the interior of K. We can then choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $0 \le \phi \le 1$, $\mathrm{supp}(\phi) \subset K$ and $\phi = 1$ on $\mathrm{supp}(u)$ and

$$B = (1 - \phi) + \sum_i A_i^* A_u \in \Psi^0_\infty(\mathbb{R}^n)$$

is globally elliptic in $\Psi^0_\infty(\mathbb{R}^n)$ and $Bu \in H^s(\mathbb{R}^n)$ by construction (since $(1 - \phi)u = 0$). Thus $u \in H^s(\mathbb{R}^n)$. Applying this argument to ψu for each $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ for $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ we see that $\mathrm{WF}_s(u) = \emptyset$ implies $\psi u \in H^s(\mathbb{R}^n)$ and hence $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$. $\square$

Of course if $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ then $\mathrm{WF}_s(u) = \emptyset$ is equivalent to $u \in H^s(\mathbb{R}^n)$.

It also follows directly from this definition that pseudodifferential operators are 'appropriately' microlocal given their order.

Lemma 5.8. *If $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ then*

$$\mathrm{WF}(u) \supset \bigcup_s \mathrm{WF}_s(u). \tag{5.100}$$

and coversely if $\gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is an open cone then

$$\gamma \cap \mathrm{WF}_s(u) = \emptyset\ \forall\ s \Longrightarrow \gamma \cap \mathrm{WF}(u) = \emptyset. \tag{5.101}$$

The combination of these two statements is that

$$\mathrm{WF}(u) = \overline{\bigcup_s \mathrm{WF}_s(u)}. \tag{5.102}$$

Note that there is not in general equality in (5.100).

Proof. If $(\bar{x}, \bar{\xi}) \in \mathrm{WF}_s(u)$ for some s then by definition there exists $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\psi(\bar{x}) \neq 0$ and $A \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\bar{x}, \bar{\xi})$ and is such that $A(\psi u) \notin H^s(\mathbb{R}^n)$. This certainly implies that $(\bar{x}, \bar{\xi}) \in \mathrm{WF}(u)$ proving (5.100).

To prove the partial converse if suffices to assume that $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and to fix a point $(\bar{x}, \bar{\xi}) \in \gamma$ and deduce from (5.101) that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$. Since γ is an open cone we may choose $\epsilon > 0$ such that $G = \{(x, \xi); |x - \bar{x}| \le \epsilon, |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{\bar{\xi}}| \le \epsilon\} \subset \gamma$. Now for each s the covering argument in the proof of Lemma 5.7 shows that we may find $A_s \in \Psi^0_\infty(\mathbb{R}^n)$ such that $A_s(u) \in H^s(\mathbb{R}^n)$ and $G \cap \Sigma_0(A_s) = \emptyset$. Now choose one $A \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\bar{x}, \bar{\xi})$ and has $\mathrm{WF}'(A) \subset \{(x,\xi); |x - \bar{x}| < \epsilon, |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{\bar{\xi}}| < \epsilon\}$, which is the interior of G. Since A_s has a microlocal parametrix in a neighbourhood of G, $B_s A_s = \mathrm{Id} + E_s$, $\mathrm{WF}'(E_s) \cap G = \emptyset$ it follows that

$$Au = A(B_s A_s - E_s)u = (AB_s)A_s u - AE_s u \in H^s(\mathbb{R}^n) \ \forall\ s, \tag{5.103}$$

since $AE_s \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Thus $Au \in \mathbb{S}(\mathbb{R}^n)$ (since u is assumed to have compact support) so $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$, proving (5.101). □

Lemma 5.9. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $A \in \Psi^m_\infty(\mathbb{R}^n)$ then*

$$\mathrm{WF}_{s-m}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}_s(u) \ \forall\ s \in \mathbb{R}. \tag{5.104}$$

Proof. See the proof of the absolute version, Proposition 5.6. This shows that if $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}'(A)$ then $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(Au)$, so certainly $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}_{s-m}(Au)$. Similarly, if $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}_s(u)$ then there exists $B \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\bar{x}, \bar{\xi})$ and such that $Bu \in H^s(\mathbb{R}^n)$. If $G \in \Psi^0_\infty(\mathbb{R}^n)$ is a microlocal parametrix for B at $(\bar{x}, \bar{\xi})$ then $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}'(GB - \mathrm{Id})$ so by the first part $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}_{s-m}(A(GB - \mathrm{Id})u)$ and on the other hand, $AGBu \in H^{s-m}(\mathbb{R}^n)$, so (5.104) follows. □

Now, we can state a relative version of Theorem 5.1:-

Theorem 5.2 (Hörmander's propagation theorem, L^2, local version). *Suppose $P \in \Psi^1_\infty(M)$ has real principal symbol, that $c : [a,b] \longrightarrow \Sigma_m(P)$ is an interval of a null bicharacteristic curve (meaning $c_*(\frac{d}{dt}) = H_p$) and that $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ satisfies*

$$c([a,b]) \cap \mathrm{WF}_{\frac{1}{2}}(Pu) = \emptyset \ \textit{(eventually } c([a,b]) \cap \mathrm{WF}_0(Pu) = \emptyset) \tag{5.105}$$

then

$$\begin{cases} \textit{either} & c([a,b]) \cap \mathrm{WF}_0(u) = \emptyset \\ \textit{or} & c([a,b]) \subset \mathrm{WF}_0(u). \end{cases} \tag{5.106}$$

Proof that Theorem 5.1 follows from Theorem 5.2. The basic idea is to apply (5.101), remembering that there is not equality (in general) in (5.100) – the necessary uniformity here comes from the geometry so let us check that first.

Lemma 5.10. *First, we can act on P on the left with some elliptic operator with positive principal symbol, such as $\langle D \rangle^{-m+1}$ which changes the order of P to 1. This does not change $\Sigma(P)$ as the principal symbol changes from p to ap where $a > 0$, and only scales the Hamilton vector field on $\Sigma(P)$ since*

$$H_{ap} = aH_p + pH_a \tag{5.107}$$

and the second term vanishes on $\Sigma(P)$. Thus it suffices to consider the case $m = 1$.

If p is real and homogeneous of degree 1, Γ is an open conic neighbourhood of a bicharacteristic segment $c([a,b])$ such that dp and the canonical 1-form $\alpha = \xi \cdot dx$ are independent at $c(a)$ and γ is an open conic neighbourhood of $c(t)$ for some $t \in [a,b]$ then there is an open conic neighbourhood G of $c([a,b])$, $G \subset \Gamma$ such that $G \cap \Sigma(P)$ is a union of (null) bicharacteristic intervals $c^q(a_q, b_q))$ which intersect γ.

Proof. If dp and α are linearly dependent at a some point $(\bar{x}, \bar{\xi}) \in \Sigma(P)$ then $H_p = c\xi \cdot \partial_\xi$ is a multiple of the radial vector field at that point. By homogeneity the same must be true at $(\bar{x}, s\bar{\xi})$ for all $s > 0$ so the integral curve of H_p through

$(\bar{x}, \bar{\xi})$ must be contained in the ray through that point. Thus the condition that dp and $\xi \cdot dx$ are linearly independent at $c(a)$ implies that this must be true on all points of $c([a, b])$ and hence in a neighbourhood of this interval.

Thus it follows that H_p and $\xi \cdot \partial_\xi$ are linearly independent near $c([a, b])$. Since p is homogeneous of degree 1, H_p is homogeneous of degree 0. It follows that there are local coordinates $\Xi \neq 0$ homogeneous of degree 1 and y_k, homogeneous of degree 0, in a neighbourhood of $c([a, b])$ in terms of which $H_p = \partial_{y_1}$. These can be obtained by integrating along H_p to solve

$$H_p y_1 = 1,\ H_p y_k = 0,\ k > 1,\ H_p \Xi = 0 \tag{5.108}$$

with appropriate initial conditions on a conic hypersurface transversal to H_p. Then the integral curves, including $c([a, b])$ must just be the y_1 lines for which the conclusion is obvious, noting that ∂_{y_1} must be tangent to $\Sigma(P)$. $\square$

Now, returning to the proof note that we are assuming that Theorem 5.2 has been proved for *all* first order pseudodifferential operators with real principal symbol. Suppose we have the same set up but assume that

$$c([a, b]) \cap \mathrm{WF}_{s+\frac{1}{2}}(Pu) = \emptyset \text{ (eventually } c([a, b]) \cap \mathrm{WF}_s(Pu) = \emptyset) \tag{5.109}$$

in place of (5.96). Then we can simply choose a globally invertible elliptic operator of order s, say $Q_s = \langle D \rangle^s$ and rewrite the equation as

$$P_s v = Q_s f,\ P_s = Q_s P Q_{-s},\ v = Q_s u \tag{5.110}$$

Then (5.109) implies that

$$c([a, b]) \cap \mathrm{WF}_{\frac{1}{2}}(P_s v) = \emptyset \tag{5.111}$$

and $P_s \in \Psi^0_\infty(\mathbb{R}^n)$ is another operator with real principal symbol – in fact the same as before, so we get (5.97) which means that for each s we have the alternatives

$$\begin{cases} \text{either} & c([a, b]) \cap \mathrm{WF}_s(u) = \emptyset \\ \text{or} & c([a, b]) \subset \mathrm{WF}_s(u). \end{cases} \tag{5.112}$$

Now the hypothesis in (5.96) implies (5.109) for each s and hence for each s we have the alternatives (5.112). Of course if the second condition holds for any one s then it holds for all larger s and in particular implies that the second case in (5.97) (but for the compact interval) holds. So, what we really need to show is that if the first case in (5.112) holds for all s then

$$c([a, b]) \cap \mathrm{WF}(u) = \emptyset. \tag{5.113}$$

This is where we need to get some uniformity. However, consider nearby points and bicharacteristics. Our assumption is that for some $t \in [a, b]$, $c(t) \notin \mathrm{WF}(u)$ – otherwise we are in the second case. Since the set $\mathrm{WF}(u)$ is closed and conic, this implies that some open cone γ containing $c(t)$ is also disjoint from $\mathrm{WF}(u)$. Thus it follows that $\gamma \cap \mathrm{WF}_s(u) = \emptyset$ for all s. This is where the geometry comes in to show that there is a fixed open conic neighbourhood G of $c([a, b])$ such that

$$G \cap \mathrm{WF}_s(u) = \emptyset\ \forall\ s \in \mathbb{R}. \tag{5.114}$$

Namely we can take G to be a small neighbourhood as in Lemma 5.10. Since one point on each of the null bicharacteristic intervals forming $G \cap \Sigma(P)$ meets a point of γ, the first alternative in (5.112) must hold for all these intervals, for all s. That is,

$$G \cap \mathrm{WF}_s(u) = \emptyset \ \forall \ s. \tag{5.115}$$

Now (5.101) applies to show that $G \cap \mathrm{WF}(u) = \emptyset$ so in particular we are in the first case in (5.97) and the theorem follows. $\square$

Finally we further simplify Theorem 5.2 to a purely local statement.

Proposition 5.9. *Under the hypotheses of Theorem 5.2 if $t \in (a, b)$ and $\mathrm{WF}_0(u) \cap c((t \pm \epsilon)) = \emptyset$ for some $\epsilon > 0$ then $c(t) \notin \mathrm{WF}_0(u)$.*

Derivation of Theorem 5.2 from Proposition 5.9. The dicotomy in (5.97) amounts to the statement that if $c(t) \notin \mathrm{WF}_0(u)$ for some $t \in [a, b]$ then $C = \{t' \in [a, b]; c(t) \in \mathrm{WF}_0(u)\}$ must be empty. Since $\mathrm{WF}(u)$ is closed, C is also closed. Applying the Proposition to $\sup(C \cap [a, t))$ shows that it cannot be in C and neither can $\inf(C \cap (t, b])$ so both these sets must be empty and hence C itself must be empty. $\square$

5.15. Proof of Proposition 5.9

Before we finally get down to the analysis let me note some more simiplifications. We can actually assume that $c(t) = a = 0$ and that the interval is $[0, \delta]$ for some $\delta > 0$. Indeed this is just changing the parameter in the case of the positive sign. In the case of the negative sign reversing the sign of P leaves the hypotheses unchanged but reverses the parameter along the integral curve. Thus our hypotheses are that

$$c([0,\delta]) \cap \mathrm{WF}_{\frac{1}{2}}(Pu) = \emptyset \text{ (eventually just } c([0,\delta]) \cap \mathrm{WF}_0(Pu) = \emptyset) \text{ and } c((0,\delta]) \cap \mathrm{WF}_0(u) = 0 \tag{5.116}$$

and we wish to conclude that

$$c(0) \notin \mathrm{WF}_0(u). \tag{5.117}$$

We can also assume that

$$c(0) \notin \mathrm{WF}_{-\frac{1}{2}}(u). \tag{5.118}$$

In fact, if (5.118) does not hold, then there is in fact some $s < -\frac{1}{2}$ such that $c(0) \notin \mathrm{WF}_s(u)$ but $c(0) \in \mathrm{WF}_t(u)$ for some $t \le \max(-\frac{1}{2}, s+\frac{1}{2})$. Indeed, u itself is in some Sobolev space. Now we can apply the argument used earlier to deduce (5.112) from (5.97). Namely, replace P by $\langle D\rangle^{s+\frac{1}{2}} P \langle D\rangle^{-s-\frac{1}{2}}$ and u by $u' = \langle D\rangle^{s+\frac{1}{2}} u$. Then (5.118) is satisfied by u' and if the argument to prove (5.117) works, we conclude that $c(0) \notin \mathrm{WF}_s(u)$ which is a contradiction. Thus, proving that (5.117) follows form (5.116) and (5.117) suffices to prove everything.

Okay, now to the construction. What we will first do is find a 'test' operator $A \in \Psi^0_\infty(\mathbb{R}^n)$ which has

$$\mathrm{WF}'(A) \subset N(c(0)),\ A^* = A \tag{5.119}$$

for a preassigned conic neighbourhood $N(c(0))$ of the point of interest. Then we want in addition to arrange that for a preassigned conic neigbourhood $N(c(\delta/2))$,

$$\begin{gathered}\frac{1}{i}(AP-P^*A)=B^2+E_0+E_1,\\ B\in\Psi^0_\infty(\mathbb{R}^n),\ B^*=B\text{ is elliptic at }c(0),\\ E_0\in\Psi^0_\infty(\mathbb{R}^n),\ \mathrm{WF}'(E_0)\subset N(c(\frac{\delta}{2}))\\ \text{and }E_1\in\Psi^{-1}_\infty(\mathbb{R}^n).\end{gathered}\tag{5.120}$$

Before checking that we can arrange (5.120) let me comment on why it will help! In fact there is a flaw in the following argument which will be sorted out below. Given (5.120) let us apply the identity to u and then take the L^2 pairing with u which would give

$$-2Im\langle u,APu\rangle=-i\langle u,APu\rangle+i\langle APu,u\rangle=\|Bu\|^2+\langle u,E_0\rangle+\langle u,E_1u\rangle.\tag{5.121}$$

where I have illegally integrated by parts, which is part of the flaw in the argument. Anyway, the idea is that APu is smooth – at least it would be if we assumed that $N(c(0))\cap\mathrm{WF}(Pu)=\emptyset$ – so the left side is finite. Similarly by the third line of (5.120), $\mathrm{WF}'(E_0)$ is confined to a region where u is known to be well-behaved and the order of E_1 allows us to use (5.118). So with a little luck we can show, and indeed we will, that

$$Bu\in L^2(\mathbb{R}^n)\Longrightarrow c(0)\notin\mathrm{WF}_0(u)\tag{5.122}$$

which is what we are after. The problems with this argument are of the same nature that are met in discussions of elliptic regularity and the niceties are discussed below.

So, let us now see that we can arrange (5.120). First recall that we have normalized P to be of order 1 with real principal symbol. So

$$P^*=P+iQ,\ Q\in\Psi^0_\infty(\mathbb{R}^n),\ Q=Q^*.$$

Thus the left side of the desired identity in (5.120) can be written

$$-i[A,P]+QA\in\Psi^0_\infty(\mathbb{R}^n),\ \sigma_0(-i[A,P]+QA)=-H_pa+qa\tag{5.123}$$

where q is the principal symbol of q etc. Since E_1 in (5.120) can include any terms of order -1 we just need to arrange the principal symbol identity

$$-H_pa+qa=b^2+e.\tag{5.124}$$

Notice that p is by assumption a function which is homogeneous of degree 1 so the vector field H_p is homogeneous of degree 0. We can further assume that

$$H_p\neq 0\text{ on }c([0,\delta]).\tag{5.125}$$

Indeed, if $H_p=0$ at $c(0)$ then the whole integral curve through $c(0)$ consists of the point and the result is trivial. So we can assume that $H_p\neq 0$ at $c(0)$ and then (5.125) follows by shrinking δ. As noted above we can now introduce coordinates t $s\in\mathbb{R}^{2n-2}$ and $\Theta>0$, homogeneous respectively of degrees 0, 0 and 1, in terms of which $H_p=\frac{\partial}{\partial t}$, $c(0)=(0,1)$ so the integral curve is just $(t,0,1)$ and the differential equation (5.124) only involves the t variable and the s variables as parameters (ξ_j disappears because of the assumed homogeneity)

$$-\frac{d}{dt}a+qa=b^2+e.\tag{5.126}$$

So, simply choose $b = \phi(t)\phi(|s|)$ for some cut-off function $\phi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ which is 1 near 0 and has small support in $|x| \le \delta'$ which will be chosen small. Then solve

$$-\frac{d}{dt}\tilde{a} + q\tilde{a} = b^2 \Longrightarrow \tilde{a}(t,s) = -\phi^2(|s|)e^{-Q(t,s)}\int_{-\infty}^{t} e^{Q(t',s)}\phi^2(t',s)dt' \tag{5.127}$$

where Q is a primitive of q. Integrating from $t << 0$ ensures that the support of a' is confined to $|s| \le \delta'$ and $t \ge -\delta'$. Now simply choose a function $\psi \in \mathcal{C}^\infty(\mathbb{R})$ which is equal to 1 in $t < \frac{1}{2}\delta - \delta'$ and equal to 0 in $t > \frac{1}{2}\delta + \delta'$. Then setting $a(t,s) = \psi(t)\tilde{a}(t,s)$ gives a solution of (5.126) with the desired support properties. Namely if we simply cut a and b off in Θ near zero to make them into smooth symbols and select operators B and A self-adjoint and with these principal symbols then (5.120) follows where the supports behave as we wish when δ' is made small.

So, what is the problem with the derivation of (5.121). For one thing the integration by parts, but for another the pairing which we do not know to make sense. In particular the norm $\|Bu\|$ which we wish to show to be finite certainly has to be for this argument to be possible. The solution to these problems is simply to regularize the operators.

So, now choose a sequence $\mu_n(\mathbb{R})$ where the variable will be Θ. We want

$$\mu_n \in \mathcal{C}_c^\infty(\mathbb{R}),\ \mu_n \text{ bounded in } S^0(\mathbb{R}) \text{ and } \mu_n \to 1 \in S^\epsilon(\mathbb{R})\ \forall\ \epsilon > 0. \tag{5.128}$$

This is easily arranged, for instance taking $\mu \in \mathcal{C}_c^\infty(\mathbb{R})$ equal to 1 near 0 and setting $\mu_n(\Theta) = \mu(\Theta/n)$. Since we have arranged that the homogeneous variable Θ is annihilated by $H_p = \frac{d}{dt}$ we can simply multiply through the equation and get a similar family of solutions to (5.124)

$$-H_p a_n + q a_n = b_n^2 + e_n \tag{5.129}$$

where all terms are bounded in $S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ (and have compact support in the base variables). Now if we take operators A_n, B_n with these full symbols, and then their self-adjoint parts, we conclude that A_n, $B_n \in \Psi^0_\infty(\mathbb{R}^n)$ have left symbols bounded in S^0 and we get a sequence of solutions to the identity (5.120) with uniformity. Let's check that we know precisely what this means. Namely for all $\epsilon > 0$,

(5.130)

$$\begin{gathered}A_n \text{ is bounded in } \Psi^0_\infty(\mathbb{R}^n),\ A_n \to A \text{ in } \Psi^\epsilon_\infty(\mathbb{R}^n),\ \mathrm{WF}'(A_n) \subset N(c(\delta)) \text{ is uniform},\\ \frac{1}{i}(A_nP - P^*A_n) = B_n^2 + E_{0,n} + E_{1,n},\\ B_n^* = B_n \in \Psi^0_\infty(\mathbb{R}^n) \text{ is bounded},\ B_n \to B \text{ in } \Psi^\epsilon_\infty(\mathbb{R}^n),\ \Psi^0_\infty(\mathbb{R}^n) \ni B \text{ is elliptic at } c(0),\\ E_{0,n} \in \Psi^0_\infty(\mathbb{R}^n) \text{ is bounded},\ \mathrm{WF}'(E_{0,n}) \subset N(c(\frac{\delta}{2})) \text{ is uniform}\\ \text{and } E_{1,n} \in \Psi^{-1}_\infty(\mathbb{R}^n) \text{ is bounded}.\end{gathered}$$

where the boundedness of the sequences means that the symbols estimates on the left symbols have fixed constants independent of n and uniformity of the essential support conditions means that for instance

$$\begin{gathered}q \notin N(c(\frac{\delta}{2})) \Longrightarrow\ \exists\ R \in \Psi^0_\infty(\mathbb{R}^n) \text{ elliptic at } q\\ \text{such that } RE_{0,n} \text{ is bounded in } \Psi^{-\infty}_\infty(\mathbb{R}^n).\end{gathered} \tag{5.131}$$

All this follows from our choice of symbols.

I leave as an exercise the effect of the uniformity statement on the essential support.

LEMMA 5.11. *Suppose* A_n *is bounded in* $\Psi^m_\infty(\mathbb{R}^n)$ *for some* m *and that*

$$\mathrm{WF}'(A_n) \subset G \text{ uniformly} \tag{5.132}$$

for a closed cone G *in the sense of* (5.131)*. Then if* $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ *is such that*

$$\mathrm{WF}_m(u) \cap G = \emptyset \text{ then } A_n u \text{ is bounded in } L^2(\mathbb{R}^n). \tag{5.133}$$

Now we are in a position to finish! For finite n all the operators in the identity in (5.130) are smoothing so we can apply the operators to u and pair with u. Then the integration by parts used to arrive at (5.121) is really justified in giving

$$-2Im\langle u, A_n P u\rangle = -i\langle u, A_n P u\rangle + i\langle A_n P u, u\rangle = \|B_n u\|^2 + \langle u, E_{0,n} u_n\rangle + \langle u, E_{1,n} u\rangle. \tag{5.134}$$

We have arranged that $\mathrm{WF}'(A_n)$ is uniformly concentrated near (the cone over) $c([0, \frac{\delta}{2}])$ and, from (5.116), that $\mathrm{WF}_{\frac{1}{2}}(Pu)$ does not meet such a set. Thus Lemma 5.11 shows us that $A_n P u$ is bounded in $H^{\frac{1}{2}}(\mathbb{R}^n)$. Since we know that $\mathrm{WF}_{-\frac{1}{2}}(u)$ does not meet $c([0, \delta])$ we conclude (always taking the parameter δ' determining the size of the supports small enough) that

$$|\langle u, A_n P u\rangle| \text{ is bounded} \tag{5.135}$$

as $n \to \infty$. Similarly $|\langle u, E_{0,n}\rangle|$ is bounded since $E_{0,n}$ is bounded in $\Psi^0_\infty(\mathbb{R}^n)$ and has essential support uniformly in the region where u is known to be in $L^2(\mathbb{R}^n)$ and $|\langle u, E_{1,n}\rangle|$ is bounded since $E_{1,n}$ is uniformly of order -1 and has essential support (uniformly) in the region where u is known to be in $H^{-\frac{1}{2}}(\mathbb{R}^n)$. Thus indeed, $\|B_n u\|_{L^2}$ is bounded. Thus $B_n u$ is bounded in $L^2(\mathbb{R}^n)$, hence has a weakly convergent subsequence, but this must converge to Bu when paired with test functions. Thus in fact $Bu \in L^2(\mathbb{R}^n)$ and (5.117) follows.

5.16. Hörmander's propagation theorem

There are still some global issues to settle. Theorem 5.1, which has been proved above, can be immediately globalized and microlocalized at the same time. It is also coordinate invariant – see the discussion in Chapter 6, so can be transferred to any manifold as follows.

THEOREM 5.3. *If* $P \in \Psi^m(M)$ *has real principal symbol and is properly supported then for any distribution* $u \in \mathcal{C}^{-\infty}(M)$,

$$\mathrm{WF}(u) \setminus \mathrm{WF}(Pu) \subset \Sigma(P) \tag{5.136}$$

is a union of maximally extended null bicharacteristics in $\Sigma(P) \setminus \mathrm{WF}(Pu)$.

Some consequences of this in relation to the wave equation are discussed below, and extension of it in Chapter 7.

As already noted, the strengthened assumption on the regularity of Pu in (5.96) is not necessary to deduce (5.97), or correspondingly (5.116) for (5.117). This is not important in the proof of Theorem 5.1 since we are making a much stronger assumption on the regularity of Pu anyway. However, to get the more refined version of Theorem 5.2, as stated 'eventually' we only need to prove (5.117) using the corresponding form of (5.116). This in turn involves a more careful choice of $\phi(x)$ using the following sort of division result.

Lemma 5.12. *There exist a function $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with support in $[0,\infty)$ which is strictly positive in $(0,\infty)$ and such that for any $0 < f \in \mathcal{C}^\infty(\mathbb{R})$,*

$$\int_{-\infty}^t f(t')\phi^2(t')dt' = \phi(t)a(t),\ a \in \mathcal{C}^\infty(\mathbb{R}),\ \operatorname{supp}(a) \subset [0,\infty). \tag{5.137}$$

Proof. This is true for $\phi = \exp(-1/t)$ in $t > 0$, $\phi(t) = 0$ in $t \le 0$. Indeed the integral is then bounded by

$$|\int_{-\infty}^t f(t')\exp(-2/t')dt'| \le C\exp(-2/t),\ t \le 1. \tag{5.138}$$

This shows that $a(t)$, defined as the quotient for $t > 0$ and 0 for $t < 0$ is bounded by $C\phi(t)$. A similar argument show that each of the derivatives are also uniformly bounded by $t^{-N}\phi(t)$ and is therefore also bounded. □

Taking ϕ to be such a function in the discussion above (near the lower bound of its support) allows the symbol a defined by integration, and then a_n, to be decomposed as

$$a_n = b_n g_n + a'_n \tag{5.139}$$

where a'_n is uniformly supported in $t < \delta'/10$ and g_n is also a uniformly bounded sequence of symbols of order 0. This results in a similar decomposition for the operators

$$A_n = B_n G_n + A'_n + R'_n \tag{5.140}$$

where R'_n is uniformly of order -1, G_n is uniformly of order 0 and A'_n, also uniformly of order 0 is uniformly supported in the region where we already know that $u \in L^2(\mathbb{R}^n)$. The previous estimate (5.135) on the left side of (5.134) can then be replaced by

$$|\langle u, A_n Pu\rangle| \le |\langle B_n u, G_n Pu\rangle| + |\langle u, A'_n Pu\rangle| + |\langle u, R'_n Pu\rangle| \le C\|B_n u\| + C' \tag{5.141}$$

using only the 'eventual' estimate in (5.116) to control the third term. The other terms in (5.134) behave as before which results in an estimate

$$\|B_n u\|^2 \le C'\|B_n u\| + C'' \tag{5.142}$$

which still implies that $\|B_n u\|$ is bounded, so the argument can be completed as before. This then proves the 'eventual' form of Theorem 5.2 and hence, after reinterpretation, Theorem 5.3.

5.17. Elementary calculus of wavefront sets

We want to achieve a reasonable understanding, in terms of wavefront sets, of three fundamental operations. These are

$$\text{Pull-back: } F^* u \tag{5.143}$$

$$\text{Push-forward: } F_* u \text{ and} \tag{5.144}$$

$$\text{Multiplication: } u_1 \cdot u_2. \tag{5.145}$$

In order to begin to analyze these three operations we shall first introduce and discuss some other more "elementary" operations:

$$\text{Pairing: } (u,v) \longrightarrow \langle u,v\rangle = \int u(x)\overline{v(x)}dx \tag{5.146}$$

$$\text{Projection: } u(x,y) \longmapsto \int u(x,y)dy \tag{5.147}$$

$$\text{Restriction: } u(x,y) \longmapsto u(x,0) \tag{5.148}$$

$$\text{Exterior product: } (u,v) \longmapsto (u \boxtimes v)(x,y) = u(x)v(y) \tag{5.149}$$

$$\text{Invariance: } F^*u, \text{ for } F \text{ a diffeomorphism.} \tag{5.150}$$

Here (5.148) and (5.150) are special cases of (5.143), (5.147) of (5.144) and (5.149) is a combination of (5.145) and (5.143). Conversely the three fundamental operations can be expressed in terms of these elementary ones. We can give direct definitions of the latter which we then use to analyze the former. We shall start with the pairing in (5.146).

5.18. Pairing

We know how to 'pair' a distribution and a $\mathcal{C}^\infty$ function. If both are $\mathcal{C}^\infty$ and have compact supports then

$$\langle u_1,u_2\rangle = \int u_1(x)\overline{u_2(x)}dx \tag{5.151}$$

and in general this pairing extends by continuity to either $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)\times\mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathcal{C}^\infty(\mathbb{R}^n)\times\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ Suppose *both* u_1 and u_2 are distributions, when can we pair them?

Proposition 5.10. *Suppose u_1, $u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfy*

$$\mathrm{WF}(u_1)\cap\mathrm{WF}(u_2)=\emptyset \tag{5.152}$$

then if $A\in\Psi^0_\infty(\mathbb{R}^n)$ has

$$\mathrm{WF}(u_1)\cap\mathrm{WF}'(A)=\emptyset,\ \mathrm{WF}(u_2)\cap\mathrm{WF}'(\mathrm{Id}-A)=\emptyset \tag{5.153}$$

the bilinear form

$$\langle u_1,u_2\rangle = \langle Au_1,u_2\rangle + \langle u_1,(\mathrm{Id}-A^*)u_2\rangle \tag{5.154}$$

is independent of the choice of A.

Notice that A satisfying (5.153) does indeed exist, just choose $a\in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ to be identically 1 on $\mathrm{WF}(u_2)$, but to have $\operatorname{cone\,supp}(a)\cap\mathrm{WF}(u_1)=\emptyset$, possible because of (5.152), and set $A=q_L(a)$.

Proof. Of course (5.154) makes sense because Au_1, $(\mathrm{Id}-A^*)u_2\in\mathcal{C}^\infty(\mathbb{R}^n)$ by microlocality and the fact that $\mathrm{WF}'(A)=\mathrm{WF}'(A^*)$. To prove that this definition is independent of the choice of A, suppose A' also satisfies (5.153). Set

$$\langle u_1,u_2\rangle' = \langle A'u_1,u_2\rangle + \langle u_1,(\mathrm{Id}-A')^*u_2\rangle. \tag{5.155}$$

Then

$$\mathrm{WF}'(A-A')\cap\mathrm{WF}(u_1)=\mathrm{WF}'((A-A')^*)\cap\mathrm{WF}(u_2)=\emptyset. \tag{5.156}$$

The difference can be written

$$\langle u_1,u_2\rangle' - \langle u_1,u_2\rangle = \langle (A-A')u_1,u_2\rangle - \langle u_1,(A-A')^*u_2\rangle. \tag{5.157}$$

Naturally we expect this to be zero, but this is not quite obvious since u_1 and u_2 are both distributions. We need an approximation argument to finish the proof.

Choose $B \in \Psi^0_\infty(\mathbb{R}^n)$ with

$$\begin{gathered}\mathrm{WF}'(B) \cap \mathrm{WF}(u_1) = \mathrm{WF}'(B) \cap \mathrm{WF}(u_2) = \emptyset \\ \mathrm{WF}'(\mathrm{Id} - B) \cap \mathrm{WF}(A - A') = \emptyset\end{gathered} \tag{5.158}$$

If $v_n \longrightarrow u_2$, in $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$, $v_n \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then

$$w_n = \phi\big[(\mathrm{Id} - B)\, v_n + Bu_2\big] \longrightarrow u_2 \tag{5.159}$$

if $\phi \equiv 1$ in a neighbourhood of $\mathrm{supp}(u_2)$, $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Here $Bu_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$, so

$$(A - A')w_n = (A - A')\phi(\mathrm{Id} - B) \cdot v_n + (A - A')\phi Bu_2 \longrightarrow (A - A')u_2 \text{ in } \mathcal{C}^\infty(\mathbb{R}^n), \tag{5.160}$$

since $(A - A')\phi(\mathrm{Id} - B) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Thus

$$\begin{gathered}\langle (A - A')\, u_1, u_2\rangle \longrightarrow \langle (A - A')\, u_1, u_2\rangle \\ \langle u_1, (A - A')^*\, w_n\rangle \longrightarrow \langle u_1, (A - A')^*\, u_2\rangle,\end{gathered}$$

since $w_n \longrightarrow u_2$ in $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and $(A - A')^* w_n \longrightarrow (A - A')^*\, u_2$ in $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus

$$\langle u_1, u_2\rangle' - \langle u_1, u_2\rangle = \lim_{n\to\infty}\big[\langle (A - A')\, u_1, w_n\rangle - \langle u_1, (A - A')^*\, w_n\big] = 0. \tag{5.161}$$

□

Here we are using the *complex* pairing. If we define the real pairing by

$$(u_1, u_2) = \langle u_1, \overline{u}_2\rangle \tag{5.162}$$

then we find

PROPOSITION 5.11. *If u_1, $u_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ satisfy*

$$(x, \xi) \in \mathrm{WF}(u_1) \Longrightarrow (x, -\xi) \notin \mathrm{WF}(u_2) \tag{5.163}$$

then the real pairing, defined by

$$(u_1, u_2) = (Au_1, u_2) + (u_1, (\mathrm{Id} - A^t)u_2), \tag{5.164}$$

where A satisfies (5.153)*, is independent of A.*

PROOF. Notice that

$$\mathrm{WF}(\overline{u}) = \big\{(x, -\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0); (x, \xi) \in \mathrm{WF}(u)\big\}. \tag{5.165}$$

We can write (5.163), using (5.162), as

$$(u_1, u_2) = \langle Au_1, \overline{u}_2\rangle + \langle u_1, \overline{(\mathrm{Id} - A^t)u_2}\rangle. \tag{5.166}$$

Since, by definition, $\overline{A^t u_2} = A^*\overline{u}_2$,

$$(u_1, u_2) = \langle Au_1, \overline{u}_2\rangle + \langle u_1, (\mathrm{Id} - A^*)\overline{u}_2\rangle = \langle u_1, \overline{u}_2\rangle \tag{5.167}$$

is defined by (5.154), since (5.163) translates to (5.152). □

5.19. Multiplication of distributions

The pairing result (5.164) can be used to define the *product* of two distributions under the same hypotheses, (5.163).

PROPOSITION 5.12. *If $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfy*

$$(x,\xi) \in \mathrm{WF}(u_1) \Longrightarrow (x,-\xi) \notin \mathrm{WF}(u_2) \tag{5.168}$$

then the product of u_1 and $u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ is well-defined by

$$u_1u_2(\phi) = (u_1, \phi u_2) = (\phi u_1, u_2)\ \forall\ \phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n) \tag{5.169}$$

using (5.164).

PROOF. We only need to observe that if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $A \in \Psi_\infty^m(\mathbb{R}^n)$ has $\mathrm{WF}'(A) \cap \mathrm{WF}(u) = \emptyset$ then for any fixed $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\|\psi A\phi u\|_{C^k} \le C\|\phi\|_{C^p} \quad p = k + N \tag{5.170}$$

for some N, depending on m. This implies the continuity of $\phi \longmapsto u_1u_2(\phi)$ defined by (5.169). $\square$

5.20. Projection

Here we write $\mathbb{R}^n_z = \mathbb{R}^p_x \times \mathbb{R}^k_y$ and define a continuous linear map, which we write rather formally as an integral

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \int u(x,y)dy \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p) \tag{5.171}$$

by pairing. If $\phi \in \mathcal{C}^\infty(\mathbb{R}^p)$ then

$$\pi_1^*\phi \in \mathcal{C}^\infty(\mathbb{R}^n), \quad \pi_1:\ \mathbb{R}^n \ni (x,y) \longmapsto x \in \mathbb{R}^p \tag{5.172}$$

and for $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ we define the formal 'integral' in (5.171) by

$$(\int u(x,y)dy, \phi) = ((\pi_1)_*u, \phi) := u(\pi_1^*\phi). \tag{5.173}$$

In this sense we see that the projection is dual to pull-back (on functions) under π_1, so is "push-forward under π_1," a special case of (5.144). The support of the projection satisfies

$$\mathrm{supp}\left((\pi_1)_*u\right) \subset \pi_1\left(\mathrm{supp}(u)\right)\ \ \forall\ u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n), \tag{5.174}$$

as follows by duality from

$$\mathrm{supp}(\pi_1^*\phi) \subset \pi_1^{-1}\left(\mathrm{supp}\,\phi\right). \tag{5.175}$$

PROPOSITION 5.13. *Let $\pi_1 : \mathbb{R}^{p+k} \longrightarrow \mathbb{R}^p$ be projection, then for every $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k})$*

$$\begin{aligned} \mathrm{WF}\left((\pi_1)_*u\right) \subset \big\{&(x,\xi) \in \mathbb{R}^p \times (\mathbb{R}^p\backslash 0)\,; \\ &\exists\ y \in \mathbb{R}^k \text{ with } (x,y,\xi,0) \in \mathrm{WF}(u)\big\}. \end{aligned} \tag{5.176}$$

PROOF. First notice that

$$(\pi_1)_*:\ \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p). \tag{5.177}$$

Combining this with (5.174) we see that

$$\mathrm{sing\,supp}\left((\pi_1)_*u\right) \subset \pi_1\left(\mathrm{sing\,supp}\,u\right) \tag{5.178}$$

which is at least consistent with Proposition 5.13. To prove the proposition in full let me restate the local characterization of the wavefront set, in terms of the Fourier transform:

LEMMA 5.13. *Suppose $K \subset\subset \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^n \setminus 0$ is a* closed *cone, then*

$$u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \mathrm{WF}(u) \cap (K \times \Gamma) = \emptyset,\ A \in \Psi_\infty^m(\mathbb{R}^n),\ \mathrm{WF}'(A) \subset K \times \Gamma \\ \Longrightarrow Au \in \mathcal{S}(\mathbb{R}^n). \tag{5.179}$$

In particular

$$u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \mathrm{WF}(u) \cap (K \times \Gamma) = \emptyset,\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n), \mathrm{supp}(\phi) \subset K \\ \Longrightarrow \widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma. \tag{5.180}$$

Conversely suppose $\Gamma \subset \mathbb{R}^n \setminus 0$ is a closed cone and $u \in \mathcal{S}'(\mathbb{R}^n)$ is such that for some $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma \tag{5.181}$$

then

$$\mathrm{WF}(u) \cap \{x \in \mathbb{R}^n; \phi(x) \neq 0\} \times \mathrm{int}(\Gamma) = \emptyset. \tag{5.182}$$

With these local tools at our disposal, let us attack (5.176). We need to show that

$$(\bar{x}, \bar{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0) \text{ s.t. } (\bar{x}, y, \bar{\xi}, 0) \notin \mathrm{WF}(u)\ \forall\ y \in \mathbb{R}^n \\ \Longrightarrow (\bar{x}, \bar{\xi}) \notin \mathrm{WF}\left((\pi_1)_* u\right). \tag{5.183}$$

Notice that, $\mathrm{WF}(u)$ being conic and $\pi(\mathrm{WF}(u))$ being compact, $\mathrm{WF}(u) \cap (\mathbb{R}^n \times S^{n-1})$ is compact. The hypothesis (5.183) is the statement that

$$\{\bar{x}\} \times \mathbb{R}^k \times \mathbb{S}^{n-1} \times \{0\} \cap \mathrm{WF}(u) = \emptyset. \tag{5.184}$$

Thus $\bar{x}$ has an open neighbourhood, W, in $\mathbb{R}^p$, and $(\bar{\xi}, 0)$ a conic neighbourhood γ_1 in $(\mathbb{R}^n \backslash 0)$ such that

$$(W \times \mathbb{R}^k \times \gamma_1) \cap \mathrm{WF}(u) = \emptyset. \tag{5.185}$$

Now if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ is chosen to have support in W

$$\widehat{(\pi_1^* \phi) u}(\xi, \eta) \text{ is rapidly decreasing in } \gamma_1. \tag{5.186}$$

Set $v = \phi(\pi_1)_* u$. From the definition of projection and the identity

$$v = \phi(\pi_1)_* u = (\pi_1)_*[(\pi_1^* \phi) u], \tag{5.187}$$

we have

$$\hat{v}(\xi) = v(e^{-ix \cdot \xi}) = (\widehat{(\pi_1^* \phi) u})(\xi, 0). \tag{5.188}$$

Now (5.186) shows that $\hat{v}(\xi)$ is rapidly decreasing in $\gamma_1 \cap (\mathbb{R}^p \times \{0\})$, which is a cone around $\bar{\xi}$ in $\mathbb{R}^p$. Since $v = \phi(\pi_1)_* u$ this shows that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}\left((\pi_1)_* u\right)$, as claimed. □

Before going on to talk about the other operations, let me note a corollary of this which is useful and, even more, helps to explain what is going on:

COROLLARY 5.1. *If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and*

$$\mathrm{WF}(u) \cap \left\{(x, y, \xi, 0); x \in \mathbb{R}^p, y \in \mathbb{R}^k, \xi \in \mathbb{R}^p \backslash 0\right\} = \emptyset \tag{5.189}$$

then $(\pi_1)_*(u) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

PROOF. Indeed, (5.176) says $\mathrm{WF}\left((\pi_1)_* u\right) = \emptyset$. □

Here, the vectors $(x, y, \xi, 0)$ are the ones "normal" (as we shall see, really *conormal*) to the surfaces over which we are integrating. Thus Lemma 5.13 and Corollary 5.1 both state that the only singularities that survive integration are the ones which are conormal to the surface along which we integrating; the ones even partially in the direction of integration are wiped out. This in particular fits with the fact that if we integrate in *all variables* then there are no singularities left.

5.21. Restriction

Next we wish to consider the *restriction* of a distribution to a subspace

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto u \restriction \{y = 0\} \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p). \tag{5.190}$$

This is *not* always defined, i.e. no reasonable map (5.190) exists for all distributions. However under an appropriate condition on the wavefront set we can interpret (5.190) in terms of pairing, using our definition of products. Thus let

$$\iota : \mathbb{R}^p \ni x \longmapsto (x, 0) \in \mathbb{R}^n \tag{5.191}$$

be the *inclusion* map. We want to think of $u \restriction \{y = 0\}$ as $\iota^* u$. If $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then for any $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the identity

$$\iota^* u(\iota^* \phi') = u\left(\phi' \delta(y)\right) \tag{5.192}$$

holds.

The restriction map $\iota^* : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p)$ is surjective. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies the condition

$$\mathrm{WF}(u) \cap \left\{(x, 0, 0, \eta); x \in \mathbb{R}^p, \eta \in \mathbb{R}^{n-p}\right\} = \emptyset \tag{5.193}$$

then we can interpret the pairing

$$\begin{gathered} \iota^* u(\phi) = u\left(\phi' \delta(y)\right) \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^p) \\ \text{where } \phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } \iota^* \phi' = \phi \end{gathered} \tag{5.194}$$

to define $\iota^* u$. Indeed, the right side makes sense by Proposition 5.12.

Thus we have directly proved the first part of

PROPOSITION 5.14. *Set* $\mathcal{R} = \left\{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); (5.193) \text{ *holds*}\right\}$ *then* (5.194) *defines a linear* restriction *map* $\iota^* : \mathcal{R} \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^p)$ *and*

$$\mathrm{WF}(\iota^* u) \subset \left\{(x, \xi) \in \mathbb{R}^p \times (\mathbb{R}^p \setminus 0); \ \exists \ \eta \in \mathbb{R}^n \text{ with } (x, 0, \xi, \eta) \in \mathrm{WF}(u)\right\}. \tag{5.195}$$

PROOF. First note that (5.193) means *precisely* that

$$\hat{u}(\xi, \eta) \text{ is rapidly decreasing in a cone around } \{0\} \times \mathbb{R}^k \backslash 0. \tag{5.196}$$

When $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ taking Fourier transforms in (5.192) gives

$$\widehat{\iota^* u}(\xi) = \frac{1}{(2\pi)^k} \int \hat{u}(\xi, \eta) d\eta. \tag{5.197}$$

In general (5.196) ensures that the integral in (5.197) converges, it will then hold by continuity.

We actually apply (5.197) to a localized version of u; if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ then

$$\widehat{\psi \iota^*(u)}(\xi) = (2\pi)^{-k} \int \widehat{\psi}(\xi) \widehat{u}(\xi, \eta) d\eta. \tag{5.198}$$

Thus suppose $(\overline{x},\overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p\backslash 0)$ is such that $(\overline{x},0,\overline{\xi},\eta) \notin \mathrm{WF}(u)$ for *any* η. If ψ has support close to $\overline{x}$ and $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^{n-p})$ has support close to 0 this means

$$\widehat{\psi\zeta u}(\xi,\eta) \text{ is rapidly decreasing in a cone around each } (\overline{\xi},\eta). \tag{5.199}$$

We also have rapid decrease around $(0,\eta)$ from (5.196) (make sure you understand this point) as

$$\widehat{\psi\zeta u}(\xi,\eta) \text{ is rapidly decreasing in } \gamma \times \mathbb{R}^p \tag{5.200}$$

for a cone, γ, around $\overline{\xi}$. From (5.197)

$$\widehat{\psi\iota^*(\zeta u)}(\xi) \text{ is rapidly decreasing in } \gamma. \tag{5.201}$$

Thus $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(\iota^*(\zeta u))$. Of course if we choose $\zeta(y) = 1$ near 0, $\iota^*(\zeta u) = \iota^*(u)$ so $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$, provided $(\overline{x},0,\overline{\xi},\eta) \notin \mathrm{WF}(u)$, for all η. This is what (5.195) says. □

Try to picture what is going on here. We can restate the main conclusion of Proposition 5.14 as follows.

Take $\mathrm{WF}(u) \cap \big\{(x,0,\xi,\eta) \in \mathbb{R}^p \times \{0\} \times (\mathbb{R}^n\backslash 0)\big\}$ and let Z denote projection *off* the η variable:

$$\mathbb{R}^p \times \{0\} \times \mathbb{R}^p \times \mathbb{R}^k \xrightarrow{Z} \mathbb{R}^p \times \mathbb{R}^p \tag{5.202}$$

then

$$\mathrm{WF}(\iota^* u) \subset Z(\mathrm{WF}(u) \cap \{y = 0\}). \tag{5.203}$$

We will want to think more about these operations later.

5.22. Exterior product

This is maybe the easiest of the elementary operators. It is always defined

$$(u_1 \boxtimes u_2)(\phi) = u_1\left(u_2(\phi(x,\cdot)\right) = u_2(u_1(\phi(\cdot,y)). \tag{5.204}$$

Moreover we can easily compute the Fourier transform:

$$\widehat{u_1 \boxtimes u_2}(\xi,\eta) = \hat{u}_1(\xi)\hat{u}_2(\eta). \tag{5.205}$$

Proposition 5.15. *The (exterior) product*

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^p) \times \mathcal{C}_c^{-\infty}(\mathbb{R}^k) \longleftarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k}) \tag{5.206}$$

is a bilinear map such that

$$\begin{aligned}\mathrm{WF}(u_1 \boxtimes u_2) \subset {} & [(\mathrm{supp}(u_1) \times \{0\}) \times \mathrm{WF}(u_2)] \\ & \cup [\mathrm{WF}(u_1) \times (\mathrm{supp}(u_2) \times \{0\})] \cup [\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)].\end{aligned} \tag{5.207}$$

Proof. We can localize near any point $(\overline{x},\overline{y})$ with $\phi_1(x)\phi_2(y)$, where ϕ_1 is supported near $\overline{x}$ and ϕ_2 is supported near $\overline{y}$. Thus we only need examine the decay of

$$\phi_1 \widehat{u_1 \boxtimes \phi_2} u_2 = \widehat{\phi_1 u_1}(\xi) \cdot \widehat{\phi_2 u_2}(\eta). \tag{5.208}$$

Notice that if $\widehat{\phi_1 u_1}(\xi)$ is rapidly decreasing around $\overline{\xi} \neq 0$ then the product is rapidly decreasing around *any* $(\overline{\xi},\eta)$. This gives (5.207). □

5.23. Diffeomorphisms

We next turn to the question of the extension of F^*, where $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map, from $\mathcal{C}^\infty(\Omega_2)$ to some elements of $\mathcal{C}^{-\infty}(\Omega_2)$. The simplest example of pull-back is that of transformation by a diffeomorphism.

We have already noted how pseudodifferential operators behave under a diffeomorphism: $F : \Omega_1 \longrightarrow \Omega_2$ between open sets of $\mathbb{R}^n$. Suppose $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel of *compact* support in $\Omega_1 \times \Omega_1$ then we define

$$A_F : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_2) \tag{5.209}$$

by $A_F = G^* \cdot A \cdot F^*$, $G = F^{-1}$. In § 5.4 we showed that $A_F \in \Psi^m_\infty(\mathbb{R}^n)$. In fact we showed much more, namely we computed a (very complicated) formula for the full symbols. Recall the definition of the *cotangent bundle* of $\mathbb{R}^n$

$$T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \tag{5.210}$$

identified as pairs of points $(\overline{x}, \overline{\xi})$, where $\overline{x} \in \mathbb{R}^n$ and

$$\overline{\xi} = df(\overline{x}) \text{ for some } f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{5.211}$$

The *differential* $df(\overline{x})$ of f at $\overline{x} \in \mathbb{R}^n$ is just the equivalence class of $f(x) - f(\overline{x}) \in \mathcal{I}_{\overline{x}}$ modulo $\mathcal{I}^2_{\overline{x}}$. Here

$$\begin{cases} \mathcal{I}_{\overline{x}} = \left\{ g \in \mathcal{C}^\infty(\mathbb{R}^n);\ g(\overline{x}) = 0 \right\} \\ \mathcal{I}^2_{\overline{x}} = \left\{ \sum\limits_{\text{finite}} g_i h_i,\ g_i, h_i \in \mathcal{I}_{\overline{x}} \right\}. \end{cases} \tag{5.212}$$

The identification of $\overline{\xi}$, given by (5.210) and (5.211), with a point in $\mathbb{R}^n$ is obtained using Taylor's formula. Thus if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$

$$f(x) = f(\overline{x}) + \sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j + \sum_{i,j=1} g_{ij}(x) x_i x_j. \tag{5.213}$$

The double sum here is in $\mathcal{I}^2_{\overline{x}}$, so the residue class of $f(x) - f(\overline{x})$ in $\mathcal{I}_{\overline{x}}/\mathcal{I}^2_{\overline{x}}$ is the same as that of

$$\sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j. \tag{5.214}$$

That is, $d(x - \overline{x})_j = dx_j$, $j = 1, \dots, n$ form a basis for $T^*_{\overline{x}}\mathbb{R}^n$ and in terms of this basis

$$df(\overline{x}) = \sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x}) dx_j. \tag{5.215}$$

Thus the entries of $\overline{\xi}$ are just $\left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}\right)$ for some f. Another way of saying this is that the *linear functions* $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 \cdots \xi_n x_n$ have differentials spanning $T^*_x\mathbb{R}^n$.

So suppose $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map. Then

$$F^* : T^*_{\overline{y}}\Omega_2 \longrightarrow T^*_{\overline{x}}\Omega_1,\ \overline{y} = F(\overline{x}) \tag{5.216}$$

is defined by $F^*df(\overline{y}) = d(F^*f)(\overline{x})$ since $F^* : \mathcal{I}_{\overline{y}} \longrightarrow \mathcal{I}_{\overline{x}}$, $F^* : \mathcal{I}^2_{\overline{y}} \longrightarrow \mathcal{I}^2_{\overline{x}}$. In coordinates $F(x) = y \Longrightarrow$

$$\frac{\partial}{\partial x_j}(F^*f(x)) = \frac{\partial}{\partial y}f(F(x)) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(y)\frac{\partial F_k}{\partial x_j} \tag{5.217}$$

i.e. $F^*(\eta \cdot dy) = \xi \cdot dx$ if

$$\xi_j = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_j}(x) \cdot \eta_k. \tag{5.218}$$

Of course if F is a diffeomorphism then the Jacobian matrix $\frac{\partial F}{\partial x}$ is invertible and (5.218) is a linear isomorphism. In this case

$$\begin{gathered} F^* : T^*_{\Omega_2}\mathbb{R}^n \longleftrightarrow T^*_{\Omega_1}\mathbb{R}^n \\ (x, \xi) \longleftrightarrow (F(x), \eta) \end{gathered} \tag{5.219}$$

with ξ and η connected by (5.218). Thus $(F^*)^* : \mathcal{C}^\infty(T^*\Omega_1) \longrightarrow \mathcal{C}^\infty(T^*\Omega_2)$.

Proposition 5.16. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism of open sets of $\mathbb{R}^n$ and $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel with compact support in $\Omega_1 \times \Omega_2$ then*

$$\sigma_m(A_F) = (F^*)^*\sigma_m(A) \tag{5.220}$$

and

$$F^*\left(\mathrm{WF}'(A_F)\right) = \mathrm{WF}'(A). \tag{5.221}$$

It follows that symbol $\sigma_m(A)$ of A is well-defined as an element of $S^{m-[1]}_\infty(T^*\mathbb{R}^n)$ independent of coordinates and $\mathrm{WF}'(A) \subset T^*\mathbb{R}^n\backslash 0$ is a well-defined closed conic set, independent of coordinates. The elliptic set and the characteristic set Σ_m are therefore also well-defined complementary conic subsets of $T^*\Omega\backslash 0$.

Proof. Look at the formulae. □

The main use we make of this invariance result is the freedom it gives us to *choose* local coordinates adapted to a particular problem. It also suggests that there should be neater ways to write various formulae, e.g. the wavefront sets of push-forward and pull-backs.

Proposition 5.17. *If $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ has $\mathrm{supp}(u) \subset \Omega_2$ and $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism then*

$$\mathrm{WF}(F^*u) \subset \big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash 0); (F(x),\eta) \in \mathrm{WF}(u), \eta_j = \sum_i \frac{\partial F_i}{\partial x_j}(x)\xi_i\big\}. \tag{5.222}$$

Proof. Just use the standard definition

$$\mathrm{WF}(F^*u) = \bigcap \big\{\Sigma(A);\ A(F^*u) \in \mathcal{C}^\infty\big\}. \tag{5.223}$$

To test the wavefront set of F^*u it suffices to consider A's with kernels supported in $\Omega_1 \times \Omega_1$ since $\mathrm{supp}(F^*u \Subset \Omega_1$ and for a general pseudodifferential operator A'

there exists A with kernel supported in Ω_1 such that $A'u - Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then $AF^*u \in \mathcal{C}_c^\infty(\Omega_1) \Longleftrightarrow A_F u \in \mathcal{C}_c^\infty(\Omega_2)$. Thus

$$\mathrm{WF}(F^*u) = \bigcap\big\{\Sigma(A);\ A_F u \in \mathcal{C}^\infty\big\} \tag{5.224}$$

$$= \bigcap\big\{F^*(\Sigma(A_F)); A_F u \in \mathcal{C}^\infty\big\} \tag{5.225}$$

$$= F^*\,\mathrm{WF}(u) \tag{5.226}$$

since, for u, it is enough to consider operators with kernels supported in $\Omega_2\times\Omega_2$. □

5.24. Products

Although we have discussed the definition of the product of two distributions we have not yet analyzed the wavefront set of the result.

Proposition 5.18. *If $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ are such that*

$$(x,\xi) \in \mathrm{WF}(u_1) \Longrightarrow (x,-\xi) \notin \mathrm{WF}(u_2) \tag{5.227}$$

then the product $u_1u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$, defined by Proposition 5.12 satisfies

$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset \big\{&(x,\xi); x\in \operatorname{supp}(u_1) \text{ and } (x,\xi)\in \mathrm{WF}(u_2)\big\}\\ &\cup\big\{(x,\xi); x\in\operatorname{supp}(u_2) \text{ and } (x,\xi)\in\mathrm{WF}(u_1)\big\}\\ &\cup\big\{(x,\xi); \xi=\eta_1+\eta_2, (x,\eta_i)\in\mathrm{WF}(u_i), i=1,2\big\}.\end{aligned} \tag{5.228}$$

Proof. We can represent the product in terms of three 'elementary' operations.

$$u_1u_2(x) = \iota^*\big[F^*(u_1\boxtimes u_2)\big] \tag{5.229}$$

where $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is the linear transformation

$$F(x,y) = (x+y, x-y) \tag{5.230}$$

and $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^n\times\{0\} \subset \mathbb{R}^{2n}$ is inclusion as the first factor. Thus (5.229) corresponds to the 'informal' notation

$$u_1u_2(x) = u_1(x+y)u_2(x-y) \restriction \{y=0\} \tag{5.231}$$

and will follow by continuity once we analyse the wavefront set properties.

We know from Proposition 5.15 that

$$\begin{aligned}\mathrm{WF}\,(u_1\boxtimes u_2) \subset &\big\{(X,Y,\Xi,H)\,; X\in\operatorname{supp}(u_1), \Xi=0, (Y,H)\in\mathrm{WF}(u_2)\big\}\\ \cup&\big\{(X,Y,\Xi,H)\,;\ (X,\Xi)\in\mathrm{WF}(u_1),\ Y\in\operatorname{supp}(u_2), H=0\big\}\\ \cup&\big\{(X,Y,\Xi,H)\,;\ (X,\Xi)\in\mathrm{WF}(u_1),\ (Y,H)\in\mathrm{WF}(u_2)\big\}.\end{aligned} \tag{5.232}$$

Since F is a diffeomorphism, by Proposition 5.17,

$$\begin{aligned}\mathrm{WF}(F^*(u_1\boxtimes u_2)) = \big\{(x,y,\xi,\eta); (F^t(x,y),\Xi,H)\in\mathrm{WF}(u_1\boxtimes u_2),&\\ (\xi,\eta)=A^t(\Xi,H)\big\}&.\end{aligned}$$

where F^t is the transpose of F as a linear map. In fact $F^t = F$, so

$$\begin{aligned}&\mathrm{WF}(F^*(u_1\boxtimes u_2)) \subset\\ &\quad\big\{(x,y,\xi,\eta); x+y\in\operatorname{supp}(u_1),\ \xi+\eta=0, (x-y,\tfrac{1}{2}(\xi-\eta))\in\mathrm{WF}(u_2)\big\}\\ &\quad\cup\big\{(x,y,\xi,\eta); (x+y,\tfrac{1}{2}(\xi+\eta))\in\mathrm{WF}(u_1),\ (x-y,\tfrac{1}{2}(\xi-\eta))\in\mathrm{WF}(u_2)\big\}\end{aligned}$$

and so using Proposition 5.14

$$\begin{aligned}\mathrm{WF}(F^*(u_1 \boxtimes u_2)) \restriction \{y=0\} \\ \subset \big\{(x,0,\xi,-\xi); x \in \operatorname{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(x,0,\xi,\eta); (x \in \operatorname{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(x,0,\xi,\eta); (x, \frac{1}{2}(\xi+\eta)) \in \mathrm{WF}(u_2), (x, \frac{1}{2}(\xi-\eta)) \in \mathrm{WF}(u_1)\big\}\end{aligned}$$

Notice that

(5.233)
$$(x,0,0,\eta) \in \mathrm{WF}\left(F^*(u_1 \boxtimes u_2)\right) \Longrightarrow (x, \frac{1}{2}\eta) \in \mathrm{WF}(u_1) \text{ and } (x, \frac{1}{2}\eta)\, \mathrm{WF}(u_2)$$

which introduces the assumption under which $u_1 u_2$ is defined. Finally then we see that

(5.234)
$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset \big\{(x,\xi); x \in \operatorname{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup\big\{(x,\xi); x \in \operatorname{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_1)\big\} \\ \cup\big\{(x,\xi); (x,\eta_1) \in \mathrm{WF}(u_1), (x,\eta_2) \in \mathrm{WF}(u_2) \text{ and } \xi = \eta_1 + \eta_2\big\}.\end{aligned}$$

which is another way of writing the conclusion of Proposition 5.18. □

5.25. Pull-back

Now let us consider a general $\mathcal{C}^\infty$ map

(5.235) $$F : \Omega_1 \longrightarrow \Omega_2,\ \Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m.$$

Thus even the dimension of domain and range spaces can be different. When can we *define* F^*u, for $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ and what can we say about $\mathrm{WF}(F^*u)$? For a general map F it is not possible to give a sensible, i.e. consistent, definition of F^*u for all distributions $u \in \mathcal{C}^{-\infty}(\Omega_2)$.

For smooth functions we have defined

(5.236) $$F^* : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$$

but in general $F^*\phi$ does not have compact support, even if ϕ does. We therefore impose the condition that F be *proper*

(5.237) $$F^{-1}(K) \Subset \Omega_2 \ \forall\ K \Subset \Omega_2,$$

(mostly just for convenience). In fact if we want to understand F^*u near $\overline{x}_1 \in \Omega_1$ we only need to consider u near $F(\overline{x}_1) \in \Omega_2$.

The problem is that the map (5.235) may be rather complicated. However *any* smooth map can be decomposed into a product of simpler maps, which we can analyze locally. Set

(5.238) $$\operatorname{graph}(F) = \big\{(x,y) \in \Omega_1 \times \Omega_2;\ y = F(x)\big\} \xrightarrow{\iota_F} \Omega_1 \times \Omega_2.$$

This is *always* an embedded submanifold of $\Omega_1 \times \Omega_2$ the functions $y_i - F_i(x)$, $i = 1, \dots, N$ are independent defining functions for $\operatorname{graph}(F)$ and $x_1, \dots, x_n$ are coordinates on it. Now we can write

(5.239) $$F = \pi_2 \circ \iota_F \circ g$$

where $g : \Omega_1 \longleftrightarrow \operatorname{graph}(F)$ is the diffeomorphism onto its range $x \longmapsto (x, F(x))$. This decomposes F as a projection, an inclusion and a diffeomorphism. Now consider

$$F^*\phi = g^* \cdot \iota_F^* \cdot \pi_2^*\phi \tag{5.240}$$

i.e. $F^*\phi$ is obtained by pulling ϕ back from Ω_2 to $\Omega_1 \times \Omega_2$, restricting to $\operatorname{graph}(F)$ and then introducing the x_i as coordinates. We have directly discussed $(\pi_2^*\phi)$ but we can actually write it as

$$\pi_2^*\phi = 1 \boxtimes \phi(y), \tag{5.241}$$

so the result we have proved can be applied to it. So let us see what writing (5.240) as

$$F^*\phi = g^* \circ \iota_F^*(1 \boxtimes \phi) \tag{5.242}$$

tells us. If $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ then

$$\operatorname{WF}(1 \boxtimes u) \subset \big\{(x, y, 0, \eta); (y, \eta) \in \operatorname{WF}(u)\big\} \tag{5.243}$$

by Proposition 5.15. So we have to discuss $\iota_F^*(1 \boxtimes u)$, i.e. restriction to $y = F(x)$. We can do this by making a diffeomorphism:

$$T_F(x, y) = (x, y + F(x)) \tag{5.244}$$

so that $T_F^{-1}(\operatorname{graph}(F)) = \{(x, 0)\}$. Notice that $g \circ T_F = \pi_1$, so

$$F^*\phi = \iota_{\{y=0\}}^* \left(T_F^*(1 \boxtimes u)\right). \tag{5.245}$$

Now from Proposition 5.17 we know that

$$\begin{aligned} \operatorname{WF}(T_F^*(1 \boxtimes u)) &= T_F^*(\operatorname{WF}(1 \boxtimes u)) \\ &= \big\{(X, Y, \Xi, H); (X, Y + F(X), \xi, \eta) \in \operatorname{WF}(1 \boxtimes u), \\ &\qquad\qquad \eta = H, \xi_i = \Xi_i + \Sigma \frac{\partial F_j}{\partial x_i} H_j\big\} \end{aligned} \tag{5.246}$$

i.e.

$$\operatorname{WF}(T_F^*(1 \boxtimes u)) = \big\{(x, y, \xi, \eta); \xi_i = \sum_j \frac{\partial F_j}{\partial x_j}(x)\eta_j, (F(x), \eta) \in \operatorname{WF}(u)\big\}. \tag{5.247}$$

So consider our existence condition for restriction to $y = 0$, that $\xi \neq 0$ on $\operatorname{WF}(T_F^*(1 \boxtimes u))$ i.e.

$$(F(x), \eta) \in \operatorname{WF}(u) \Longrightarrow \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j \neq 0. \tag{5.248}$$

If (5.248) holds then, from (5.246) and Proposition 5.14

$$\operatorname{WF}(F^*u) \subset \big\{(x, \xi);\ \exists\ (F(x), \eta) \in \operatorname{WF}(u) \text{ and } \xi_j = \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j\big\}. \tag{5.249}$$

We can reinterpret (5.248) and (5.249) more geometrically. The differential of F gives a map

$$\begin{gathered} F^* : T_{F(x)}^*\Omega_2 \longrightarrow T_x^*\Omega_1\ \forall\ x \in \Omega_1 \\ (F(x), \eta) \longmapsto (x, \xi) \text{ where } \xi_i = \Sigma \frac{\partial F_j}{\partial x_i}\eta_j. \end{gathered} \tag{5.250}$$

Thus (5.248) can be restated as:

$$\begin{gathered}\forall\ x \in \Omega_1,\ \text{the null space of } F_x^* : T^*_{F(x)}\Omega_2 \longrightarrow T^*_x\Omega_1 \\ \text{does not meet } \operatorname{WF}(u)\end{gathered} \tag{5.251}$$

and then (5.249) becomes

$$\operatorname{WF}(F^*u) \subset \bigcup_{x\in\Omega_1} F_x^*[\operatorname{WF}(u) \cap T^*_{F(x)}\Omega_2] = F^*(\operatorname{WF}(u)) \tag{5.252}$$

(proved we are a little careful in that F^* is *not* a map; it is a *relation* between $T^*\Omega_2$ and $T^*\Omega_1$) and in this sense (5.251) holds. Notice that (5.249) is a sensible "consequence" of (5.251), since otherwise $\operatorname{WF}(F^*u)$ would contain some zero directions.

PROPOSITION 5.19. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a proper $\mathcal{C}^\infty$ map then F^* extends (by continuity) from $\mathcal{C}^\infty_c(\Omega_2)$ to*

$$\big\{u \in \mathcal{C}^{-\infty}_c(\Omega_2); F^*(\operatorname{WF}(u)) \cap (\Omega_1 \times 0) = \emptyset \textit{ in } T^*\Omega_1\big\} \tag{5.253}$$

and (5.252) *holds.*

5.26. The operation F_*

Next we will look at the dual operation, that of push-forward. Notice the basic properties of pull-back:

$$\text{Maps } \mathcal{C}^\infty_c \text{ to } \mathcal{C}^\infty_c (\text{if } F \text{ is proper}) \tag{5.254}$$

$$\textit{Not} \text{ always defined on distributions.} \tag{5.255}$$

Dually we find

PROPOSITION 5.20. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map of an open subset of $\mathbb{R}^n$ into an open subset of $\mathbb{R}^n$ then for any $u \in \mathcal{C}^{-\infty}_c(\Omega_1)$*

$$F_*(u)(\phi) = u(F^*\phi) \tag{5.256}$$

is a distribution of compact support and

$$F_* :\ \mathcal{C}^{-\infty}_c(\Omega_1) \longrightarrow \mathcal{C}^{-\infty}_c(\Omega_2) \tag{5.257}$$

has the property:

$$\begin{aligned}\operatorname{WF}(F_*u) \subset \big\{(y,\eta); y \in F(\operatorname{supp}(u)), y = F(x), F_x^*\eta = 0\big\}\cup \\ \big\{(y,\eta); y = F(x), (x, F_x^*\eta) \in \operatorname{WF}(u)\big\}.\end{aligned} \tag{5.258}$$

PROOF. Notice that the 'opposite ' of (5.254) and (5.255) hold, i.e. F_* is *always* defined but even if $u \in \mathcal{C}^\infty_c(\Omega_1)$ in general $F_*u \notin \mathcal{C}^\infty_c(\Omega_2)$. All we really have to prove is (5.258). As usual we look for a formula in terms of elementary operations. So suppose $u \in \mathcal{C}^\infty_c(\Omega_1)$

$$\begin{aligned}F_*u(\phi) &= u(F^*\phi) \quad \phi \in \mathcal{C}^\infty_c(\Omega_2) \\ &= \int u(x)\ \phi(F(x))\ dx \\ &= \int u(x)\delta(y - F(x))\ \phi(y)\ dydx.\end{aligned} \tag{5.259}$$

Thus, we see that

$$F_*u = \pi_* H^*(u \boxtimes \delta) \tag{5.260}$$

where $\delta=\delta(y)\in\mathcal{C}_c^{-\infty}(\mathbb{R}^m)$, H is the diffeomorphism

$$H(x,y)=(x,y-F(x)) \tag{5.261}$$

and $\pi:\mathbb{R}^{n+m}\longrightarrow\mathbb{R}^m$ is projection *off* the first factor.

Thus (5.260) is the desired decomposition into elementary operations, since $u\boxtimes\delta\in\mathcal{C}_c^{-\infty}(\mathbb{R}^{n+m})$, $\pi_*H^*(u\boxtimes\delta)$ is always defined and indeed the map is continuous, which actually proves (5.260).

So all we need to do is estimate the wavefront set using our earlier results. From Proposition 5.15 it follows that

$$\begin{aligned}\operatorname{WF}(u\boxtimes\delta)\subset\big\{(x,0,\xi,\eta);x\in\operatorname{supp}(u),\xi=0\big\}&\cup\big\{(x,0,\xi,0);(x,\xi)\in\operatorname{WF}(u)\big\}\\ &\cup\big\{(x,0,\xi,\eta);(x,\xi)\in\operatorname{WF}(u)\big\}\\ =\big\{(x,0,\xi,\eta);x\in\operatorname{supp}(u),\xi=0\big\}&\cup\big\{(x,0,\xi,\eta);(x,\xi)\in\operatorname{WF}(u)\big\}.\end{aligned} \tag{5.262}$$

Then consider what happens under H^*. This is a diffeomorphism so the wavefront set transforms under the pull-back:

$$\begin{aligned}&\operatorname{WF}(H^*(u\boxtimes\delta))=\operatorname{WF}(u(x)\delta(y-F(x))\\ &=\big\{(x,F(x),\Xi,\eta);\Xi_i=\xi_i-\sum_j\frac{\partial F_j}{\partial x_i}(x)\eta_j,(x,0,\xi,\eta)\in\operatorname{WF}(u\boxtimes\delta)\big\}\\ &=\big\{(x,F(x),\Xi,\eta);x\in\operatorname{supp}(u),\Xi_i=-\sum_j\frac{\partial F_j}{\partial x_i}(x)\eta_j)\big\}\\ &\quad\cup\big\{(x,F(x),\Xi,\eta);\eta\in\mathbb{R}^m,(x,\xi)\in\operatorname{WF}(u)),\Xi_i=\xi_i-\sum_j\frac{\partial F_i}{\partial x_j}\eta_j\big\}.\end{aligned} \tag{5.263}$$

Finally recall the behaviour of wavefront sets under projection, to see that

$$\begin{aligned}&\operatorname{WF}(F_*u)\subset\big\{(y,\eta);\ \exists\ (x,y,0,\eta)\in\operatorname{WF}(H^*(u\boxtimes\delta))\big\}\\ &=\big\{(y,\eta);y=F(x)\text{ for some }x\in\operatorname{supp}(u)\text{ and}\\ &\qquad\qquad\sum_j\frac{\partial F_j}{\partial x_i}\eta_j=0,\ i=1,\dots,n\big\}\\ &\cup\big\{(y,\eta);y=F(x)\text{ for some }(x,\xi)\in\operatorname{WF}(u)\text{ and}\\ &\qquad\qquad\xi_i=\sum_j\frac{\partial F_i}{\partial x_i}\eta_j,i=1,\dots,n)\big\}.\end{aligned}$$

This says

$$\operatorname{WF}(F_*u)\subset\big\{(y,\eta);y\in F(\operatorname{supp}(u))\text{ and }F_x^*(\eta)=0\big\} \tag{5.264}$$

$$\cup\big\{(y,\eta);y=F(x)\text{ with }(x,F_x^*\eta)\in\operatorname{WF}(u)\big\} \tag{5.265}$$

which is just (5.258). □

As usual one should note that the two terms here are "really the same".

Now let us look at F_* as a linear map,

$$F_*:\ \mathcal{C}_c^{\infty}(\Omega_1)\longrightarrow\mathcal{C}_c^{-\infty}(\Omega_2). \tag{5.266}$$

As such it has a Schwartz kernel, indeed (5.260) is just the usual formula for an operator in terms of its kernel:

$$F_*u(y) = \int K(y,x)u(x)dx,\ \ K(y,x) = \delta(y - F(x)). \tag{5.267}$$

So consider the wavefront set of the kernel:

$$\mathrm{WF}(\delta(y - F(x)) = \mathrm{WF}(H^*\delta(y)) = \big\{(y,x;\eta,\xi); y = F(x), \xi = F_x^*\eta\big\}. \tag{5.268}$$

Now changing the order of the factors we can regard this as a subset

$$\mathrm{WF}'(K) = \big\{((y,\eta),(x,\xi)); y = F(x), \xi = F^*\eta\big\} \subset (\Omega_2 \times \mathbb{R}^m) \times (\Omega_1 \times \mathbb{R}^n). \tag{5.269}$$

As a subset of the product we can regard $\mathrm{WF}'(K)$ as a *relation*: if $\Gamma \subset \Omega_2 \times (\mathbb{R}^n\backslash 0)$ set

$$\mathrm{WF}'(K) \circ \Gamma = \\ \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^m\backslash 0); \exists\ ((y,\eta)),(x,\xi)) \in \mathrm{WF}'(K) \text{ and } (x,\xi) \in \Gamma\big\}$$

Indeed with this definition

$$\mathrm{WF}(F_*u) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u), \quad K = \text{ kernel of } F_*. \tag{5.270}$$

5.27. Wavefront relation

One serious application of our results to date is:

THEOREM 5.4. *Suppose $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ are open and $A \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2)$ has proper support, in the sense that the two projections*

$$\begin{tikzcd} & \mathrm{supp}(A) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Omega_1 & & \Omega_2 \end{tikzcd} \tag{5.271}$$

are proper, then A defines a linear map

$$A : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1) \tag{5.272}$$

and extends by continuity to a linear map

$$A : \big\{u \in \mathcal{C}_c^{-\infty}(X); \mathrm{WF}(u) \cap \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^n\backslash 0); \tag{5.273}$$

$$\exists\ (x,0,y,-\eta) \in \mathrm{WF}(K)\big\} = \emptyset\big\} \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1) \tag{5.274}$$

for which

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u), \tag{5.275}$$

where

$$\begin{aligned} \mathrm{WF}'(A) = \big\{((x,\xi),(y,\eta)) \in &(\Omega_1 \times \mathbb{R}^n) \times (\Omega_2 \times \mathbb{R}^m); (\xi,\eta) \neq 0 \\ &\textit{and } (x,y,\xi,-\eta) \in \mathrm{WF}(K)\big\}. \end{aligned} \tag{5.276}$$

PROOF. The action of the map A can be written in terms of its Schwartz kernel as

$$Au(x) = \int K(x,y)u(y)dy = (\pi_1)_*(K \cdot (1 \boxtimes u)). \tag{5.277}$$

Here $1 \boxtimes u$ is always defined and

$$\mathrm{WF}(1 \boxtimes u) \subset \big\{(x,y,0,\eta); (y,\eta) \in \mathrm{WF}(u)\big\}. \tag{5.278}$$

So the main question is, when is the product defined? Our sufficient condition for this is:

$$(x,y,\xi,\eta)\in \mathrm{WF}(K)\Longrightarrow (x,y,-\xi,-\eta)\notin \mathrm{WF}(1\boxtimes u) \tag{5.279}$$

which is

$$(x,y,0,\eta)\in \mathrm{WF}(K)\Longrightarrow (x,y,0,-\eta)\notin \mathrm{WF}(1\boxtimes u) \tag{5.280}$$

$$\text{i.e. } (y,-\eta)\notin \mathrm{WF}(u) \tag{5.281}$$

This of course is (5.274):

$$Au \text{ is defined (by continuity) if} \tag{5.282}$$

$$\big\{(y,\eta)\in \mathrm{WF}(u);\ \exists\ (x,0,y,-\eta)\in \mathrm{WF}(A)\big\}=\emptyset. \tag{5.283}$$

Then from our bound on the wavefront set of a product

$$\begin{aligned}\mathrm{WF}\,(K\cdot(1\boxtimes u))\subset&\\ \big\{(x,y,\xi,\eta);(\xi,\eta)=&(\xi',\eta')+(0,\eta'')\text{ with}\\ &(x,y,\xi',\eta')\in \mathrm{WF}(K)\text{ and }(x,\eta'')\in \mathrm{WF}(u)\big\}\\ \cup\big\{(x,y,\xi,\eta);(x,&y,\xi,\eta)\in \mathrm{WF}(K),\ y\in \operatorname{supp}(u)\big\}\\ \cup\big\{(x,y,0,\eta);&(x,y)\in \operatorname{supp}(A)(y,\eta)\in \mathrm{WF}(u)\big\}.\end{aligned} \tag{5.284}$$

This gives the bound

$$\mathrm{WF}\,(\pi_*(K\cdot(1\boxtimes u)))\subset \big\{(x,\xi);(x,y,\xi,0)\in \mathrm{WF}(K\cdot(1\boxtimes u))\text{ for some } y\big\} \tag{5.285}$$

$$\subset \mathrm{WF}'(A)\circ \mathrm{WF}(u). \tag{5.286}$$

□

5.28. Applications

Having proved this rather general theorem, let us note some examples and applications.

First, for pseudodifferential operators we know that

$$\mathrm{WF}'(A)\subset\{(x,x,\xi,\xi)\} \tag{5.287}$$

i.e. corresponds to the identity relation (which is a map). Then (5.275) is the microlocality of pseudodifferential operators. The next result also applies to all pseudodifferential operators.

Corollary 5.2. *If* $K\in\mathcal{C}^{-\infty}(\Omega_1\times\Omega_2)$ *has proper support and*

$$\mathrm{WF}'(K)\cap\{(x,y,\xi,0)\}=\emptyset \tag{5.288}$$

then the operator with Schwartz kernel K *defines a continuous linear map*

$$A:\mathcal{C}^\infty_c(\Omega_2)\longrightarrow\mathcal{C}^\infty_c(\Omega_1). \tag{5.289}$$

If

$$\mathrm{WF}'(K)\cap\{(x,y,0,\eta)\}=\emptyset \tag{5.290}$$

then A *extends by continuity to*

$$A:\mathcal{C}^{-\infty}_c(\Omega_2)\longrightarrow\mathcal{C}^{-\infty}_c(\Omega_1). \tag{5.291}$$

Proof. Immediate from (5.272)-(5.291). □

5.29. Problems

PROBLEM 5.9. Show that the general definition (5.52) reduces to

$$\mathrm{WF}(u) = \bigcap \big\{ \Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n) \big\},\ u \in \mathcal{S}'(\mathbb{R}^n) \tag{5.292}$$

and prove the basic result of 'microlocal elliptic regularity:'

$$\begin{gathered} \text{If } u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } A \in \Psi^m_\infty(\mathbb{R}^n) \text{ then} \\ \mathrm{WF}(u) \subset \Sigma(A) \cup \mathrm{WF}(Au). \end{gathered} \tag{5.293}$$

PROBLEM 5.10. Compute the wavefront set of the following distributions:

$$\begin{gathered} \delta(x) \in \mathcal{S}'(\mathbb{R}^n),\ |x| \in \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ \chi_{\mathbb{B}^n}(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1. \end{cases} \end{gathered} \tag{5.294}$$

PROBLEM 5.11. Let $\Gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ be an open cone and define

$$\mathcal{C}^{-\infty}_{c,\Gamma}(\mathbb{R}^n) = \big\{ u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n); Au \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{5.295}$$

$$\forall\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ with } \mathrm{WF}'(A) \cap \Gamma = \emptyset \big\}. \tag{5.296}$$

Describe a complete topology on this space with respect to which $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is a dense subspace.

PROBLEM 5.12. Show that, for any pseudodifferential operator $A \in \Psi^m_\infty(\mathbb{R}^n)$, $\mathrm{WF}'(A) = \mathrm{WF}'(A^*)$.

PROBLEM 5.13. Give an alternative proof to Lemma 5.5 along the following lines (rather than using Lemma 2.75). If $\sigma_L(A)$ is the left reduced symbol then for $\epsilon > 0$ small enough

$$b_0 = \gamma_\epsilon \big/ \sigma_L(A) \in S^{-m}_\infty(\mathbb{R}^n; \mathbb{R}^n). \tag{5.297}$$

If we choose $B_0 \in \Psi^{-m}_\infty(\mathbb{R}^n)$ with $\sigma_L(B_0) = b_0$ then

$$\mathrm{Id} - A \circ B_0 = G \in \Psi^0_\infty(\mathbb{R}^n) \tag{5.298}$$

has principal symbol

$$\sigma_0(G) = 1 - \sigma_L(A) \cdot b_0. \tag{5.299}$$

From (5.67)

$$\gamma_{\epsilon/4}\sigma_0(G) = \gamma_{\epsilon/4}. \tag{5.300}$$

Thus we conclude that if $\sigma_L(C) = \gamma_{\epsilon/4}$ then

$$G = (\mathrm{Id} - C)G + CG \text{ with } CG \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{5.301}$$

Thus (5.298) becomes

$$\mathrm{Id} - AB_0 = CG + R_1 \quad \mathrm{WF}'(R_1) \not\ni z. \tag{5.302}$$

Let $B_1 \sim \sum\limits_{j \ge 1} (CG)^j$, $B_1 \in \Psi^{-1}$ and set

$$B = B_0\,(\mathrm{Id} + B_1) \in \Psi^{-m}_\infty(\mathbb{R}^n). \tag{5.303}$$

From (5.302)

$$AB = AB_0(I + B_1) \tag{5.304}$$

$$= (\mathrm{Id} - CG)\,(I + B_1) - R_1\,(\mathrm{Id} + B_1) \tag{5.305}$$

$$= \mathrm{Id} + R_2, \quad \mathrm{WF}'(R_2) \not\ni z. \tag{5.306}$$

Thus B is a right microlocal parametrix as desired. Write out the construction of a left parametrix using the same method, or by finding a right parametrix for the adjoint of A and then taking adjoints using Problem 5.12.

Problem 5.14. Essential uniqueness of left and right parametrices.

Problem 5.15. If $(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is a given point, construct a distribution $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ which has

$$\mathrm{WF}(u) = \left\{(\bar{x}, t\bar{\xi}); t > 0\right\} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0). \tag{5.307}$$

Problem 5.16. Suppose that $A \in \Psi_\infty^m(\mathbb{R}^n)$ has Schwartz kernel of compact support. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ use the four 'elementary operations' (and an earlier result on the wavefront set of kernels) to investigate under what conditions

$$\kappa(x, y) = K_A(x, y)u(y) \text{ and then } \gamma(x) = (\pi_1)_*\kappa \tag{5.308}$$

make sense. What can you say about $\mathrm{WF}(\gamma)$?

Problem 5.17. Consider the projection operation under $\pi_1 : \mathbb{R}^p \times \mathbb{R}^k \longrightarrow \mathbb{R}^p$. Show that $(\pi_1)_*$ can be extended to some distributions which do not have compact support, for example

$$\left\{u \in \mathcal{S}'(\mathbb{R}^n); \mathrm{supp}(u) \cap K \times \mathbb{R}^k \text{ is compact for each } K \subset\subset \mathbb{R}^n\right\}. \tag{5.309}$$

Problem 5.18. As an exercise, check the Jacobi identify for the Poisson bracket

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \ \forall \ f, g, h \in \mathcal{C}^\infty(\mathbb{R}^{2n}). \tag{5.310}$$

Problem 5.19. The fact that (5.90) determines H_h uniquely is equivalent to the non-degeneracy of ω, that

$$\omega(v, w) = 0 \ \forall \ w \Longrightarrow v = 0. \tag{5.311}$$

Show that if ω is a non-degenerate form and (5.90) is used to define the Poisson bracket by

$$\{f, g\} = \omega(H_f, H_g) = dg(H_f) = H_f g \tag{5.312}$$

then the Jacobi identity (5.310) holds if and only if ω is closed as a 2-form.

Problem 5.20. Check that a finite number of regions (5.94) cover the complement of a neighbourhood of 0 in $\mathbb{R}^n$ and that if a is smooth and has compact support in x then the estimates (5.95) is such neighbourhoods imply that $a \in S_{\mathrm{c}}^M(\mathbb{R}^n; \mathbb{R}^n)$ and conversely.

CHAPTER 6

Pseudodifferential operators on manifolds

In this chapter the notion of a pseudodifferential on a manifold is discussed. Some preliminary material on manifolds is therefore necessary. However the discussion of the basic properties of differentiable manifolds is kept to a bare minimum. For a more leisurely treatement the reader might well consult XX or YY. Our main aims here are first, to be able to prove the Hodge theorem (given the deRham theorem). Then we describe some global object which are very useful in applications, namely a global quantization map, the structure of complex powers and the zeta function.

6.1. $\mathcal{C}^\infty$ structures

Let X be a paracompact Hausdorff topological space. A $\mathcal{C}^\infty$ structure on X is a subspace

$$\mathcal{F} \subset C^0(X) = \{u : X \longrightarrow \mathbb{R} \ \text{ continuous } \} \tag{6.1}$$

with the following property:

For each $\overline{x} \in X$ there exists elements $f_1, \dots, f_n \in \mathcal{F}$ such that for some open neighbourhood $\Omega \ni \overline{x}$

$$F : \Omega \ni x \longmapsto (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n \tag{6.2}$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$ and *every* $f \in \mathcal{F}$ satisfies

$$f \restriction \Omega = g \circ F \quad \text{ for some } g \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{6.3}$$

The map (6.2) is a coordinate system near $\overline{x}$. Two $\mathcal{C}^\infty$ structures $\mathcal{F}_1$ and $\mathcal{F}_2$ are 'compatible' if $\mathcal{F}_1 \cup \mathcal{F}_2$ is also a $\mathcal{C}^\infty$ structure. Compatibility in this sense is an equivalence relation on $\mathcal{C}^\infty$ structures. It therefore makes sense to say that:

Definition 6.1. *A $\mathcal{C}^\infty$ manifold is a (connected) paracompact Hausdorff topological space with a* maximal *$\mathcal{C}^\infty$ structure.*

The maximal $\mathcal{C}^\infty$ structure is conventionally denoted

$$\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X). \tag{6.4}$$

It is necessarily an algebra. If we let $\mathcal{C}_c^\infty(W) \subset \mathcal{C}^\infty(X)$ denote the subspace of functions which vanish outside a compact subset of W then any local coordinates (6.2) have the property

$$F^* : \mathcal{C}_c^\infty(F(\Omega)) \longleftrightarrow \big\{u \in \mathcal{C}^\infty(X);\ u = 0 \ \text{ on } X \backslash K, K \subset\subset \Omega\big\}. \tag{6.5}$$

Futhermore $\mathcal{C}^\infty(X)$ is *local*:

$$\begin{gathered} u : X \longrightarrow \mathbb{R} \text{ and } \ \forall\, \overline{x} \in X\ \exists\ \Omega_{\overline{x}} \text{ open},\ \Omega_{\overline{x}} \ni \overline{x}, \\ \text{s.t. } u - f_{\overline{x}} = 0 \text{ on } \Omega_{\overline{x}} \text{ for some } f_{\overline{x}} \in \mathcal{C}^\infty(X) \Longrightarrow u \in \mathcal{C}^\infty(X). \end{gathered} \tag{6.6}$$

A map $G : X \longrightarrow Y$ between $\mathcal{C}^\infty$ manifolds X and Y is $\mathcal{C}^\infty$ if

$$G^* : \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X) \tag{6.7}$$

i.e. $G \circ u \in \mathcal{C}^\infty(X)$ for all $u \in \mathcal{C}^\infty(Y)$.

6.2. Form bundles

A vector bundle is a triple $\pi : V \longrightarrow X$ consisting of two manifolds, X and V, and a surjective $\mathcal{C}^\infty$ map π with each

$$V_x = \pi^{-1}(x) \tag{6.8}$$

having a *linear structure* such that

$$\mathcal{F} = \big\{u : V \longrightarrow \mathbb{R}, u \text{ is linear on each } V_x\big\} \tag{6.9}$$

is a $\mathcal{C}^\infty$ structure on V compatible with $\mathcal{C}^\infty(V)$ (i.e. contained in it, since it is maximal).

The basic example is the *cotangent bundle* which we defined before for open sets in $\mathbb{R}^n$. The same definition works here. Namely for each $\overline{x} \in X$ set

$$\begin{gathered}
\mathcal{I}_{\overline{x}} = \big\{u \in \mathcal{C}^\infty(X); u(\overline{x}) = 0\big\} \\
\mathcal{I}^2_{\overline{x}} = \big\{u = \sum_{\text{finite}} u_i u_i';\ u_i, u_i' \in \mathcal{I}_{\overline{x}}\big\} \\
T^*_{\overline{x}}X = \mathcal{I}_{\overline{x}}/\mathcal{I}^2_{\overline{x}},\ T^*X = \bigcup_{\overline{x}\in X} T^*_{\overline{x}}X.
\end{gathered} \tag{6.10}$$

So $\pi : T^*X \longrightarrow X$ just maps each $T^*_{\overline{x}}X$ to $\overline{x}$. We need to give T^*X a $\mathcal{C}^\infty$ structure so that "it" (meaning $\pi : T^*X \longrightarrow X$) becomes a vector bundle. To do so note that the *differential* of any $f \in \mathcal{C}^\infty(X)$

$$df : X \longrightarrow T^*X \quad df(\overline{x}) = [f - f(\overline{x})] \in T^*_{\overline{x}}X \tag{6.11}$$

is a section ($\pi \circ df = \mathrm{Id}$). Put

$$\mathcal{F} = \big\{u : T^*X \longrightarrow \mathbb{R}; u \circ df : X \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^\infty\ \forall\ f \in \mathcal{C}^\infty(X)\big\}. \tag{6.12}$$

Then $\mathcal{F} = \mathcal{C}^\infty(T^*X)$ is a maximal $\mathcal{C}^\infty$ structure on T^*X and

$$\mathcal{F}_{\text{lin}} = \big\{u : T^*X \longrightarrow \mathbb{R},\ \text{linear on each } T^*_{\overline{x}}X; u \in \mathcal{F}\big\}$$

is therefore compatible with it. Clearly df is $\mathcal{C}^\infty$.

Any (functorial) operation on finite dimensional vector spaces can be easily seen to generate new vectors bundles from old. Thus *duality*, *tensor product*, *exterior powers* all lead to new vector bundles:

$$T_xX = (T^*_xX)^*\,,\ TX = \bigcup_{x\in X} T_xX \tag{6.13}$$

is the tangent bundle

$$\Lambda^k_x X = \big\{u : \overbrace{T_xX \times \cdots \times T_xX}^{k \text{ factors}} \longrightarrow \mathbb{R}; u \text{ is multilinear and antisymmetric }\big\}$$

leads to the k-form bundle

$$\Lambda^k X = \bigcup_{x\in X} \Lambda^k_x X,\ \Lambda^1 X \simeq T^*X$$

where equivalence means there exists (in this case a *natural*) $\mathcal{C}^\infty$ diffeomorphism mapping fibres to fibres linearly (and in this case projecting to the identity on X).

A similar construction leads to the *density bundles*

$$\Omega_x^\alpha X = \big\{u : \overset{n=\dim X\quad\text{factors}}{T_xX \wedge \cdots \wedge T_xX} \longrightarrow \mathbb{R};\ \text{absolutely homogeneous of degree } \alpha\big\}$$

that is

$$u(tv_1 \wedge \dots v_n) = |t|^\alpha u(v_1 \wedge \cdots \wedge v_n).$$

These are important because of integration. In general if $\pi : V \longrightarrow X$ is a vector bundle then

$$\mathcal{C}^\infty(X;V) = \big\{u :\ X \longrightarrow V;\ \pi \circ u = \mathrm{Id}\big\}$$

is the space of sections. It has a natural linear structure. Suppose $W \subset X$ is a coordinate neighbourhood and $u \in \mathcal{C}^\infty(X;\Omega)$, $\Omega = \Omega^1X$, has compact support in W. Then the coordinate map gives an identification

$$\Omega_x^* X \longleftrightarrow \Omega_{F(x)}^*\mathbb{R}^n \qquad \forall\ \alpha$$

and

$$\int u = \int_{\mathbb{R}^n} g_u(x), \quad u = g_u(x)|dx| \tag{6.14}$$

is defined independent of coordiantes. That is the integral

$$\int : \mathcal{C}_c^\infty(X;\Omega) \longrightarrow \mathbb{R} \tag{6.15}$$

is well-defined.

6.3. Pseudodifferential operators

We will start with a definition of pseudodifferential operators on a (not necessarily compact) manifold which has lots of properties but may be a bit hard to verify in practice.

Definition 6.2. *If X is a $\mathcal{C}^\infty$ manifold and $\mathcal{C}_c^\infty(X) \subset \mathcal{C}^\infty(X)$ is the space of $\mathcal{C}^\infty$ functions of compact support, then, for any $m \in \mathbb{R}$, $\Psi^m(X)$ is the space of linear operators*

$$A : \mathcal{C}_c^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \tag{6.16}$$

with the following properties. First,

$$\begin{gathered}\text{if } \phi,\ \psi \in \mathcal{C}^\infty(X) \text{ have disjoint supports then } \exists\ K \in \mathcal{C}^\infty(X^2;\Omega_R)\\ \text{such that } \forall\ u \in \mathcal{C}_c^\infty(X)\ \phi A\psi u = \int_X K(x,y)u(y),\end{gathered} \tag{6.17}$$

and secondly if $F : W \longrightarrow \mathbb{R}^n$ is a coordinate system in X and $\psi \in \mathcal{C}_c^\infty(X)$ has support in W then

$$\begin{gathered}\exists\ B \in \Psi_\infty^m(\mathbb{R}^n),\ \mathrm{supp}(B) \subset F(W) \times F(W)\ \text{s.t.}\\ \psi A \psi u \restriction W = F^*(B((F^{-1})^*(\psi u)))\ \forall\ u \in \mathcal{C}_c^\infty(X).\end{gathered}$$

This seems a pretty horrible definition, since it requires us to check *every* coordinate system, at least in principle. In practice the coordinate-invariance we proved earlier (see Proposition 5.4) means that this is *not* necessary and also that there are plenty of examples as we proceed to see.

LEMMA 6.1. *The space $\Psi^{-\infty}(X) = \bigcap_m \Psi^m(X)$ contains all the smoothing operators on X, those with kernels $K \in \mathcal{C}^\infty(X^2; \Omega_R)$.*

In fact there is equality between $\Psi^{-\infty}(X)$ and the space of smoothing operators but it is easier to see this after a little more thought!

PROOF. Smoothing operators, having smooth kernels, satisfy the first part of the definition and also the second since smoothing operators with compactly supported kernels are pseudodifferential operators on $\mathbb{R}^n$. $\square$

LEMMA 6.2. *If $G : U \longrightarrow \mathbb{R}^n$ is a coordinate patch on X and $B \in \Psi^m_\infty(\mathbb{R}^n)$ has kernel with support $\operatorname{supp}(B) \Subset G(U) \times G(U)$ then*

$$Au = G^* B (G^{-1})^* (u\big|_U) \text{ defines } A \in \Psi^m(X). \tag{6.18}$$

PROOF. Since the kernel of a pseudodifferential operator is smooth outside the diagonal the first part of the definition holds for A – indeed if ϕ, $\psi \in \mathcal{C}^\infty(X)$ then

$$\phi A \psi = G^* B' (G^{-1})^* (u\big|_U),\ B' = ((G^{-1})^*\phi) B ((G^{-1})^*\psi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \tag{6.19}$$

since $(G^{-1})^*\phi$, $(G^{-1})^*\psi \in \mathcal{C}^\infty(G(U))$ have disjoint supports. Similarly for the second part, the identity (6.19) still holds and if ϕ and ψ are both supported in some other coordinate patch $F : W \longrightarrow \mathbb{R}^n$ then the support of the kernel of B' is contained in $G(U \cap W) \times G(U \cap W)$ and $H = F \circ G^{-1}$ is a diffeomorphism from $G(U \cap W)$ to $F(U \cap W)$. The local coordinate invariance in Proposition 2.11 shows that $B'' = H^* B' (H^{-1})^* \in \Psi^m(\mathbb{R}^n)$ has kernel with support in $F(U \cap W) \times F(U \cap W)$ and (6.19) becomes

$$\phi A \psi = F^* B'' (F^{-1})^* (u\big|_W) \tag{6.20}$$

which implies the second condition. $\square$

Thus there are lots of examples – if $B \in \Psi^m_\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(X)$ has support in a coordinate patch with image ψ' in the local coordinates then applying (6.18) to $\psi' B \psi'$ gives an element of $\Psi^m(X)$. In fact each pseudodifferential operator is a sum of a smoothing operator and terms of this type. To see this, first note the following elementary result. Any open cover of a $\mathcal{C}^\infty$ manifold has a partition of unity subordinate to it, i.e. if $A_r \subset X$ are open sets for $r \in R$ and

$$X = \bigcup_{r \in R} A_r \tag{6.21}$$

there exists $\phi_i \in \mathcal{C}^\infty_c(X)$, all non-negative with locally finite supports:

$$\forall\ i\ \operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset \text{ for a finite set of indices } j, \tag{6.22}$$

where each $\operatorname{supp}(\phi_i) \subset A_r$ for some $r = r(i)$ and

$$\sum_i \phi_i(x) = 1 \quad \forall\ x. \tag{6.23}$$

In fact one can do slightly better than this.

LEMMA 6.3. *Given an open cover U_a of X there exists a partition of unity ϕ_i (so with locally finite supports) and*

$$\forall\ i, j\ \exists\ a \text{ such that } \operatorname{supp}(\phi_j) \cap \operatorname{supp}_j \subset U_a. \tag{6.24}$$

Taking $i = j$ shows that the partition of unity is subordinate to the given open cover and the condition (6.24) is automatically satisfied if the intersection of supports is empty.

PROOF. Take any partition of unity ψ_a subordinate to the cover U_a and indexed so that $\operatorname{supp}(\psi_a) \subset U_a$. Thus, the support of each $\operatorname{supp}(\psi_a)$ is compact and only meets finitely many of the others. It follows that each point $p \in \operatorname{supp}(\phi_a)$ has a neighbourhood $V(p)$ which is contained in the intersection of all of the U_b such that $p \in \operatorname{supp}(\psi_b)$. For each a take a partition of unity of X subordinate to the cover by such $V(p)$'s and $X \setminus \operatorname{supp}(\psi_a)$. Then replace ψ_a by the finitely many non-zero products with this partition of unity (any term from a factor with support in $X \setminus \operatorname{supp}(\psi_a)$) gives zero. Taken together all the resulting (non-zero) functions give a partition of unity as desired since when two of the supports intersect they are contained in one of the $V(p)$'s. $\square$

PROPOSITION 6.1. *If ϕ_i is a partition of unity subordinate to a coordinate covering of X satisfying the condition of Lemma 6.3 and for each pair i, j such that $\operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset$ $F_{ij} : \Omega_{ij} \longrightarrow \mathbb{R}^n$ is a coordinate system in a neighbourhood Ω_{ij} of this set, then an operator $A : \mathcal{C}^\infty_c(X) \longrightarrow \mathcal{C}^\infty(X)$ is a pseudodifferential operator on X if and only if*

$$\begin{aligned}(6.25)\quad \phi_i A \phi_j \text{ has smooth kernel if } &\operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) = \emptyset \\ &\text{and otherwise is of the form } F^*_{ij} A_{ij} (F^{-1}_{ij})^* \text{ with } A_{ij} \in \Psi^m_\infty(\mathbb{R}^n) \\ &\qquad\qquad \text{and kernel supported in } F(\Omega_{ij}) \times F(\Omega_{ij}).\end{aligned}$$

PROOF. The necessity of these conditions follows directly from the definition. Conversely if A satisifies all these conditions then for each $\phi,\ \psi \in \mathcal{C}^\infty_c(X)$ $\phi A \psi$ is a finite sum (by local finiteness of the partition of unity) of terms to which either Lemma 6.1 or Lemma 6.2 applies. Thus it is an element of $\Psi^m(X)$. $\square$

So, this means that the original defintion can be replaced by the same one with respect to any given cover by coordinate patches – meaning that a pseudodifferential operator is just a (locally finite) sum of a smoothing operator plus pseudodifferential operators acting in a cover by coordinate patches $F_i : \Omega_i \longrightarrow \mathbb{R}^n$:

$$\begin{aligned}(6.26)\quad A \in \Psi^m(X) \Longrightarrow A = A' + \sum_i A_i,\ A' \in \mathcal{C}^\infty(X^2; \Omega_R),\ A_i = F^*_i B_i (F^{-1}_i)^*, \\ B_i \in \Psi^m_\infty(\mathbb{R}^n),\ \operatorname{supp}(B_i) \Subset_i (\Omega_i) \times F_i(\Omega_i).\end{aligned}$$

THEOREM 6.1. *Let X be a compact $\mathcal{C}^\infty$ manifold then the pseudodifferential operators $\Psi^*(X)$ form an order filtered ring.*

PROOF. The main point of course is that they form a ring, the order-filtering means that

$$\Psi^m(X) \circ \Psi^{m'}(X) \subset \Psi^{m+m'}(X). \tag{6.27}$$

Since X is compact, $\mathcal{C}^\infty_c(X) = \mathcal{C}^\infty(X)$ and all the operators act on $\mathcal{C}^\infty(X)$, so the product is well-defined. From the remarks above, it suffices to consider the four cases of products $A \circ B$ where A and B are either smoothing operators or pseudodifferential operators with supports in a coordinate patch. In fact using a partition of unity as in Lemma 6.3 corresponding to a coordinate cover and then applying Proposition 6.1 if they are *both* pseudodifferential operators we can assume they have support in the same coordinate patch. Then the result follows from the local composition theorem of Chapter 2. So it is enough to suppose that at least one of the operators is a smoothing operator. If both are smoothing then this follows from the fact that the kernel of the composite is given in terms of the kernels of the factors by

$$(A \circ B)(p,p') = \int_X A(p,\cdot)B(\cdot,p') \in \mathcal{C}^\infty(X^2;\Omega_R). \tag{6.28}$$

When one factor is smoothing and the other is a local pseudodifferential the composte is smoothing since it is given by the action of the pseudodifferential operator (or its transpose) on the kernel of the smoothing operator, in one of the variables. $\square$

Note that if X is not compact we cannot in general compose pseudodifferential operators, since the first one maps $\mathcal{C}^\infty_c(X)$ into $\mathcal{C}^\infty(X)$ and the second may not act on $\mathcal{C}^\infty(X)$. This is sorted out below.

Now, it is most important to show that the symbol maps still makes sense and has at leat most of the properties it had on $\mathbb{R}^n$. This is not quite obvious because of the non-uniqueness inherent in a presentation such as (6.25). First however we need to check that there is a place for the symbol to take values.

Recall that for an open set $\Omega \subset \mathbb{R}^n$ we defined the symbol spaces $S^m_\infty(\Omega;\mathbb{R}^p)$ as consisting of the smooth functions satisfying (2.1). Let $\pi : W \longrightarrow X$ be a real vector bundle over a manifold X. So X is covered by local coordinate patches Ω_i over which W is trivial, meaning there is a diffeomorphism

$$F_i : \pi^{-1}(\Omega_i) \longrightarrow \Omega'_i \times \mathbb{R}^p \tag{6.29}$$

which maps each fibre $\pi^{-1}(p)$ to the corresponding $\{p'\} \times \mathbb{R}^q$ and is a linear map. Then if we choose a partition of unity subordinate to the cover we can set

$$S^m(W) = \left\{ a : W \longrightarrow \mathbb{C}; a = \sum_i \phi_i F_i^* a_i \text{ for some } a_i \in S^m(\Omega'_i \times \mathbb{R}^p) \right\} \tag{6.30}$$

provided we show this is independent of choices.

Proposition 6.2. *If $W \longrightarrow X$ is a real vector bundle over a smooth manifold X then the space, $S^m(W)$, of symbols on W is well-defined for each $m \in \mathbb{R}$ by* (6.30).

Proof. We need to check to things here, what happens under changes of coordinate covering and changes of local trivializations. Notice that can move the ϕ into local coordinates to get $\phi'_i \in \mathcal{C}^\infty_c(\Omega'_i)$ and write (6.30) as

$$S^m(W) = \left\{ a : W \longrightarrow \mathbb{C}; a = \sum_i F_i^* \phi'_i a_i \text{ for some } a_i \in S^m(\Omega'_i \times \mathbb{R}^p) \right\}. \tag{6.31}$$

Then $\phi'_i a_i$ actually has compact support in the base variables, so is a global symbol on $\mathbb{R}^n \times \mathbb{R}^p$. If ψ_j is a partition of unity subordinate to another coordinate patch then we can lift these functions under π to W and write $a \in S^m(W)$ as

$$a = \sum_{i,j} \psi_j F_i^* \phi'_i a_i = \sum_{i,j} F_i^* \psi'_j \phi'_i a_i.$$

Thus each $\psi_j\phi_i$ is supported in the intersection of the two coordinate patches. Thus it suffices to show that if

$$F : \omega \times \mathbb{R}^p \longrightarrow \Omega' \times \mathbb{R}^p,\ F(x,\xi) = (f(x), A(x)\xi) \tag{6.32}$$

is a diffeomorphism, so f is a diffeomorphism and $A(x)$ is smooth and invertible, then $a \in S^m(\Omega';\mathbb{R}^p)$, $\operatorname{supp}(a) \subset K \times \mathbb{R}^p$ imples that $F^*a \in S^m(\Omega;\mathbb{R}^p)$. We can do this in two steps since $F = F' \circ (f, \mathrm{Id})$ where F' is of the same form with $f = \mathrm{Id}$. The second map amounts to a coordinate change and it is easy to see that the estimates in (2.1) are preserved by such a transformation. Thus it suffices to show that if $a \in S^m(\mathbb{R}^m;\mathbb{R}^p)$ has support in $K \times \mathbb{R}^p$ for some compact K and $A : \Omega \longleftrightarrow \mathrm{GL}(p,\mathbb{R})$ is a smooth map in an open neighbourhood of $\Omega \supset K$ then

$$a(x, A(x)\xi) \in S^m(\mathbb{R}^n;\mathbb{R}^p). \tag{6.33}$$

The basic symbol estimate

$$|a(x, A(x)\xi)| \le C \sup_K \langle A(x)\xi\rangle^m \le C'\langle\xi\rangle^m$$

therefore follows from the invertibility of $A(x)$ and the fact that a vanishes outside $K \times \mathbb{R}^p$. $V_{ij}\xi_i D_{\xi_j}$ and D_{x_k}. The symbol estimates on a function b just amount to requiring the estimate

$$|P(x, V, D_x)b(x,\xi)| \le C\langle\xi\rangle^m \tag{6.34}$$

for all polynomials P with smooth coefficient in x (since b vanishes outside $K \times \mathbb{R}^p$. The diffeomorphism $(x,\xi) \longmapsto (x, A(x)\xi)$ maps the space of these differential operators into itself, so the symbol estimates carry over. □

Suppose $A \in \Psi^m(X)$ and ρ_i is a square partition of unity subordinate to a coordinate cover $F_i : \Omega_i \longrightarrow \mathbb{R}^b$, so we can suppose

$$\operatorname{supp}(\rho_i) \subset \Omega_i,\ \sum_i \rho_i^2 = 1. \tag{6.35}$$

Then

$$A - \sum_i \rho_i A \rho_i \in \Psi^{m-1}(X) \tag{6.36}$$

since $[A, \rho_i] \in \Psi^{m-1}(X)$ as follows from (6.26) and the corresponding local property. This lead us to set

$$\sigma_m(\tau)(A) = \sum_{\{i, \pi(\tau) \in \operatorname{supp}(\rho_i)\}} b_i(x^{(i)}, \xi^{(i)}) \tag{6.37}$$

where

$$\tau = F_i^*(\sum_j \xi_j^{(i)} \cdot dx_j) \quad \xi^{(i)} \cdot dx \in T^*_{x^{(i)}}\mathbb{R}^n,\ x^{(i)} = F_i(\pi(\tau)) \tag{6.38}$$

and the b_i are representatives of the symbols of the $\rho_i A \rho_i$. This defines a function on $T^*X\backslash 0$, in fact the equivalence class

$$\sigma_m(A) \in S^{m-[1]}(T^*X) = S^m(T^*X)/S^{m-1}(T^*X) \tag{6.39}$$

is well-defined.

Proposition 6.3. *The principal symbol map in* (6.39)*, defined as in* (6.37)*, gives a short exact sequence:*

$$0 \hookrightarrow \Psi^{m-1}(X) \hookrightarrow \Psi^m(X) \xrightarrow{\sigma_m} S^{m-[1]}(T^*X) \longrightarrow 0. \tag{6.40}$$

PROOF. First we need to check that $\sigma_m(A)$ is indeed well-defined. This involves checking what happens under a change of coordinate covering and a change of partition of unity subordinate to it. For a change of coordinate covering for a fixed square partition of unity it suffices to use the transformation law for the principal symbol under a diffeomorphism of $\mathbb{R}^n$.

Now, if ρ_j' is another square partition of unity, subordinate to the same covering note that

$$\sum_j \rho_j' \rho_i A \rho_j' \rho_i \equiv \rho_i A \rho_i$$

where equality is modulo Ψ^{m-1}, since $[\phi, \Psi^m] \subset \Psi^{m-1}$ for any $\mathcal{C}^\infty$ function ϕ. It follows from (6.40) that the principal symbols, defined by (6.37), for the two partitions are the same.

The principal symbol is therefore well defined. Moreover, it follows that if $\phi \in \mathcal{C}^\infty(X)$ then

$$\sigma_m(\phi A) = \phi \sigma_m(A) \text{ since } \rho_i(\phi A)\rho_i = \phi(\rho_i A \rho_i). \tag{6.41}$$

Certainly if $A \in \Psi^{m-1}(X)$ then $\sigma_m(N) \equiv 0$. Moreover if $A \in \Psi^m(X)$ and $\sigma_m(A) \equiv 0$ then it follows from (6.41) that $\sigma_m(\rho_i A \rho_i) = 0$ and hence, from the properties of operators on $\mathbb{R}^n$ that $\rho_i A \rho_i$ is actually of order $m-1$. This proves that the null space of σ_m is exactly $\Psi^{m-1}(X)$.

Thus it only remains to show that the map σ_m is surjective. If $a \in S^m(T^*X)$ choose $A_i \in \Psi^m_\infty(\mathbb{R}^n)$ by

$$\sigma_L(A_i) = \rho_i(x)(F^*)^{-1} a_i \rho_i(y) \in S^m_\infty(\mathbb{R}^n \times \mathbb{R}^n) \tag{6.42}$$

and set

$$A = \sum_i F_i^* A_i G_i^* \qquad G_i = F_i^{-1}. \tag{6.43}$$

Then, from (6.37) $\sigma_m(A) \equiv a$ by invariance of the principal symbol. □

6.4. The symbol calculus

The other basic properties of the calculus on a compact manifold are easily established. For example to check that

$$\sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_m(B) \tag{6.44}$$

if $A \in \Psi^m(X)$, $B \in \Psi^{m'}(X)$ note that

$$AB = \sum_{i,j} \rho_i^2 A \rho_j^2 B = \sum_{i,j} \rho_i A \rho_i \cdot \rho_j B \rho_i \qquad \text{mod } \Psi^{m+m'-1}. \tag{6.45}$$

In § 5.9 we used the symbol calculus to construct a left and right parametrix for an elliptic element of $\Psi^m(X)$, where X is compact, i.e. an element $B \in \Psi^{-m}(X)$, such that

$$AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty}(X). \tag{6.46}$$

As a consequence of this construction note that:

PROPOSITION 6.4. *If $A \in \Psi^m(X)$ is elliptic, and X is compact, then*

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \tag{6.47}$$

is Fredholm*, i.e. has finite dimensional null space and closed range with finite dimensional complement. If* ν *is a non-vanishing* $\mathcal{C}^\infty$ *measure on* X *and a generalized inverse of* A *is defined by*

$$\begin{aligned} &Gu = f \text{ if } u \in \operatorname{Ran}(A), Af = u, f \perp_\nu \operatorname{Nul}(A) \\ &Gu = 0 \text{ if } u \perp_\nu \operatorname{Ran}(A) \end{aligned} \tag{6.48}$$

then $G \in \Psi^{-m}(X)$ *satisfies*

$$\begin{aligned} GA &= \operatorname{Id} - \pi_N \\ AG &= \operatorname{Id} - \pi_R \end{aligned} \tag{6.49}$$

where π_N *and* π_R *are* ν*-orthogonal projections onto the null space of* A *and the* ν*-orthocomplement of the range of* A *respectively.*

PROOF. The main point to note is that $E \in \Psi^{-\infty}(X)$ is smoothing,

$$E : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^\infty(X) \quad \forall\ E \in \Psi^{-\infty}(X). \tag{6.50}$$

Such a map is *compact* on $L^2(X)$, i.e. maps bounded sets into precompact sets by the theorem of Ascoli and Arzela. The second thing to recall is that a Hilbert space with a compact unit ball is finite dimensional. Then

$$\operatorname{Nul}(A) = \{u \in \mathcal{C}^\infty(X); Au = 0\} = \{u \in L^2(X); Au = 0\} \tag{6.51}$$

since, from (6.51) $Au = 0 \Longrightarrow (BA - \operatorname{Id})u = -Eu$, $E \in \Psi^\infty(X)$, so $Au = 0$, $u \in \mathcal{C}^{-\infty}(X) \Longrightarrow u \in \mathcal{C}^\infty(X)$. Then

$$\operatorname{Nul}(A) = \{u \in L^2(X); Au = 0 \int |u|^2\nu = 1\} \subset L^2(X) \tag{6.52}$$

is compact since it is closed (A is continuous) and so $\operatorname{Nul}(A) = E(\operatorname{Nul}(A))$ is *precompact*. Thus $\operatorname{Nul}(A)$ is finite dimensional.

Next let us show that $\operatorname{Ran}(A)$ is closed. Suppose $f_j = Au_j \longrightarrow f$ in $\mathcal{C}^\infty(X)$, $u_j \in \mathcal{C}^\infty(X)$. By what we have just shown we can assume that $u_j \perp_\nu \operatorname{Nul}(A)$. Now if B is the parametrix

$$u_j = Bf_j + Eu_j,\ E \in \Psi^{-\infty}(X). \tag{6.53}$$

Suppose, along some subsequence, $\|u_j\|_\nu \longrightarrow \infty$. Then

$$\frac{u_j}{\|u_j\|_\nu} = B\left(\frac{f_j}{\|u_j\|_\nu}\right) + E\left(\frac{u_j}{\|u_j\|_\nu}\right) \tag{6.54}$$

shows that $\frac{u_j}{\|u_j\|_\nu}$ lies in a precompact subset of L^2, $\frac{u_j}{\|u_j\|_\nu} \longrightarrow u$. This is a contradiction, since $Au = 0$ but $\|u\| = 1$ and $u \perp_\nu \operatorname{Nul}(A)$. Thus the norm sequence $\|u_j\|$ is bounded and therefore the sequence has a weakly convergent subsequence, which we can relabel as u_j. The parametrix shows that $u = Bf_j + Eu_j$ is strongly convergent with limit u, which satisfies $Au = f$.

Finally we have to show that $\operatorname{Ran}(A)$ has a finite dimensional complement. If π_R is orthogonal projection off $\operatorname{Ran}(A)$ then from the second part of (6.46) $f = \pi_R E' f$ for some smoothing operator E. This shows that the orthocomplement has compact unit ball, hence is finite dimensional. □

Notice that it follows that the two projections in (6.49) are both smoothing operators of finite rank.

6.5. Pseudodifferential operators on vector bundles

Perhaps unwisely I have carried through the discussion above for pseudodifferential operators acting on functions. The extension to operators between sections of vector bundles is mainly notational.

THEOREM 6.2. *If $W \longrightarrow Y$ is a $\mathcal{C}^\infty$ vector bundle and $F : X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map then $F^*W \longrightarrow X$ is a well-defined $\mathcal{C}^\infty$ vector bundle over X with total space*

$$F^*W = \bigcup_{x\in X} W_{F(x)}; \tag{6.55}$$

if $\phi \in \mathcal{C}^\infty(Y; W)$ then $F^*\phi$ is a section of F^*W and $\mathcal{C}^\infty(X; F^*W)$ is spanned by $\mathcal{C}^\infty(X) \cdot F^*\mathcal{C}^\infty(Y; W)$.

Distributional sections of any $\mathcal{C}^\infty$ vector bundle can be defined in two equivalent ways:

$$\text{“Algebraically” } \mathcal{C}^{-\infty}(X; W) = \mathcal{C}^{-\infty}(X) \bigotimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; W) \tag{6.56}$$

or as the dual space

$$\text{“Analytically” } \mathcal{C}^{-\infty}(X; W) = \left[\mathcal{C}_c^\infty(X; \Omega \otimes W')\right]' \tag{6.57}$$

where W' is the dual bundle and Ω the density bundle over X. In order to use (6.57) we need to define a topology on $\mathcal{C}_c^\infty(X; U)$ for any vector bundle U over X. One can do this by reference to local coordinates.

We have just shown that any elliptic pseudodifferential operator, $A \in \Psi^m(X)$ on a compact manifold X has a generalized inverse $B \in \Psi^{-m}(X)$, meaning

$$\begin{aligned} BA &= \mathrm{Id} - \pi_N \\ AB &= \mathrm{Id} - \pi_R \end{aligned} \tag{6.58}$$

where π_N and π_R are the orthogonal projections onto the null space of A and the orthocomplement of the range of A with respect to a prescribed $\mathcal{C}^\infty$ positive density ν, both are elements of $\Psi^{-\infty}(X)$ and have finite rank. To use this theorem in geometric situations we need first to make the “trivial” extension to operators on sections of vector bundles.

As usual there are two ways (at least) to approach this extension; the high road and the low road. The “low” road is to go back to the definition of $\Psi^m(X)$ and to generalize to $\Psi^m(X; V, W)$. This just requires to take the definition, following (6.16), but using a covering with respect to which the bundles V, W are *both* locally trivial. The local coordinate representatives of the pseudodifferential operator are then matrices of pseudodifferential opertors. The symbol mapping becomes

$$\Psi^m(X; V, W) \longrightarrow S^{m-[1]}\left(T^*X; \mathrm{Hom}(V, W)\right) \tag{6.59}$$

where $\mathrm{Hom}(V, W) \simeq V \otimes W'$ is the bundle of homomorphisms from V to W and the symbol space consists of symbolic sections of the lift of this bundle to T^*X. We leave the detailed description and proof of these results to the enthusiasts.

So what is the “high” road. This involves only a little sheaf-theoretic thought. Namely we want to define the space $\Psi^m(X; V, W)$ using $\Psi^m(X)$ by:

$$\Psi^m(X; V, W) = \Psi^m(X) \bigotimes_{\mathcal{C}^\infty(X^2)} \mathcal{C}^\infty(X^2; V \boxtimes W'). \tag{6.60}$$

To make sense of this we first note that $\Psi^m(X)$ is a $\mathcal{C}^\infty(X^2)$-module as is the space $\mathcal{C}^\infty(X^2; V \boxtimes W')$ where $V \boxtimes W'$ is the "exterior" product:

$$(V \boxtimes W')_{(x,y)} = V_x \otimes W'_y. \tag{6.61}$$

The tensor product in (6.60) means that

$$A \in \Psi^m(X; V, W) \text{ is of the form } A = \sum_i A_i \cdot G_i \tag{6.62}$$

where $A_i \in \Psi^m(X)$, $G_i \in \mathcal{C}^\infty(X^2; V \boxtimes W')$ and equality is fixed by the relation

$$\phi A \cdot G - A \cdot \phi G \equiv 0. \tag{6.63}$$

Now what we really need to note is:

Proposition 6.5. *For any compact $\mathcal{C}^\infty$ manifold Y and any vector bundle U over Y*

$$\mathcal{C}^{-\infty}(Y; U) \equiv \mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y; U). \tag{6.64}$$

Proof. $\mathcal{C}^{-\infty}(Y; U) = (\mathcal{C}^\infty(Y; \Omega \otimes U'))'$ is the definition. Clearly we have a mapping

$$\mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y; U) \ni \sum_i A_i \cdot g_i \longrightarrow \mathcal{C}^{-\infty}(Y; U) \tag{6.65}$$

given by

$$\sum_i u_i \cdot g_i(\psi) = \sum_i u_i(g_i \cdot \psi) \tag{6.66}$$

since $g_i\psi \in \mathcal{C}^\infty(Y; \Omega)$ and linearity shows that the map descends to the tensor product. To prove that the map is an isomorphism we construct an inverse. Since Y is compact we can find a finite number of sections $g_i \in \mathcal{C}^\infty(Y; U)$ such that *any* $u \in \mathcal{C}^\infty(Y; U)$ can be written

$$u = \sum_i h_i g_i \quad h_i \in \mathcal{C}^\infty(Y). \tag{6.67}$$

By reference to local coordinates the same is true of distributional sections with

$$h_i = u \cdot q_i \quad q_i \in \mathcal{C}^\infty(Y; U'). \tag{6.68}$$

This gives a left and right inverse. □

Theorem 6.3. *The calculus extends to operators on sections of vector bundles over any compact $\mathcal{C}^\infty$ manifold.*

6.6. Hodge theorem

The identification of the deRham cohomology of a compact manifold with the finite dimensional vector space of harmonic forms goes back to Hodge in the algebraic setting and to Hermann Weyl in the general case. It is a rather direct consequence of the Fredholm properties on smooth sections of the Laplacian. In fact this has nothing much to do with the explicit form of the deRham complex, so let's do it in the natural context of an elliptic complex over a compact manifold M.

Thus let E_i, $i = 0, \dots, N$ be complex vector bundles and suppose $d_i \in \mathrm{Diff}^1(M; E_i, E_{i+1})$, $i < N$, form a complex of differential operators, meaning that for each $i < N$ d_{i+1} annihilates the range of d_i which means just that

$$d_{i+1}d_i = 0 \in \mathrm{Diff}^1(M; E_i; E_{i+2}),\ i < N. \tag{6.69}$$

Such a complex is said to be exact (on $\mathcal{C}^\infty$ sections) if

(6.70)

$$\mathcal{C}^\infty(M; E_0) \xrightarrow{d_0} \mathcal{C}^\infty(M; E_1) \xrightarrow{d_1} \dots \qquad \mathcal{C}^\infty(M; E_{N-1}) \xrightarrow{d_{N-1}} \mathcal{C}^\infty(M; E_N)$$

is exact, meaning that conversely

$$\mathrm{null}(d_{i+1}) = d_i\mathcal{C}^\infty(M : E_i)\ \forall\ i < N. \tag{6.71}$$

The principal symbol $\sigma_i(d_i) \in \mathcal{C}^\infty(T^*M; \pi^* \hom(E_i, E_{i+1})$ is a homogeneous polynomial of degree 1 and from (6.69) these bundle maps for a complex over T^*M. Of course the all vanish at the zero section so, excluding that, we say the original complex is *elliptic* if the symbol complex

(6.72)

$$\mathcal{C}^\infty(T^*M \setminus 0; \pi^*E_0) \xrightarrow{\sigma_1(d_0)} \mathcal{C}^\infty(T^*M \setminus 0; \pi^*E_1) \xrightarrow{\sigma_1(d_1)} \dots$$

$$\mathcal{C}^\infty(T^*M \setminus 0; \pi^*E_{N-1}) \xrightarrow{\sigma_1(d_{N-1})} \mathcal{C}^\infty(T^*M \setminus 0; \pi^*E_N)$$

is exact.

Now, choose an Hermitian inner product on each of the E_i and a smooth density on M so that we can define the adjoints δ_i of the d_{i-1} (so that the subscript corresponds to the subscript of the vector space on which the operator acts)

$$\delta_i = (d_{i-1})^* \in \mathrm{Diff}^1(M; E_i, E_{i-1}),\ i = 1, \dots, N. \tag{6.73}$$

Then form the Hodge operator and the Laplacian

$$(d+\delta)_i \in \mathrm{Diff}^1(M; E_i, E_{i-1} \oplus E_{i+1}),\ \varDelta_i = \delta_{i+1}d_i + d_{i-1}\delta_i^2 \in \mathrm{Diff}^2(M; E_i). \tag{6.74}$$

We can also take the direct sum of all the terms in the complex and set

$$d = \oplus_i d_i \in \mathrm{Diff}^1(M; E_*),\ \delta = \oplus_i \delta_i \in \mathrm{Diff}^1(M; E_*). \tag{6.75}$$

Then (6.69) and the induced identity $\delta_{i-1}\delta_i = 0$ together show that

$$(d+\delta)^2 = \oplus_i \varDelta_i \in \mathrm{Diff}^2(M; E_*) \tag{6.76}$$

since applied to $\mathcal{C}^\infty(M; E_i)$

(6.77)

$$(d+\delta)^2\big|_{\mathcal{C}^\infty(M;E_i)} = (d+\delta)d_i + (d+\delta)\delta_i = (d_{i+1} + \delta_{i+1})d_i + (d_{i-1} + \delta_{i-1})\delta_i = \varDelta_i.$$

THEOREM 6.4. *For an elliptic complex the operators $d+\delta$ and all the $\varDelta_i$ are elliptic,*

$$\mathrm{null}(\varDelta_i) = \{u \in \mathcal{C}^\infty(M; E_i); d_iu = 0,\ \delta_iu = 0\} \tag{6.78}$$

and the inclusion of this space into the null space of d_i induces an isomorphism of vector spaces

$$\mathrm{null}(\varDelta_i) \simeq \{u \in \mathcal{C}^\infty(M; E_i); d_iu = 0\} / d_{i-1}\mathcal{C}^\infty(M; E_i). \tag{6.79}$$

In particular the vector spaces on the right in (6.79) are finite dimensional; these are the (hyper-)cohomology spaces of the original complex.

PROOF. The symbol of Δ_i is exactly

$$\sigma_2(\Delta_i) = \sigma_1(\delta_{i+1})\sigma_1(d_i) + \sigma_1(d_{i-1})\sigma_1(\delta_i). \tag{6.80}$$

Over points of $T^*M \setminus 0$ we can use the (pointwise) inner product on the E_i's and the fact that $\sigma_1(\delta_i) = (\sigma_1(d^*_{i-1})$ to see that

$$\langle f, \sigma_1(\Delta_i)f\rangle = \langle f, \sigma_1(\delta_{i+1})\sigma_1(d_i)f\rangle + \langle f, \sigma_1(d_{i-1})\sigma_1(\delta_i)f\rangle = |\sigma_1(d_i)f|^2 + |\sigma_1(\delta_i)f|^2.$$

Thus an element of the null space of $\sigma_2(\Delta_i)$ is in the intersection of the null spaces of $\sigma_1(d_i)$ and $\sigma_1(\delta_i)$. The null space of the latter is precisely the orthocomplement to the range of the former, so (by the assumed ellipticity) $\sigma_2(\Delta_i)$ is injective and hence an isomorphism. As an elliptic operator the null space of Δ_i, even acting on distributions, consists of elements of $\mathcal{C}^\infty(M; E_i)$. Moreover integration by parts then gives

$$\Delta_i u = 0 \Longrightarrow \int_M \langle u, \Delta_i u\rangle \nu = \|d_i u\|^2_{L^2} + \|\delta_i u\|^2_{L^2} \Longrightarrow d_i u = 0,\ \delta_i u = 0. \tag{6.81}$$

The converse is obvious, so this proves (6.78).

We know that any elliptic operator on a compact manifold is Fredholm. Moreover Δ_i is self-adjoint, directly from the definition in (6.74). Thus the range of Δ_i is precisely the orthocomplement (with respect to the L^2 inner product) of its own null space:

$$\mathcal{C}^\infty(M; E_i) = \operatorname{null}(\Delta_i) \oplus \Delta\mathcal{C}^\infty(M; E_i). \tag{6.82}$$

Now expanding out Δ_i we can decompose each element of the second term as

$$\Delta u = d_{i-1}\delta_i u + \delta_{i+1} d_i u = d_{i-1} v_{i-1} + \delta_{i+1} w_{i+1}. \tag{6.83}$$

The two terms here are orthogonal in $L^2(M; E_i)$ and this allows us to rewrite (6.82) as *The Hodge Decomposition*

$$\mathcal{C}^\infty(M; E_i) = \operatorname{null}(\Delta_i) \oplus d_{i-1}\mathcal{C}^\infty(M; E_{i-1}) \oplus \delta_{i+1}\mathcal{C}^\infty(M; E_{i+1}). \tag{6.84}$$

Indeed, all three terms here are orthogonal as follows by integration by parts and the fact that $d^2 = 0$ and hence there must be equality in (6.84) since each element has such a decomposition, as follows from (6.82) and (6.83).

Now if $u \in \mathcal{C}^\infty(M; E_i)$ satisfies $d_i u = 0$, consider its Hodge decomposition

$$u = u_0 + du_1 + \delta v. \tag{6.85}$$

The last term must vanish since applying d to (6.85), $d\delta v = 0$ and then integrating by parts

$$\int_M \langle v, d\delta v\rangle \nu = \|\delta v\|^2_{L^2} = 0. \tag{6.86}$$

The map $u\rangle u_0$ therefore takes the left side of (6.79) to the right. It is injective, since $u_0 = 0$ means that u is 'exact' and it is surjective since u_0 is itself closed and the decomposition (6.85) is unique, so it is mapped to u_0. This gives the *Hodge isomorphism* (6.79). □

In fact the same argument works with distributional sections of the various bundles. We know that, as an elliptic operator

$$\Delta_i : \mathcal{C}^{-\infty}(M; E_i) \longrightarrow \mathcal{C}^{-\infty}(M; E_i) \tag{6.87}$$

also has range exactly the annihilator of the null space of its adjoint, also Δ_i, on $\mathcal{C}^\infty$ sections. Thus we get a distributional decomposition

$$\mathcal{C}^{-\infty}(M;E_i) = \mathrm{null}(\Delta_i) \oplus \Delta\mathcal{C}^{-\infty}(M;E_i) \tag{6.88}$$

which we can still think of as 'orthogonal' since the pairing exists between the smooth harmonic forms and the general distibutional sections. A distributional form of the Hodge decomposition follows as before which we can write as

$$\mathcal{C}^{-\infty}(M;E_i) = \mathrm{null}(\Delta_i) \oplus \left(d_{i-1}\mathcal{C}^{-\infty}(M;E_{i-1}) \dot{+} \delta_{i+1}\mathcal{C}^{-\infty}(M;E_{i+1})\right). \tag{6.89}$$

Here the second two terms do not formally 'pair' under extension of the L^2 inner product so we just claim that the intersection is empty. This follows from the fact that an element of the intersection is harmonic and hence smooth and thus, from (6.88), vanishes. This lead immediately to a distributional Hodge isomorphism

$$\mathrm{null}(\Delta_i) = \left\{u \in \mathcal{C}^{-\infty}(M;E_i); d_i u = 0\right\} / d_{i-1}\mathcal{C}^{-\infty}(M;E_i) \tag{6.90}$$

completely analogous to (6.78). The proof is almost the same. A closed distributional form has a decomposition as in (6.89), $u = u_0 + du' + \delta v$ where u_1 and v are now distributional sections. However applying d we see that $d\delta v = 0$ and $\delta\delta v = 0$ so δv is harmonic, hence smooth, and the integration by parts argument as before shows that $\delta v = 0$ (not of course that $v = 0$). This gives a map from right to left in (6.90) which is an isomorphism just as before.

In particular this shows that the 'distributional deRham' and 'smooth deRham' cohomologies are isomorphic. In fact the isomorphism is natural, even though both isomorphisms (6.78) and (6.90) depend on the choice of inner product and smooth density (since of course the harmonic forms depend on these choices). Namely the isomorphism is induced by the natural 'inclusion map'

$$\{u \in \mathcal{C}^\infty(M;E_i); d_i u = 0\} / d_{i-1}\mathcal{C}^\infty(M;E_i) \longrightarrow \left\{u \in \mathcal{C}^{-\infty}(M;E_i); d_i u = 0\right\} / d_{i-1}\mathcal{C}^{-\infty}(M;E_i). \tag{6.91}$$

In many applications in differential geometry it is important to go a little further than this. The Hodge theorem above identifies the null space of the Laplacian with the intersections of the null spaces of d and δ. More generally consider the spectral decomposition associated to the Δ_i.

Proposition 6.6. *If $(\mathcal{C}^\infty(M;E_i))^+$ is the orthocomplement to* $\mathrm{null}(\Delta_i)$ *for each i then the d_i induce and exact complex*

$$(\mathcal{C}^\infty(M;E_0))^+ \xrightarrow{d_0} (\mathcal{C}^\infty(M;E_1))^+ \xrightarrow{d_1} \dots \qquad (\mathcal{C}^\infty(M;E_{N-1}))^+ \xrightarrow{d_{N-1}} (\mathcal{C}^\infty(M;E_N))^+ \tag{6.92}$$

which restricts to an exact finite-dimensional complex on the subspaces which are eigenspaces of Δ_i for a fixed $\lambda > 0$.

Proof. All the null spaces vanish and exactness follows from the Hodge decomposition. ∎

Of course the adjoint complex is the one for δ and the same result holds for distributional sections. Note that this means that the eigenspaces of Δ_i, corresponding to non-zero eigenvalues, can be decomposed into exact and coexact parts. Thus even though the Hodge operator $d + \delta$ mixes form degrees, all its eigenvectors are can be decomposed into eigenvectors of Δ which have 'pure degree'.

6.7. Sobolev spaces and boundedness

[Following discussions with Sheel Ganatra]

In the discussion above, I have shown that elliptic pseudodifferential operators are Fredholm on the spaces of $\mathcal{C}^\infty$ sections directly from the existence of parameterices, rather than using the more standard argument on Sobolev spaces. However, let me now recall this starting with operators of order 0. In fact it is convenient to define the Sobolev spaces for other orders so that boundedness is 'obvious' and then check that the definition is sensible.

LEMMA 6.4. *On any compact manifold M each $A \in \Psi^0(M; V, W)$ for vector bundles V and W extends by continuity from $\mathcal{C}^\infty(M; V)$ to a bounded operator*

$$A : L^2(M; V) \longrightarrow L^2(M; W). \tag{6.93}$$

PROOF. There are two obvious alternatives here. The first is to use the same construction of approximate square roots as before. That is, using the symbol calculus onie can see that if A is as above and we choose inner products on V and W and a smooth volume form on M so that A^* is defined then for a large positive constant C there exists $B \in \Psi^0(M; V)$ so that

$$C - B^*B = A^*A + G,\ G \in \Psi^{-\infty}(M; V). \tag{6.94}$$

This starts by solving the equation at a symbolic level, so showing that $\sigma_0(A)$ exists such that

$$C - \sigma_0(B)^*\sigma_0(B) = \sigma_0(A)^2,\ \sigma_0^*(A) = \sigma_0(A). \tag{6.95}$$

Thus $\sigma_0(A)$ is the square root of the positive definite matrix $C - \sigma_0^*(B)\sigma_0(B)$. Then one can proceed inductively using the symbol calculus, as before, to solve the problem modulo smoothing.

Alternatively we can simply use the known boundedness of smoothing operators on M and of pseudodifferential operators on $\mathbb{R}^n$. Thus the local (matrices of) operators, or order 0, as in (6.18) are bounded on $L^2(\mathbb{R}^n)$ and since $u \in L^2(M; V)$ is equivalent to $(F_i)^*\psi_i u_j \in L^2(\mathbb{R}^n)$ for a partition of unity ψ_i subordinate to a coordinate cover over each element of which the bundle is trivial, the boundedness (6.93) follows. Of course we are also using the density of $\mathcal{C}^\infty(M; W)$ in $L^2(M; W)$ which follows from the same argument. □

defines

DEFINITION 6.3. *On a compact manifold and for a vector bundle W we set*

$$H^s(M; W) = \left\{u \in \mathcal{C}^{-\infty}(M; W); Au \in L^2(M; V)\ \forall\ A \in \Psi^{-s}(M; W, V)\right\},\ s \in \mathbb{R}. \tag{6.96}$$

Here we are demanding this for all pseudodifferential operators and all vector bundles V. This of course is gross overkill.

PROPOSITION 6.7. *For each $s \in \mathbb{R}$, $\mathcal{C}^\infty(M; V)$ is dense in $H^s(M; V)$, every element $A \in \Psi^m(M; V, W)$ extends by continuity to a bounded linear operator*

$$A : H^s(M; V) \longrightarrow H^{s-m}(M; W)\ \forall\ s \in \mathbb{R},\ \forall\ s \in \mathbb{R} \tag{6.97}$$

and if $A \in \Psi^m(M; V, W)$ is elliptic then

$$Au \in H^s(M; V) \Longrightarrow u \in H^{s+m}(M; W). \tag{6.98}$$

PROOF. Since I have not quite fixed the topology on $H^s(M;V)$, the density statement is to be interpreted as meaning that if $u \in H^s(M;V)$ then there is a sequence $u_n \in \mathcal{C}^\infty(M;V)$ such that $Pu_n \to Pu$ in $L^2(M;W)$ for every $P \in \Psi^s(M;V,W)$. In fact the simplest thing to prove is that, with the ugly definition (6.96) of $H^s(M;V)$ that

$$P \in \Psi^s(M;V,W),\ u \in H^s(M;V) \Longrightarrow Pu \in L^2(M;W) \tag{6.99}$$

since this is precisely what the definition requires. Conversely we can see that

$$P \in \Psi^s(M;V,W),\ u \in L^2(M;V) \Longrightarrow Pu \in H^{-s}(M;W) \tag{6.100}$$

Here we are using the action of pseudodifferential operators on distributions. Indeed, if $A \in \Psi^{-s}(M;W,U)$ for some other vector bundle U then we just need to show that $APu \in L^2(M;U)$. However, by the composition theorem, $AP \in \Psi^0(M;V,U)$ so this follows from Lemma 6.4.

Combining these two special cases of (6.97) we can get the general case. Note that there is always an elliptic element $P_s \in \psi^s(M;V)$ for any $s \in \mathbb{R}$ and any vector bundle V. There is certainly an elliptic symbol, say $(1+|\xi|^2)^{\frac{s}{2}}\operatorname{Id}_V$ where $|\cdot|$ is some Riemannian metric. The surjectivity of the symbol maps shows that there is in fact a pseudodifferential operator P_s with this symbol, which is therefore elliptic. By the elliptic construction above this operator has a parameterix $Q_s \in \Psi^{-s}(M;V)$ which is also elliptic and satisfies

$$Q_sP_s - \operatorname{Id},\ P_sQ_s - \operatorname{Id} \in \Psi^{-\infty}(M;V). \tag{6.101}$$

Now, given a general $A \in \Psi^m(M;W,V)$ composing with this identity shows that

$$A = (AQ_s)P_s + G = BP_s + G,\ B = AQ_s \in \Psi^{m-s}(M;V,W),\ G \in \Psi^{-\infty}(M;V,W). \tag{6.102}$$

A smoothing operator certainly satisfies (6.97) (since $\mathcal{C}^\infty(M;V) \subset H^s(M;V)$ for all s) so it suffices to consider BP_s in place of A. Applying (6.99) to P_s and (6.100) to B, with s replaced by $m-s$ shows that

$$H^s(M;V) \xrightarrow{P_s} L^2(M;V) \xrightarrow{B} H^{s-m}(M;W) \tag{6.103}$$

which gives (6.97).

If A is elliptic then (6.98) follows since if $Q \in \Psi^{-m}(M;W,V)$ is a parametrix for A then

$$QA = \operatorname{Id} - G,\ G \in \Psi^{-\infty}(M;V),\ Au \in H^s(M;W) \Longrightarrow u = QAu + Gu \in H^{s+m}(M;V). \tag{6.104}$$

This means that the original definition can be written in the much simpler form

$$H^s(M;W) = \left\{u \in \mathcal{C}^{-\infty}(M;W); P_{-s}u \in L^2(M;W) \text{ for some elliptic } P_{-s}s \in \Psi^{-s}(M;W)\right\}. \tag{6.105}$$

Here of course 'some' means for any one elliptic element.

Finally then the density also follows. Namely, if $u \in H^s(M;V)$ then

$$u = Q_s(P_su) + Gu,\ P_s \in \Psi^s(M;V),\ Q_s \in \Psi^{-s}(M;V),\ G \in \Psi^{-\infty}(M;V). \tag{6.106}$$

Thus $P_su \in L^2(M;V)$. Let $v_n \to P_su$ in $L^2(M;V)$ then $Gu \in \mathcal{C}^\infty(M;V)$ and

$$Q_su_n + Gu \to u \in H^s(M;V) \text{ since } PQ_su_n + PGu \to Gu \in L^2(M;W)\ \forall\ P \in \Psi^s(M;V,W). \tag{6.107}$$

$\square$

Using the Fredholm properties of elliptic operators we see that if $P_{s/2} \in \Psi^{s/2}(M;V)$ is elliptic then, if $s > 0$,

$$B_s = P_{s/2}^* P_{s/2} + 1 \in \Psi^s(M;V) \tag{6.108}$$

is an isomorphism. Indeed, it is elliptic so we know that any element u of its null space is in $\mathcal{C}^\infty(M;V)$. However integration by parts is then justified and shows that

$$B_s u = 0 \Longrightarrow \langle P_{s/2}u, P_{s/2}u\rangle + \|u\|^2_{L^2} = 0 \Longrightarrow u \equiv 0. \tag{6.109}$$

Thus its null space consists of $\{0\}$ and since it is (formally) self-adjoint, the same is true of the null space of its adjoint. Thus, being Fredholm, it is an isomorphism. In fact its inverse

$$B_s^{-1} \in \Psi^{-s}(M;V) \tag{6.110}$$

is also invertible. We have already shown (6.110) since B_s^{-1} is the generalized inverse.

Thus we have shown the main part of

Proposition 6.8. *For any compact manifold M and any vector bundle V over M there is an invertible element $B_s \in \Psi^s(M;V)$ for each s and then*

$$H^s(M;V) = \{u \in \mathcal{C}^{-\infty}(M;V); B_s u \in L^2(M;V)\},\ \|u\|_s = \|B_s u\|_{L^2} \tag{6.111}$$

shows that $H^s(M;V)$ is a Hilbert space. Moreover $\psi_i u$ has entries in $H^s(\mathbb{R}^n)$ for any covering of M by coordinate patches over each of which the bundle is trivial and for any partition of unity subordinate to it.

Proof. The last part just follows by looking at the local coordinate representative of B_s. Namely $\psi_i u$ is a (vector of) compactly supported distributions in the coordinate patch and $(1 + |D|^2)^{-s/2}\psi_i u \in L^2(\mathbb{R}^n)$ since it is smooth outside the image of the support of ψ_i by pseudolocality and inside the coordinate patch by the boundedness of pseudodifferential operators discussed above. □

Proposition 6.9. *The L^2 pairing with respect to an inner product and smooth volume form extends by continuity to a non-degenerate pairing*

$$H^s(M;V) \times H^{-s}(M;V) \longrightarrow \mathbb{C} \tag{6.112}$$

which allows $H^{-s}(M;V)$ to be identified with the dual of $H^s(M;V)$ for any s.

Proof. Exercise! □

6.8. Pseudodifferential projections

We are interested in constructing projections in the pseudodifferential algebra corresponding to arbitrary symbolic projections.

Theorem 6.5. *If M is compact, E is a complex vector bundle over M and $p \in \mathcal{C}^\infty(S^*M; \hom(E))$ is valued in the projections in the sense that $p^2 = p$ then there exists an element $P \in \Psi^0(M;E)$ with symbol p which is itself a projection.*

First we work modulo smoothing operators, for later applications we shall do this without assuming the compactness of M.

Lemma 6.5. *If $E \longrightarrow M$ is a complex vector bundle and $p \in \mathcal{C}^\infty(S^*M; E)$ satisfies $p^2 = p$ then there exists $Q \in \Psi^0(M; E)$ which is properly supported and such that*

$$Q^2 - Q \in \Psi^{-\infty}(M; E). \tag{6.113}$$

Proof. Of course the first step is simply to choose $Q_0 \in \Psi^0(M; E)$ which is properly supported and has $\sigma_0(Q) = p$. This gives a version of (6.113) but only modulo terms of order -1 :

$$Q_0^2 - Q_0 = E_1 \in \Psi^{-1}(M; E). \tag{6.114}$$

However note, by composing with Q_0 first on the left and then on the right, that $Q_0 E_1 = E_1 Q_0$. It follows that

$$(\operatorname{Id} - P)E_i P,\ PE_i(\operatorname{Id} - P) \in \Psi^{-i-1}(M; E) \tag{6.115}$$

for $i = 1$. Then set $Q_1 = -Q_0 E_1 Q_0 + (\operatorname{Id} - Q)E_0(\operatorname{Id} - Q_0)$ and $Q_{(1)} = Q_0 + Q_1$. It follows from (6.114) and (6.115) that

$$Q_{(i)}^2 - Q_{(i)} = E_{i+1} = E_i + Q_{(i)}Q_i + Q_i Q_{(i)} - Q_i \in \Psi^{-i-1}(M; E). \tag{6.116}$$

Thus we can proceed by induction and successively find $Q_j \in \Psi^{-j}(M; E)$, always properly supported, such that

$$Q_{(i)} = \sum_{j=1}^{i} Q_j \text{ satisfies (6.116) for all } i. \tag{6.117}$$

Then taking Q to be a properly supported asympotic sum of this series gives an operator as claimed. □

Proposition 6.10. *If M is compact, E is a complex vector bundle over M and $Q \in \Psi^0(M; E)$ is such that $Q^2 - Q \in \Psi^{-\infty}(M; E)$ then there exists $P \in \Psi^0(M; E)$ such that $P^2 = P$ and $P - Q \in \Psi^{-\infty}(M; E)$.*

Proof. As a bounded operator on $L^2(M; E)$, Q has discrete spectrum outside $\{0, 1\}$. Indeed, if $\tau \notin \{0, 1\}$ then

$$(Q - \tau \operatorname{Id})((1-\tau)^{-1}Q - \tau^{-1}(\operatorname{Id} - Q)) = \operatorname{Id} + (1-\tau)^{-1}\tau^{-1}(Q^2 - Q) \tag{6.118}$$

gives a parametrix for $Q - \tau \operatorname{Id}$. The right side is invertible for $|\tau|$ large and hence for all τ outside a discrete subset of $\mathbb{C} \setminus \{0, 1\}$ with inverse $\operatorname{Id} + S(\tau)$ where $S(\tau)$ is meromorphic with values in $\Psi^{-\infty}(M; E)$. Letting Γ be the circle of radius $\frac{1}{2} - \epsilon$ around the origin for $\epsilon > 0$ sufficiently small it follows that $Q - \tau \operatorname{Id}$ is invertible on Γ with inverse $((1-\tau)^{-1}Q - \tau^{-1}(\operatorname{Id} - Q))(\operatorname{Id} + S(\tau)$. Thus, by Cauchy's theorem,

$$\operatorname{Id} - P = \frac{1}{2\pi i}\oint_\Gamma (\tau - Q)^{-1} d\tau = \operatorname{Id} - Q + S,\ S \in \Psi^{-\infty}(M; E) \tag{6.119}$$

and moreover P is a projection since choosing Γ' to be a circle with slightly larger radius than Γ,

$$\begin{aligned} (\operatorname{Id} - P)^2 &= \frac{1}{2\pi i}\frac{2\pi i}{\oint}_{\Gamma'}\oint_\Gamma (\tau' - Q)^{-1}(\tau - Q)^{-1} d\tau' d\tau \\ &= \frac{1}{2\pi i}\frac{1}{2\pi i}\oint_{\Gamma'}\oint_\Gamma \left((\tau' - \tau)^{-1}(\tau' - Q)^{-1} + (\tau' - \tau)^{-1}(\tau - Q)^{-1}\right) d\tau' d\tau \\ &= \operatorname{Id} - P \end{aligned} \tag{6.120}$$

since in the first integral the integrand is holomorphic in τ inside Γ and in the second the τ' integral has a single pole at $\tau' = \tau$ inside Γ. $\square$

The following more qualitative version is used in the discussion of the Calderón projection .

PROPOSITION 6.11. *If M is compact, E is a complex vector bundle over M and $Q \in \Psi^0(M;E)$ is such that $Q^2 - Q \in \Psi^{-\infty}(M;E)$ and $F \subset H^s(M;E)$ is a closed subspace corresponding to which there are smoothing operators $A, B \in \Psi^{-\infty}(M;E)$ with $\operatorname{Id} - Q = A$ on F and $(Q+B)L^2(M;E) \subset F$ then there is a smoothing operator $B' \in \Psi^{-\infty}(M:E)$ such that $F = \operatorname{Ran}(Q+B')$ and $(Q+B')^2 = Q+B'$.*

PROOF. Assume first that $s = 0$, so F is a closed subspace of $L^2(X;E)$. Applying Proposition 6.10 to Q we may assume that it is a projection P, without affecting the other conditions. Consider the intersection $E = F \cap \operatorname{Ran}(\operatorname{Id} - P)$. This is a closed subspace of $L^2(M;E)$. With A as in the statement of the proposition, $E \subset \operatorname{Nul}(\operatorname{Id} - A)$. Indeed P vanishes on $\operatorname{Ran}(\operatorname{Id} - P)$ and hence on E and by hypothesis $\operatorname{Id} - P - A$ vanishes on F and hence on E. From the properties of smoothing operators, E is contained in a finite dimensional subspace of $\mathcal{C}^\infty(M;E)$, so is itself such a space. We may modify P by adding a smoothing projection onto E to it, and so assume that $F \cap \operatorname{Ran}(\operatorname{Id} - P) = \{0\}$.

Consider the sum $G = F + \operatorname{Ran}(\operatorname{Id} - P)$ and the operator $\operatorname{Id} + B = (P+B) + (\operatorname{Id} - P)$, with B as in the statement of the Proposition. The range of $\operatorname{Id} + B$ is contained in G. Thus G must be a closed subspace of $L^2(M;E)$ with a finite dimensional complement in $\mathcal{C}^\infty(M;E)$. Adding a smoothing projection onto such a complement we can, again by altering P by smoothing term, arrange that

$$L^2(M;E) = F \oplus \operatorname{Ran}(\operatorname{Id} - P) \tag{6.121}$$

is a (possibly non-orthogonal) direct sum. Since P has only been altered by a smoothing operator the hypotheses of the Proposition continue to hold. Let Π be the projection with range F and null space equal to the range of $\operatorname{Id} - P$. It follows that $P' = P + (\operatorname{Id} - P)RP$ for some bounded operator R (namely $R = (\operatorname{Id} - P)(P' - P)P$.) Then restricted to F, $P' = \operatorname{Id}$ and $P = \operatorname{Id} + A$ so $R = -A$ on F. In fact $R = AP \in \Psi^{-\infty}(M;E)$, since they are equal on F and both vanish on $\operatorname{Ran}(\operatorname{Id} - P)$. Thus P' differs from P by a smoothing operator.

The case of general s follows by conjugating with a pseudodifferential isomorphism of $H^s(M;E)$ to $L^2(M;E)$ since this preserves both the assumptions and the conclusions. $\square$

6.9. The Toeplitz algebra

6.10. Semiclassical algebra

Recall the notion of a semiclassical 1-parameter family of pseudodifferential operators (which we will nevertheless call a semiclassical operator) on Euclidean space in Section 2.19. Following the model in Section 6.3 above we can easily 'transfer' this definition to a manifold M, compact or not. The main thing to decide is what to require of the part of the kernel away from the diagonal. This however is clear from (2.210). Namely in any compact set of M^2 which does not meet the diagonal, the kernel should be smooth uniformly down to $\epsilon = 0$, including in ϵ itself, and it should vanish there to infinite order. This motivates the following definition modelled closely on Definition 6.2 and the discussion of operators between sections of vector bundles in Section 6.5. This time I have chosen to define the classical operators, of course the spaces $\Psi^m_{\text{sl-}\infty}(X,\mathbb{E})$ have a similar definition.

Definition 6.4. *If X is a $\mathcal{C}^\infty$ manifold and $\mathbb{E}=(E_+,E_-)$ is a pair of complex vector bundles over X then, for any $m\in\mathbb{R}$, $\Psi^m_{\text{sl}}(X;\mathbb{E})$ is the space of linear operators*

$$A_\epsilon : \mathcal{C}^\infty_c([0,1]\times X;E_+) \longrightarrow \mathcal{C}^\infty([0,1]\times X;E_-) \tag{6.122}$$

with the following properties. First,

$$\begin{gathered}\text{if } \phi,\ \psi\in\mathcal{C}^\infty(X) \text{ have disjoint supports then } \exists\ K_\epsilon\in\mathcal{C}^\infty([0,1]_\epsilon\times X^2;\Omega_R\otimes\operatorname{Hom}(\mathbb{E})),\\ K_\epsilon\equiv 0 \text{ at } \{\epsilon=0\} \text{ such that } \forall\ u\in\mathcal{C}^\infty_c([0,1]\times X;E_+)\ \phi A\psi u=\int\limits_X K_\epsilon(x,y)u(y),\end{gathered} \tag{6.123}$$

and secondly if $F:W\longrightarrow\mathbb{R}^n$ is a coordinate system in X over which $\mathbb{E}$ is trivial, with trivializations $h_\pm E_\pm\big|_W \longleftrightarrow W\times\mathbb{C}^{N_\pm}$, and $\psi\in\mathcal{C}^\infty_c(X)$ has support in W then

$$\begin{gathered}\exists\ B_\epsilon\in\Psi^m_{\text{sl}}(\mathbb{R}^n;\mathbb{C}^{N_+},\mathbb{C}^{N_-}),\ \operatorname{supp}(B_\epsilon)\subset[0,1]\times F(W)\times F(W)\ \textit{s.t.}\\ \psi A_\epsilon\psi u\restriction W = h_-F^*(B_\epsilon((F^{-1})^*(h_+^{-1}\psi u)))\ \forall\ u\in\mathcal{C}^\infty_c([0,1]\times X;E_+).\end{gathered}$$

A semiclassical operator (always a family of course) is said to be properly supported if its kernel has proper support in $[0,1]\times X\times X$, that is proved the two maps

$$\begin{tikzcd} & \operatorname{supp}(B_\epsilon) \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ X & & X \end{tikzcd} \tag{6.124}$$

are both proper, meaning the inverse image of a compact set is compact. Since

$$\pi_X\operatorname{supp}(B_\epsilon u)\subset\pi_L(\operatorname{supp}(B_\epsilon)\cap\pi_R^{-1}(\pi_X\operatorname{supp}(u)) \tag{6.125}$$

(where $\operatorname{supp}(u)\subset[0,1]\times X$ and π_X is projection onto the second factor) it follows that a properly supported operator satisifes

$$B_\epsilon : \mathcal{C}^\infty_c([0,1]\times X;E_+) \longrightarrow \mathcal{C}^\infty_c([0,1]\times X;E_-). \tag{6.126}$$

The same is true of the adjoint, so in fact by duality

$$B_\epsilon : \mathcal{C}^\infty([0,1]\times X;E_+) \longrightarrow \mathcal{C}^\infty([0,1]\times X;E_-). \tag{6.127}$$

The discussion above now carries over to give similar results for semiclassical families.

Proposition 6.12. *The subspaces of properly supported semiclassial operators for any manifold have short exact symbol sequences*

$$\begin{gathered}0\hookrightarrow\Psi^{m-1}_{\text{sl}}(X)\hookrightarrow\Psi^m(X)\xrightarrow{\sigma_m}S^{m-[1]}(T^*X)\longrightarrow 0,\\ 0\hookrightarrow\epsilon\Psi^m_{\text{sl}}(X)\hookrightarrow\Psi^m_{\text{sl}}(X)\xrightarrow{\sigma_m}\longrightarrow 0,\end{gathered} \tag{6.128}$$

compose as operators (6.126) *and* (6.127) *and their symbols, standard and semiclassical, compose as well:*

$$\begin{gathered}\sigma_{m+m'}(A_\epsilon B_\epsilon)=\sigma_m(A_\epsilon)\circ\sigma_{m'}(B_\epsilon),\\ \sigma_{\text{sl}}(A_\epsilon B_\epsilon)=\sigma_{\text{sl}}(A_\epsilon)\circ\sigma_{\text{sl}}(B_\epsilon).\end{gathered} \tag{6.129}$$

The L^2 boundedness in Proposition 2.14 carries over easily to the manifold case.

Proposition 6.13. *If M is compact and E is a complex vector bundle over M then $A_\epsilon \in \Psi^0_{\mathrm{sl}}(M;E)$ then*

$$\sup_{0<\epsilon\le 1} \|A_\epsilon\|_{L^2(M;E)} < \infty. \tag{6.130}$$

We are particularly interested in semiclassical operators below because they make it possible to easily 'quantize' projections.

Proposition 6.14. *Suppose $p \in \mathcal{C}^\infty(\overline{{}^{sl}T^*X};\hom(E))$ is a smooth family of projections for a compact manifold X then there exists a semiclassical family of projections $P_\epsilon \in \Psi^0_{\mathrm{sl}}(X;E)$ such that $\sigma_{\mathrm{sl}}(P_\epsilon) = p$.*

Proof. By the surjectivity of the semiclassical symbol map we can choose $A_\epsilon \in \Psi^0_{\mathrm{sl}}(X;E)$ with $\sigma_{\mathrm{sl}}(A_\epsilon) = p$ and we can arrange that $\sigma_0(A_\epsilon)$ is the constant family of projections defined by p on the sphere bundle at infinity. Then

$$A_\epsilon^2 - A_\epsilon = E_\epsilon \in \epsilon\Psi^{-1}_{\mathrm{sl}}(X;E). \tag{6.131}$$

Composing on the left in (6.131) gives the same result as composing on the right, so

$$A_\epsilon E_\epsilon = E_\epsilon A_\epsilon \Longrightarrow \sigma(e) = p\sigma(e)p + (\mathrm{Id}-p)\sigma(e)(\mathrm{Id}-p) \tag{6.132}$$

where the symbolic identity is true in both sense, for $\sigma(e) = \sigma_{\mathrm{sl}}(e)$ and $\sigma(e) = \sigma_{-1}(e)$.

Now, we wish to 'correct' A_ϵ so this error term is smoothing and vanishes to infinite order at $\epsilon = 0$. First we add the term

$$A_\epsilon^{(1)} = A_\epsilon A_\epsilon^{(1)} A_\epsilon - (\mathrm{Id}-A_\epsilon)A_\epsilon^{(1)}(\mathrm{Id}-A_\epsilon) \in \Psi^{-1}_{\mathrm{sl}}(X;E)$$

to A_ϵ. This modifies (6.131) to

$$(A_\epsilon + A_\epsilon^{(1)})^2 - A_\epsilon - A_\epsilon^{(1)} = E_\epsilon + A_\epsilon A_\epsilon^{(1)} + A_\epsilon^{(1)} A_\epsilon - A_\epsilon^{(1)} \in \epsilon^2\Psi^{-2}_{\mathrm{sl}}(X;E). \tag{6.133}$$

Repeating this step generates an asymptotic solution and summing the asymptotic series gives a solution modulo rapidly decreasing smoothing error terms.

□

6.11. Heat kernel

6.12. Resolvent

6.13. Complex powers

6.14. Problems

Problem 6.1. Show that compatibility in the sense defined before Definition 6.1 is an equivalence relation on $\mathcal{C}^\infty$ structures. Conclude that there is a unique maximal $\mathcal{C}^\infty$ structure containing any give $\mathcal{C}^\infty$ structure.

Problem 6.2. Let $\mathcal{F}$ be a $\mathcal{C}^\infty$ structure on X and let O_a, $a \in A$, be a covering of X by coordinate neighbourhoods, in the sense of (6.2) and (6.3). Show that the maximal $\mathcal{C}^\infty$ structure containing $\mathcal{F}$ consists of ALL functions on X which are of the form (6.3) on each of these coordinate patches. Conclude that the maximal $\mathcal{C}^\infty$ structure is an algebra.

PROBLEM 6.3 (Partitions of unity). Show that any $\mathcal{C}^\infty$ manifold admits partitions of unity. That is, if O_a, $a \in A$, is an open cover of X then there exist elements $\rho_{a,i} \in \mathcal{C}^\infty(X)$, $a \in A$, $i \in \mathbb{N}$, with $0 \le \rho_{a,i} \le 1$, with each $\rho_{a,i}$ vanishing outside a compact subset $K_{a,i} \subset O_a$ such that only finite collections of the $\{K_{a,i}\}$ have non-trivial intersection and for which

$$\sum_{a\in A, i\in\mathbb{N}} \rho_{a,i} = 1.$$

CHAPTER 7

Scattering calculus

7.1. Scattering pseudodifferential operators

There is another calculus of pseudodifferential operators which is 'smaller' than the traditional calculus. It arises by taking amplitudes in (2.2) which treat the base and fibre variables symmetrically, but not 'simultaneously.' Thus consider the spaces

$$
\begin{aligned}
(7.1)\quad S^{l,m}_{\infty}(\mathbb{R}^p_z,\mathbb{R}^n_\xi) = \big\{&a \in \mathcal{C}^\infty(\mathbb{R}^{p+n});\\
&\sup_{\mathbb{R}^{p+n}} (1+|z|)^{-l+|\alpha|}(1+|\xi|)^{-m+|\beta|}|D^\alpha_z D^\beta_\xi a(z,\xi)| < \infty,\ \forall\ \alpha,\beta\big\}.
\end{aligned}
$$

Observe that

$$
S^{l,m}_{\infty}(\mathbb{R}^p_z;\mathbb{R}^n_\xi) \subset (1+|z|^2)^{l/2} S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi). \tag{7.2}
$$

We can then define

$$
\begin{aligned}
(7.3)\quad A \in \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \Longleftrightarrow A = (1+|x|^2)^{l/2}B,&\\
B \in \Psi^m_\infty(\mathbb{R}^n) \text{ and } \sigma_L(B) \in S^{0,m}_\infty(\mathbb{R}^n_x,\mathbb{R}^n_\xi)&.
\end{aligned}
$$

It follows directly from this definition and the properties of the 'traditional' operators that the left symbol map is an isomorphism

$$
\sigma_L : \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \longrightarrow S^{l,m}_\infty(\mathbb{R}^n_x,\mathbb{R}^n_\xi). \tag{7.4}
$$

To prove that this is an algebra, we need first the analogue of the asymptotic completeness, Proposition 2.3, for symbols in $S^{*,*}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

Lemma 7.1. *If $a_j \in S^{l-j,m-j}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ for $j \in \mathbb{N}_0$ then there exists*

$$
a \in S^{l,m}(\mathbb{R}^p,\mathbb{R}^n) \ \textit{s.t.}\ a - \sum_{j=0}^N a_j \in S^{l-N,m-N}_\infty(\mathbb{R}^p,\mathbb{R}^n)\ \forall\ N \in \mathbb{N}_0. \tag{7.5}
$$

Even though there is some potential for confusion we write $a \sim \sum_j a_j$ for a symbol a satisfying (7.5).

Proof. We use the same strategy as in the proof of Proposition 2.3 with the major difference that there are essentially two different symbolic variables. Thus with the same notation as in (2.54) we set

$$
a = \sum_j \phi(\epsilon_j z)\phi(\epsilon_j \xi) a_j(z,\xi) \tag{7.6}
$$

and we proceed to check that if the $\epsilon_j \downarrow 0$ fast enough as $j \to \infty$ then the series converges in $S^{l,m}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ and the limit satisfies (7.5).

The first of the seminorms, for convergence, is

$$A_j = \sup_z \sup_\xi (1+|z|)^{-l}(1+|\xi|)^{-m}\phi(\epsilon_j z)\phi(\epsilon_j \xi)|a_j(z,\xi)|.$$

On the support of this function either $|z| \ge 1/\epsilon_j$ or $|\xi| \ge 1/\epsilon$. Thus

$$\begin{aligned} A_j \le \sup_z \sup_\xi (1+|z|)^{-l+j}(1+|\xi|)^{-m+j}|a_j(z,\xi)| \\ \times \sup_z \sup_\xi (1+|z|)^{-j}(1+|\xi|)^{-j}\phi(\epsilon_j z)\phi(\epsilon_j \xi) \\ \le \epsilon_j^j \sup_z \sup_\xi (1+|z|)^{-l+j}(1+|\xi|)^{-m+j}|a_j(z,\xi)| \end{aligned}$$

The last term on the right is a seminorm on $S^{l-j,m-j}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ so convergence follows by choosing the ϵ_j *eventually* smaller than a certain sequence of positive numbers. The same argument follows, as in the discussion leading to (2.56), for convergence of the series for the derivatives and also for the stronger convergence leading to (7.5). Since overall this is a countable collection of conditions, all can be arranged by diagonalization and the result follows. □

With this result on asymptotic completeness the proof of Theorem 4.1 can be followed closely to yield the analogous result on products. In fact we can also define polyhomogeneous operators. This requires a little work if we try to do it directly. However see (1.99) and Problem 1.17 which encourages us to identify

$$\begin{gathered} \mathrm{RC}_p^* \times \mathrm{RC}_n^* : S^{0,0}_{\mathrm{ph}}(\mathbb{R}^p,\mathbb{R}^n) \longleftrightarrow \mathcal{C}^\infty(\mathbb{S}^{p,1}\times\mathbb{S}^{n,1}), \\ S^{l,m}_{\mathrm{ph}}(\mathbb{R}^p,\mathbb{R}^n) = (1+|z|^2)^{l/2}(1+|\xi|^2)^{m/2}S^{0,0}_{\mathrm{ph}}(\mathbb{R}^p,\mathbb{R}^n),\ l,m\in\mathbb{R}. \end{gathered} \tag{7.7}$$

These definitions are discussed as problems starting at Problem 1.18. Thus we simply define

$$\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n) = \left\{ A \in \Psi^{l,m}_{\infty-\mathrm{sc}}; \sigma_L(A) \in S^{l,m}_{\mathrm{ph}}(\mathbb{R}^n,\mathbb{R}^n) \right\}. \tag{7.8}$$

THEOREM 7.1. *The spaces $\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$ (resp. $\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)$) of scattering (resp. polyhomogeneous scattering) pseudodifferential operators on $\mathbb{R}^n$, form an order-bifiltered $*$-algebra*

$$\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)\circ\Psi^{l',m'}_{\infty-\mathrm{sc}}(\mathbb{R}^n)\subset\Psi^{l+l',m+m'}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \tag{7.9}$$

with residual spaces

$$\bigcap_{l,m}\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) = \bigcap_{l,m}\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n}). \tag{7.10}$$

CHAPTER 8

Elliptic boundary problems

Summary

Elliptic boundary problems are discussed, especially for operators of Dirac type. We start with a discussion of manifolds with boundary, including functions spaces and distributions. This leads to the 'jumps formula' for the relationship of the action of a differential operator to the operation of cutting off at the boundary; this is really Green's formula. The idea behind Calderòn's approach to boundary problems is introduced in the restricted context of a dividing hypersurface in a manifold without boundary. This includes the fundamental result on the boundary behaviour of a pseudodifferential operator with a rational symbol. These ideas are then extended to the case of an operator of Dirac type on a compact manifold with boundary with the use of left and right parametrices to define the Calderòn projector. General boundary problems defined by pseudodifferential projections are discussed by reference to the 'Calderòn realization' of the operator. Local boundary conditions, and the corresponding ellipticity conditions, are then discussed and the special case of Hodge theory on a compact manifold with boundary is analysed in detail for absolute and relative boundary conditions.

Introduction

Elliptic boundary problems arise from the fact that elliptic differential operators on compact manifolds with boundary have infinite dimensional null spaces. The main task we carry out below is the parameterization of this null space, in terms of boundary values, of an elliptic differential operator on a manifold with boundary. For simplicity of presentation the discussion of elliptic boundary problems here will be largely confined to the case of first order systems of differential operators of Dirac type. This has the virtue that the principal results can be readily stated.

Status as of 4 August, 1998

Read through Section 8.1–Section 8.2: It is pretty terse in places! Several vital sections are still missing.

8.1. Manifolds with boundary

Smooth manifolds with boundary can be defined in very much the same was as manifolds without boundary. Thus we start with a paracompact Hausdorff space X and assume that it is covered by 'appropriate' coordinate patches with corresponding transition maps. In this case the 'model space' is $\mathbb{R}^{n,1} = [0, \infty) \times \mathbb{R}^{n-1}$, a Euclidean half-space of fixed dimension, n. As usual it is more convenient to use as models all open subsets of $\mathbb{R}^{n,1}$; of course this means *relatively open*, not open as subsets of $\mathbb{R}^n$. Thus we allow any

$$O = O' \cap \mathbb{R}^{n,1}, \quad O' \subset \mathbb{R}^n \text{ open},$$

as local models.

By a smooth map between open sets in this sense we mean a map with a smooth extension. Thus if O_i for $i = 1, 2$ are open in $\mathbb{R}^{n,1}$ then smoothness of a map F means that

$$\begin{aligned}(8.1)\quad F : O_1 \to O_2,\ \exists\ O_i' \subset \mathbb{R}^n,\ i = 1, 2,\ \text{open and } \tilde{F} : O_1' \to O_2' \\ \text{which is } \mathcal{C}^\infty \text{ with } O_i = O_i' \cap \mathbb{R}^{n,1} \text{ and } F = F'|O_1.\end{aligned}$$

It is important to note that the smoothness condition is *much* stronger than just smoothness of F on $O \cap (0, \infty) \times \mathbb{R}^{n-1}$.

By a diffeomorphism between such open sets we mean an invertible smooth map with a smooth inverse. Various ways of restating the condition that a map be a diffeomorphism are discussed below.

With this notion of local model we define a *coordinate system (in the sense of manifolds with boundary)* as a homeomorphism of open sets

$$X \supset U \xrightarrow{\Phi} O \subset \mathbb{R}^{n,1}, \quad O, U \text{ open}.$$

Thus Φ^{-1} is assumed to exist and both Φ and Φ^{-1} are assumed to be continuous. The *compatibility* of two such coordinate systems (U_1, Φ_1, O_1) and (U_2, Φ_2, O_2) is the requirement that *either* $U_1 \cap U_2 = \phi$ *or* if $U_1 \cap U_2 \neq \phi$ then

$$\Phi_{1,2} = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \to \Phi_2(U_1 \cap U_2)$$

is a diffeomorphism in the sense described above. Notice that both $\Phi_1(U_1 \cap U_2)$ and $\Phi_2(U_1 \cap U_2)$ are open in $\mathbb{R}^{n,1}$. The inverse $\Phi_{1,2}$ is defined analogously.

A $\mathcal{C}^\infty$ manifold with boundary can then be formally defined as a paracompact Hausdorff topological space which has a maximal covering by mutually compatible coordinate systems.

An alternative definition, i.e.
characterization, of a manifold with boundary is that there exists a $\mathcal{C}^\infty$ manifold $\tilde{X}$ *without boundary* and a function $f \in \mathcal{C}^\infty(\tilde{X})$ such that $df \neq 0$ on $\{f = 0\} \subset \tilde{X}$ and

$$X = \left\{p \in \tilde{X}; f(p) \geq 0\right\},$$

with coordinate systems obtained by restriction from $\tilde{X}$. The doubling construction described below shows that this is in fact an equivalent notion.

8.2. Smooth functions

As in the boundaryless case, the space of functions on a compact manifold with boundary is the primary object of interest. There are two basic approaches to defining local smoothness, the one intrinsic and the other extrinsic, in the style of the two definitions of a manifold with boundary above. Thus if $O \subset \mathbb{R}^{n,1}$ is open we can simply set

$$\begin{aligned}\mathcal{C}^\infty(O) = \{u : O \to \mathbb{C}; \exists\ \tilde{u} \in \mathcal{C}^\infty(O'), \\ O' \subset \mathbb{R}^n \text{ open}, \quad O = O' \cap \mathbb{R}^{n,1}, u = \tilde{u}|_O\right\}.\end{aligned}$$

Here the open set in the definition might depend on u. The derivatives of $\tilde{u} \in \mathcal{C}^\infty(O')$ are bounded on all compact subsets, $K \Subset 0$. Thus

$$\sup_{K \cap O^\circ} |D^\alpha u| < \infty, \quad O^\circ = O \cap ((0,\infty) \times \mathbb{R}^{n-1}). \tag{8.2}$$

The second approach is to use (8.2) as a definition, i.e.
to set

$$\mathcal{C}^\infty(O) = \{u : O^\circ \to \mathbb{C}; (8.2) \text{ holds } \forall\ K \Subset O \text{ and all } \alpha\}. \tag{8.3}$$

In particular this implies the continuity of $u \in \mathcal{C}^\infty(O)$ up to any point $p \in O \cap (\{0\} \times \mathbb{R}^{n-1})$, the boundary of O as a manifold with boundary.

As the notation here asserts, these two approaches are equivalent. This follows (as does much more) from a result of Seeley:

Proposition 8.1. *If $\mathcal{C}^\infty(O)$ is defined by* (8.3) *and $O' \subset \mathbb{R}^n$ is open with $O = O' \cap \mathbb{R}^{n,1}$ then there is a linear extension map*

$$E : \mathcal{C}^\infty(O) \to \mathcal{C}^\infty(O'), \quad Eu|_{O'} = u$$

which is continuous in the sense that for each $K' \Subset O'$, compact, there is some $K \Subset O$ such that for each α

$$\sup_{K'} |D^\alpha Eu| \le C_{\alpha,K'} \sup_{K \cap O} |D^\alpha u|.$$

The existence of such an extension map shows that the definition of a diffeomorphism of open sets O_1, O_2, given above, is equivalent to the condition that the map be invertible and that it, and its inverse, have components which are in $\mathcal{C}^\infty(O_1)$ and $\mathcal{C}^\infty(O_2)$ respectively.

Given the local definition of smoothness, the global definition should be evident. Namely, if X is a $\mathcal{C}^\infty$ manifold with boundary then

$$\mathcal{C}^\infty(X) = \left\{u : X \to \mathbb{C}; (\Phi^{-1})^*(u|_U) \in \mathcal{C}^\infty(O)\ \forall \text{ coordinate systems}\right\}.$$

This is also equivalent to demanding that local regularity, i.e.
the regularity of $(\Phi^{-1})^*(u|_O)$, hold for any one covering by admissible coordinate systems.

As is the case of manifolds without boundary, $\mathcal{C}^\infty(X)$ admits partitions of unity. In fact the proof of Lemma 6.3 applies verbatim; see also Problem 6.3.

The topology of $\mathcal{C}^\infty(X)$ is given by the supremum norms of the derivatives in local coordinates. Thus a seminorm

$$\sup_{K \Subset O} \left|D^\alpha (\Phi^{-1})^*(u|_U)\right|$$

arises for each compact subset of each coordinate patch. In fact there is a countable set of norms giving the same topology. If X is compact, $\mathcal{C}^\infty(X)$ is a Fréchet space, if it is not compact it is an inductive limit of Fréchet spaces (an LF space).

The boundary of X, ∂X, is the union of the $\Phi^{-1}(O \cap (\{0\} \times \mathbb{R}^{n-1}))$ over coordinate systems. It is a manifold without boundary. It is compact if X is compact. Furthermore, ∂X has a global defining function $\rho \in \mathcal{C}^\infty(X)$; that is $\rho \ge 0$, $\partial X = \{\rho = 0\}$ and $d\rho \ne 0$ at ∂X. Moreover if ∂X is compact then any such boundary defining function can be extended to a product decomposition of X near ∂X:

$$\exists C \supset \partial X, \text{open in } X\ \epsilon > 0 \text{ and a diffeomorphism} \varphi : C \simeq [0,\epsilon)_\rho \times \partial X. \tag{8.4}$$

If ∂X is not compact this is still possible for an appropriate choice of ρ. For an outline of proofs see Problem 8.1.

LEMMA 8.1. *If X is a manifold with compact boundary then for any boundary defining function $\rho \in \mathcal{C}^\infty(X)$ there exists $\epsilon > 0$ and a diffeomorphism* (8.4).

PROBLEM 8.1.

The existence of such a product decomposition near the boundary (which might have several components) allows the doubling construction mentioned above to be carried through. Namely, let

$$\tilde{X} = (X \cup X)/\partial X \tag{8.5}$$

be the disjoint union of two copies of X with boundary points identified. Then consider

$$\begin{aligned}\mathcal{C}^\infty(\tilde{X}) = \{(u_1, u_2) \in \mathcal{C}^\infty(X) \oplus \mathcal{C}^\infty(X);& \\ (\varphi^{-1})^*(u_1|_C) = f(\rho, \cdot), (\varphi^{-1})^*(u_2|_C) = f(-\rho, \cdot),& \\ f \in \mathcal{C}^\infty((-1,1) \times \partial X)\}\,.&\end{aligned} \tag{8.6}$$

This is a $\mathcal{C}^\infty$ structure on $\tilde{X}$ such that $X \hookrightarrow \tilde{X}$, as the first term in (8.5), is an embedding as a submanifold with boundary, so

$$\mathcal{C}^\infty(X) = \mathcal{C}^\infty(\tilde{X})|_X.$$

In view of this possibility of extending X to $\tilde{X}$, we shall not pause to discuss all the usual 'natural' constructions of tensor bundles, density bundles, bundles of differential operators, etc. They can simply be realized by restriction from $\tilde{X}$. In practice it is probably preferable to use intrinsic definitions.

The definition of $\mathcal{C}^\infty(X)$ implies that there is a well-defined restriction map

$$\mathcal{C}^\infty(X) \ni u \longmapsto u|_{\partial X} \in \mathcal{C}^\infty(\partial X).$$

It is always surjective. Indeed the existence of a product decomposition shows that any smooth function on ∂X can be extended locally to be independent of the chosen normal variable, and then cut off near the boundary.

There are important points to observe in the description of functions near the boundary. We may think of $\mathcal{C}^\infty(X) \subset \mathcal{C}^\infty(X^\circ)$ as a subspace of the smooth functions on the interior of X which describes the 'completion' (compactification if X is compact!) of the interior to a manifold with boundary. It is in this sense that the action of a differential operator $P \in \mathrm{Diff}^m(X)$

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

should be understood. Thus P is just a differential operator on the interior of X with 'coefficients smooth up to the boundary.'

Once this action is understood, there is an obvious definition of the space of $\mathcal{C}^\infty$ functions which vanish to all orders at the boundary,

$$\dot{\mathcal{C}}^\infty(X) = \{u \in \mathcal{C}^\infty(X); Pu|_{\partial X} = 0 \;\forall\; P \in \mathrm{Diff}^*(X)\}\,.$$

Having chosen a product decomposition near the boundary, Taylor's theorem gives us an isomorphism

$$\mathcal{C}^\infty(X)/\dot{\mathcal{C}}^\infty(X) \cong \bigoplus_{k \geq 0} \mathcal{C}^\infty(\partial X) \cdot [d\rho|_{\partial X}]^k.$$

8.3. Distributions

It is somewhat confusing that there are *three* (though really only *two*) spaces of distributions immediately apparent on a compact manifold with boundary. Understanding the relationship between them is important to the approach to boundary problems used here.

The *coarsest* (as it is a little dangerous to say *largest*) space is $\mathcal{C}^{-\infty}(X^\circ)$, the dual of $\mathcal{C}^\infty_c(X^\circ;\Omega)$, just the space of distributions on the interior of X. The elements of $\mathcal{C}^{-\infty}(X^\circ)$ may have unconstrained growth, and unconstrained order of singularity, approaching the boundary. They are not of much practical value here and appear for conceptual reasons.

Probably the most natural space of distributions to consider is the dual of $\mathcal{C}^\infty(X;\Omega)$ since this is arguably the direct analogue of the boundaryless case. We shall denote this space

$$\dot{\mathcal{C}}^{-\infty}(X) = (\mathcal{C}^\infty(X;\Omega))' \tag{8.7}$$

and call it the space of *supported distributions*. The 'dot' is supposed to indicate this support property, which we proceed to describe.

If $\tilde{X}$ is any compact extension of X (for example the double) then, as already noted, the restriction map $\mathcal{C}^\infty(\tilde{X};\Omega) \to \mathcal{C}^\infty(X;\Omega)$ is continuous and surjective. Thus, by duality, we get an injective 'extension' map

$$\dot{\mathcal{C}}^{-\infty}(X) \ni u \mapsto \tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X}),\ \tilde{u}(\varphi) = u(\varphi|_X). \tag{8.8}$$

We shall regard this injection as an identification $\dot{\mathcal{C}}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(\tilde{X})$; its range is easily characterized.

Proposition 8.2. *The range of the map* (8.8) *is the subspace consisting of those* $\tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X})$ *with* $\operatorname{supp}\tilde{u} \subset X$.

The proof is given below. This proposition is the justification for calling $\dot{\mathcal{C}}^{-\infty}(X)$ the space of supported distributions; the dot is support to indicate that this is the subspace of the 'same' space for $\tilde{X}$, i.e.
$\mathcal{C}^{-\infty}(\tilde{X})$, of elements with support in X.

This notation is consistent with $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(\tilde{X})$ being the subspace (by extension as zero) of elements with support in X. The same observation applies to sections of any vector bundle, so

$$\dot{\mathcal{C}}^\infty(X;\Omega) \subset \mathcal{C}^\infty(\tilde{X};\Omega)$$

is a well-defined closed subspace. We set

$$\mathcal{C}^{-\infty}(X) = (\dot{\mathcal{C}}^\infty(X;\Omega))' \tag{8.9}$$

and call this the space of *extendible distributions* on X. The inclusion map for the test functions gives by duality a restriction map:

$$\begin{gathered} R_X : \mathcal{C}^{-\infty}(\tilde{X}) \to \mathcal{C}^{-\infty}(X), \\ R_X u(\varphi) = u(\varphi)\ \forall\ \varphi \in \dot{\mathcal{C}}^\infty(X;\Omega) \hookrightarrow \mathcal{C}^\infty(\tilde{X};\Omega). \end{gathered} \tag{8.10}$$

We write, at least sometimes, R_X for the map since it has a large null space so should *not* be regarded as an identification. In fact

$$\operatorname{Nul}(R_X) = \left\{ v \in \mathcal{C}^{-\infty}(\tilde{X}); \operatorname{supp}(v) \cap X^\circ = \phi \right\} = \dot{\mathcal{C}}^{-\infty}(\tilde{X} \backslash X^\circ), \tag{8.11}$$

is just the space of distributions supported 'on the other side of the boundary'. The primary justification for calling $\mathcal{C}^{-\infty}(X)$ the space of extendible distributions is:

PROPOSITION 8.3. *If X is a compact manifold with boundary, then the space $\mathcal{C}_c^\infty(X^\circ;\Omega)$ is dense in $\dot{\mathcal{C}}^\infty(X;\Omega)$ and hence the restriction map*

$$\mathcal{C}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(X^\circ) \tag{8.12}$$

is injective, whereas the restriction map from (8.10), $R_X : \dot{\mathcal{C}}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X)$, *is surjective.*

PROOF. If V is a real vector field on $\tilde{X}$ which is inward-pointing across the boundary then

$$\exp(sV) : \tilde{X} \to \tilde{X}$$

is a diffeomorphism with $F_s(X) \subset X^\circ$ for $s > 0$. Furthermore if $\varphi \in \mathcal{C}^\infty(\tilde{X})$ then $F_s^*\varphi \to \varphi$ in $\mathcal{C}^\infty(\tilde{X})$ as $s \to 0$. The support property shows that $F_s^*\varphi \in \mathcal{C}_c^\infty(X^\circ)$ if $s < 0$ and $\varphi \in \dot{\mathcal{C}}^\infty(X)$. This shows the density of $\mathcal{C}_c^\infty(X^\circ)$ in $\dot{\mathcal{C}}^\infty(X)$. Since all topologies are uniform convergence of all derivatives in open sets. The same argument applies to densities. The injectivity of (8.12) follows by duality.

On the other hand the surjectivity of (8.10) follows directly from the Hahn-Banach theorem. □

PROOF OF PROPOSITION 8.2. For $\tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X})$ the condition that $\operatorname{supp} \tilde{u} \subset X$ is just

$$\tilde{u}(\varphi) = 0 \ \forall \ \varphi \in \mathcal{C}_c^\infty \subset (\tilde{X}\backslash X;\Omega) \subset \mathcal{C}^\infty(\tilde{X};\Omega). \tag{8.13}$$

Certainly (8.13) holds if $u \in \dot{\mathcal{C}}^{-\infty}(X)$ since $\varphi|_X = 0$. Conversely, if (8.13) holds, then by continuity and the density of $\mathcal{C}_c^\infty(\tilde{X}\backslash X;\Omega)$ in $\mathcal{C}^\infty(\tilde{X}\backslash X^\circ;\Omega)$, what follows from Proposition 8.3, $\tilde{u}$ vanishes on $\dot{\mathcal{C}}^\infty(X\backslash X^\circ)$. □

It is sometimes useful to consider topologies on the spaces of distributions $\mathcal{C}^{-\infty}(X)$ and $\dot{\mathcal{C}}^{-\infty}(X)$. For example we may consider the *weak topology*. This is given by all the seminorms $u \mapsto \|\langle u, \phi\rangle\|$, where ϕ is a test function.

LEMMA 8.2. *With respect to the weak topology, the subspace $\mathcal{C}_c^\infty(X^\circ)$ is dense in both $\dot{\mathcal{C}}^{-\infty}(X)$ and $\mathcal{C}^{-\infty}(X)$.*

8.4. Boundary Terms

To examine the precise relationship between the supported and extendible distributions consider the space of 'boundary terms'.

$$\dot{\mathcal{C}}_{\partial X}^{-\infty}(X) = \left\{ u \in \dot{\mathcal{C}}^{-\infty}(X); \operatorname{supp}(u) \subset \partial X \right\}. \tag{8.14}$$

Here the support may be computed with respect to any extension, or intrinsically on X. We may also define a map 'cutting off' at the boundary:

$$\mathcal{C}^\infty(X) \ni u \mapsto u_c \in \dot{\mathcal{C}}^{-\infty}(X), u_c(\varphi) = \int_X u\varphi \ \forall \ \varphi \in \mathcal{C}^\infty(X;\Omega). \tag{8.15}$$

Proposition 8.4. *If X is a compact manifold with boundary then there is a commutative diagram*

$$
\begin{tikzcd}
 & & & \dot{\mathcal{C}}^{\infty}(X) \arrow[d] & \\
 & & & \mathcal{C}^{\infty}(X) \arrow[dl, "()_c"'] \arrow[d] & \\
0 \arrow[r] & \dot{\mathcal{C}}^{-\infty}_{\partial X}(X) \arrow[r] & \dot{\mathcal{C}}^{-\infty}(X) \arrow[r] & \mathcal{C}^{-\infty}(X) \arrow[r] & 0
\end{tikzcd}
\tag{8.16}
$$

with the horizontal sequence exact.

Proof. The commutativity of the triangle follows directly from the definitions. The exactness of the horizontal sequence follows from the density of $\mathcal{C}^{\infty}_c(X^\circ;\Omega)$ in $\dot{\mathcal{C}}^{\infty}(X;\Omega)$. Indeed, this shows that $v \in \dot{\mathcal{C}}^{-\infty}_{\partial X}(X)$ maps to 0 in $\mathcal{C}^{-\infty}(X)$ since $v(\varphi) = 0 \ \forall \ \varphi \in \mathcal{C}^{\infty}_c(X^\circ;\Omega)$. Similarly, if $u \in \dot{\mathcal{C}}^{-\infty}(X)$ maps to zero in $\mathcal{C}^{-\infty}(X)$ then $u(\varphi) = 0$ for all $\varphi \in \mathcal{C}^{\infty}_c(X^\circ;\Omega)$, so $\operatorname{supp}(u) \cap X^\circ = \emptyset$. □

Note that both maps in (8.16) from $\mathcal{C}^{\infty}(X)$ into supported and extendible distributions are injective. We regard the map into $\mathcal{C}^{-\infty}(X)$ as an identification. In particular this is consistent with the action of differential operators. Thus $P \in \operatorname{Diff}^m(X)$ acts on $\mathcal{C}^{\infty}(X)$ and then the smoothness of the coefficients of P amount to the fact that it preserves $\mathcal{C}^{\infty}(X)$, as a subspace. The formal adjoint P^* with respect to the sesquilinear pairing for some smooth positive density, ν

$$
\langle \varphi, \psi \rangle = \int_X \varphi \overline{\psi} \nu \quad \forall \ \varphi, \psi \in \mathcal{C}^{\infty}(X) \tag{8.17}
$$

acts on $\dot{\mathcal{C}}^{\infty}(X)$:

$$
\langle P^*\varphi, \psi \rangle = \langle \varphi P\psi \rangle \quad \forall \ \varphi \in \dot{\mathcal{C}}^{\infty}(X), \psi \in \mathcal{C}^{\infty}(X), \ P^* : \dot{\mathcal{C}}^{\infty}(X) \longrightarrow \dot{\mathcal{C}}^{\infty}(X). \tag{8.18}
$$

However, $P^* \in \operatorname{Diff}^m(X)$ is fixed by its action over X°. Thus we do have

$$
\langle P^*\varphi, \psi \rangle = \langle \varphi, P\psi \rangle \quad \forall \ \varphi \in \mathcal{C}^{\infty}(X), \psi \in \dot{\mathcal{C}}^{\infty}(X). \tag{8.19}
$$

We define the action of P by duality. In view of the possibility of confusion, we denote P the action on $\mathcal{C}^{-\infty}(X)$ and by $\dot{P}$ the action on $\dot{\mathcal{C}}^{\infty}(X)$.

$$
\begin{gathered}
\langle Pu, \varphi \rangle = \langle u, P^*\varphi \rangle \quad \forall \ u \in \mathcal{C}^{-\infty}(X), \varphi \in \dot{\mathcal{C}}^{\infty}(X), \ P : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X) \\
\langle \dot{P}u, \varphi \rangle = \langle u, P^*\varphi \rangle \quad \forall \ u \in \dot{\mathcal{C}}^{-\infty}(X), \varphi \in \mathcal{C}^{\infty}(X), \ \dot{P} : \dot{\mathcal{C}}^{-\infty}(X) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X).
\end{gathered}
\tag{8.20}
$$

It is of fundamental importance that (8.19) does not hold for all $\varphi, \psi \in \mathcal{C}^{\infty}(X)$. This failure is reflected in Green's formula for the boundary terms, which appears below as the 'Jump formula'. This is a distributional formula for the difference

$$
\dot{P}u_c - (Pu)_c \in \dot{\mathcal{C}}^{-\infty}_{\partial X}, u \in \mathcal{C}^{\infty}(X)P \in \operatorname{Diff}^m(X). \tag{8.21}
$$

Recall that a product decomposition of $C \subset X$ near ∂X is fixed by an inward pointing vector field V. Let $x \in \mathcal{C}^{\infty}(X)$ be a corresponding boundary defining function, with $Vx = 0$ near ∂X, with $\chi_V : C \to \partial X$ the projection onto the boundary from the product neighborhood C. Then Taylor's formula for $u \in \mathcal{C}^\infty(X)$ becomes

$$u \sim \sum_k \frac{1}{k!} \chi_V^*(V^k u|_{\partial x}) x^k. \tag{8.22}$$

It has the property that a finite sum

$$u_N = \varphi u - \varphi \sum_{k=0}^{N} \frac{1}{k!} \chi_V^*(V^k u|_{\partial X}) x^k$$

where $\varphi \equiv 1$ near ∂X, $\operatorname{supp} \varphi \subset C$, satisfies

$$\dot{P}(u_N)_c = (Pu_N)_c, P \in \operatorname{Diff}^m(X), m < N. \tag{8.23}$$

Since $(1-\varphi)u \in \dot{\mathcal{C}}^\infty(X)$ also satisfies this identity, the difference in (8.21) can (of course) only depend on the $V^k u|_{\partial X}$ for $k \le m$, in fact only for $k < m$.

Consider the Heaviside function $1_c \in \dot{\mathcal{C}}^{-\infty}(X)$, detained by cutting off the identity function of the boundary. We define distributions

$$\delta^{(j)}(x) = V^{j+1} 1_c \in \dot{\mathcal{C}}^{-\infty}_{\partial X}, j \ge 0. \tag{8.24}$$

Thus, $\delta^{(0)}(x) = \delta(x)$ is a 'Dirac delta function' at the boundary. Clearly $\operatorname{supp} \delta(x) \subset \partial X$, so the same is true of $\delta^{(j)}(x)$ for every j. If $\psi \in \mathcal{C}^\infty(\partial X)$ we define

$$\psi \cdot \delta^{(j)}(x) = \varphi(X_V^* \psi) \cdot \delta^{(j)}(x) \in \dot{\mathcal{C}}^{-\infty}_{\partial X}(X). \tag{8.25}$$

This, by the support property of $\delta^{(j)}$, is independent of the cut off φ used to define it.

PROPOSITION 8.5. *For each $P \in \operatorname{Diff}^m(X)$ there are differential operators on the boundary $P_{ij} \in \operatorname{Diff}^{m-i-j-1}(\partial X)$, $i + j < m$, $i, j \ge 0$, such that*

$$\dot{P} u_c - (Pu)_c = \sum_{i,j} (P_{ij}(V_u^j|_{\partial X}) \cdot \delta^{(j)}(x), \ \forall \ u \in \mathcal{C}^\infty(X), \tag{8.26}$$

and $P_{0m-1} = i^{-m}\sigma(P, dx) \in \mathcal{C}^\infty(\partial X)$.

PROOF. In the local product neighborhood C,

$$P = \sum_{0 \le l \le m} P_l V^l \tag{8.27}$$

where P_l is a differential operator of the order at most $m - l$, on X be depending on x as a parameter. Thus the basic cases we need to analyze arise from the application of V to powers of x :

$$x^l \left(V^{j+1}(x^p)_c - (V^{j+1} x^p)_c\right) = \begin{cases} 0 & l + p > j \\ \frac{p!(j-p)!}{(j-p-l)!} \quad (-1)^l \delta^{(j-p-l)} & l + p \le j. \end{cases} \tag{8.28}$$

Taking the Taylor sense of the P_l,

$$P_l \sim \sum_r x^r P_{l,r}$$

and applying P to (8.22) gives

$$Pu_c - (Pu)_c = \sum_{r+k<l} (-1)^r (P_{l,r}(V^k u|_{\partial x})) \quad \delta^{(l-1-r-k)}(x). \tag{8.29}$$

This is of the form (8.26). The only term with $l-1-r-k=m-1$ arises from $l-m$, $k=r=0$ so is the operator P_m at $x=0$. This is just $i^{-m}\sigma(P,dx)$. $\square$

8.5. Sobolev spaces

As with $\mathcal{C}^\infty$ functions we may define the standard (extendible) Sobolev spaces by restriction or intrinsically. Thus, if $\tilde{X}$ is an extension of a compact manifold with boundary, X, the we can define

$$H^m(X) = H^m_{\mathrm{c}}(\tilde{X})|X,\ \forall\ m\in\mathbb{R}; H^m(X)\subset\mathcal{C}^{-\infty}(X). \tag{8.30}$$

That this is independent of the choice of $\tilde{X}$ follows from the standard properties of the Sobolev spaces, particularly their localizability and invariance under diffeomorphisms. The norm in $H^m(X)$ can be taken to be

$$\|u\|_m = \inf\left\{\|\tilde{u}\|_{H^m(\tilde{X})}; \tilde{u}\in H^m(\tilde{X}),\ u=\tilde{u}_X\right\}. \tag{8.31}$$

A more intrinsic defintion of these spaces is discussed in the problems.

There are also *supported* Sobolev spaces,

$$\dot{H}^m(X) = \left\{u\in H^m(\tilde{X}); \operatorname{supp}(u)\subset X\right\}\subset\dot{\mathcal{C}}^{-\infty}(X). \tag{8.32}$$

Sobolev space of sections of any vector bundle can be defined similarly.

Proposition 8.6. *For any $m\in\mathbb{R}$ and any compact manifold with boundary X, $H^m(X)$ is the dual of $\dot{H}^{-m}(X;\Omega)$ with respect to the continuous extension of the densely defined bilinear pairing*

$$(u,v)=\int_X uv,\ u\in\mathcal{C}^\infty(X),\ v\in\dot{\mathcal{C}}^\infty(X;\Omega).$$

Both $H^m(X)$ and $\dot{H}^m(X)$ are $\mathcal{C}^\infty(X)$-modules and for any vector bundle over X, $H^m(X;E)\equiv H^m(X)\otimes_{\mathcal{C}^\infty(X)}\mathcal{C}^\infty(X;E)$ and $\dot{H}^m(X;E)\equiv\dot{H}^m(X)\otimes_{\mathcal{C}^\infty(X)}\mathcal{C}^\infty(X;E)$.

Essentially from the definition of the Sobolev spaces, any $P\in\operatorname{Diff}^k(X;E_1,E_2)$ defines a continuous linear map

$$P: H^m(X;E_1)\longrightarrow H^{m-k}(X;E_2). \tag{8.33}$$

We write the dual (to P^* of course) action

$$\dot{P}:\dot{H}^m(X;E_1)\longrightarrow\dot{H}^{m-k}(X;E_2). \tag{8.34}$$

These actions on Sobolev spaces are consistent with the corresponding actions on distributions. Thus

$$\mathcal{C}^{-\infty}(X;E)=\bigcup_m H^m(X),\ \mathcal{C}^\infty(X;E)=\bigcap_m H^m(X),$$
$$\dot{\mathcal{C}}^{-\infty}(X;E)=\bigcup_m \dot{H}^m(X),\ \dot{\mathcal{C}}^\infty(X;E)=\bigcap_m \dot{H}^m(X).$$

8.6. Dividing hypersurfaces

As already noted, the point of view we adopt for boundary problems is that they provide a parametrization of the space of solutions of a differential operator on a space with boundary. In order to clearly indicate the method pioneered by Calderòn, we shall initially consider the restrictive context of an operator of Dirac type on a compact manifold without boundary with an embedded separating hypersurface.

Thus, suppose initially that D is an elliptic first order differential operator acting between sections of two (complex) vector bundles V_1 and V_2 over a compact manifold without boundary, M. Suppose further that $H \subset M$ is a dividing hypersurface. That is, H is an embedded hypersuface with oriented (i.e. trivial) normal bundle and that $M = M_+ \cup M_-$ where $M_\pm$ are compact manifolds with boundary which intersect in their common boundary, H. The convention here is that M_+ is on the positive side of H with respect to the orientation.

In fact we shall make a further analytic assumption, that

$$D : \mathcal{C}^\infty(M; V_1) \longrightarrow \mathcal{C}^\infty(M; V_2) \text{ is an isomorphism.} \tag{8.35}$$

As we already know, D is always Fredholm, so this implies the topological condition that the index vanish. However we only assume (8.35) to simplify the initial discussion.

Our objective is to study the space of solutions on M_+. Thus consider the map

$$\left\{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0 \text{ in } M_+^\circ\right\} \xrightarrow{b_H} \mathcal{C}^\infty(H; V_1),\ b_H u = u_{|\partial M_+}. \tag{8.36}$$

The idea is to use the boundary values to parameterize the solutions and we can see immediately that this is possible.

Lemma 8.3. *The assumption* (8.35) *imples that map b_H in* (8.36) *is injective.*

Proof. Consider the form of D in local coordinates near a point of H. Let the coordinates be $x, y_1, \dots, y_{n-1}$ where x is a local defining function for H and assume that the coordinate patch is so small that V_1 and V_2 are trivial over it. Then

$$D = A_0 D_x + \sum_{j=1}^{n-1} A_j D_{y_j} + A'$$

where the A_j and A' are local smooth bundle maps from V_1 to V_2. In fact the ellipticity of D implies that each of the A_j's is invertible. Thus the equation can be written locally

$$D_x u = Bu,\ B = -\sum_{j=1}^{n-1} A_0^{-1} D_{y_j} - A_0^{-1} A'.$$

The differential operator B is tangent to H. By assumption u vanishes when restricted to H so it follows that $D_x u$ also vanishes at H. Differentiating the equation with respect to x, it follows that all derivatives of u vanish at H. This in turn implies that the global section of V_1 over M

$$\tilde{u} = \begin{cases} u & \text{in } M_+ \\ 0 & \text{in } M_- \end{cases}$$

is smooth and satisfies $D\tilde{u} = 0$. Then assumption (8.35) implies that $\tilde{u} = 0$, so $u = 0$ in M_+ and b_H is injective. $\square$

In the proof of this Lemma we have used the strong assumption (8.35). As we show below, if it is assumed instead that D is of Dirac type then the Lemma remains true without assuming (8.35). Now we can state the basic result in this setting.

THEOREM 8.1. *If $M = M_+ \cup M_-$ is a compact manifold without boundary with separating hypersurface H as described above and $D \in \mathrm{Diff}^1(M; V_1, V_2)$ is a generalized Dirac operator then there is an element $\Pi_C \in \Psi^0(H; V)$, $V = V_1|H$, satisfying $\Pi_C^2 = \Pi_C$ and such that*

$$b_H : \{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0\} \longrightarrow \Pi_C \mathcal{C}^\infty(H; V) \tag{8.37}$$

is an isomorphism. The projection Π_C can be chosen so that

$$b_H : \{u \in \mathcal{C}^\infty(M_-; V_1); Du = 0\} \longrightarrow (\mathrm{Id} - \Pi_C) \mathcal{C}^\infty(H; V) \tag{8.38}$$

then Π_C is uniquely determined and is called the Calderòn projection.

This result remains true for a general elliptic operator of first order if (8.35) is assumed, and even in a slightly weakened form without (8.35). Appropriate modifications to the proofs below are consigned to problems.

For first order operators the jump formula discussed above takes the following form.

LEMMA 8.4. *Let D be an elliptic differential operator of first order on M, acting between vector bundles V_1 and V_2. If $u \in \mathcal{C}^\infty(M_+; V_1)$ satisfies $Du = 0$ in M_+° then*

$$Du_c = \frac{1}{i}\sigma_1(D)(dx)(b_H u) \cdot \delta(x) \in \mathcal{C}^{-\infty}(M; V_2). \tag{8.39}$$

Since the same result is true for M_-, with an obvious change of sign, D defines a linear operator

$$\begin{aligned} D : \big\{u \in L^1(M; V_1); u_\pm = u|M_\pm \in \mathcal{C}^\infty(M_\pm; V_1),\ Du_\pm = 0 \text{ in } M_\pm^\circ\big\} \longrightarrow \\ \frac{1}{i}\sigma(D)(dx)(b_H u_+ - b_H u_-) \cdot \delta(x) \in \mathcal{C}^\infty(H; V_2) \cdot \delta(x). \end{aligned} \tag{8.40}$$

To define the Calderòn projection we shall use the 'inverse' of this result.

PROPOSITION 8.7. *If $D \in \mathrm{Diff}^1(M; V_1, V_2)$ is elliptic and satisfies (8.35) then (8.40) is an isomorphism, with inverse I_D, and*

$$\Pi_C v = b_H \left(I_D \frac{1}{i} \sigma(D)(dx) v \cdot \delta(x) \right)_+,\ v \in \mathcal{C}^\infty(H; V_1), \tag{8.41}$$

satisfies the conditions of Theorem 8.1.

PROOF. Observe that the map (8.40) is injective, since its null space consists of solutions of $Du = 0$ globally on M; such a solution must be smooth by elliptic regularity and hence must vanish by the assumed invertibility of D. Thus the main task is to show that D in (8.40) is surjective.

Since D is elliptic and, by assumption, an isomorphism on $\mathcal{C}^\infty$ sections over M it is also an isomorphism on distributional sections. Thus the inverse of (8.40) must be given by D^{-1}. To prove the surjectivity it is enough to show that

$$D^{-1}(w \cdot \delta(x))|M_\pm \in \mathcal{C}^\infty(M_\pm; V_1)\ \forall\ w \in \mathcal{C}^\infty(H; V_2). \tag{8.42}$$

There can be no singular terms supported on H since $w \cdot \delta(x) \in H^{-1}(M; V_2)$ implies that $u = D^{-1}(w \cdot \delta(x)) \in L^2(M; V_1)$.

Now, recalling that $D^{-1} \in \Psi^{-1}(M; V_2, V_1)$, certainly u is $\mathcal{C}^\infty$ away from H. At any point of H outside the support of w, u is also smooth. Since we may decompose w using a partition of unity, it suffices to suppose that w has support in a small coordinate patch, over which both V_1 and V_2 are trivial and to show that u is smooth 'up to H from both sides' in the local coordinate patch. Discarding smoothing terms from D^{-1} we may therefore replace D^{-1} by any local parametrix Q for D and work in local coordinates and with components:
(8.43)

$$Q_{ij}(w_j(y) \cdot \delta(x)) = (2\pi)^{-n} \int e^{i(x-x')\xi + i(y-y')\cdot\eta} q_{ij}(x, y, \xi, \eta) w(y') \delta(x') dx' dy' d\xi d\eta.$$

For a general pseudodifferential operator, even of order -1, the result we are seeking is not true. We must use special properties of the symbol of Q, that is D^{-1}.

8.7. Rational symbols

LEMMA 8.5. *The left-reduced symbol of any local parametrix for a generalized Dirac operator has an expansion of the form*
(8.44)

$$q_{ij}(z, \zeta) = \sum_{l=1}^{\infty} g(z, \zeta)^{-2l+1} p_{ij,l}(z, \zeta) \text{ with } p_{ij,l} \text{ a polynomial of degree } 3l - 2 \text{ in } \zeta;$$

here $g(z, \zeta)$ is the metric in local coordinates; each of the terms in (8.44) *is therefore a symbol of order $-l$.*

PROOF. This follows by an inductive arument, of a now familiar type. First, the assumption that D is a generalized Dirac operator means that its symbol $\sigma_1(D)(z, \zeta)$ has inverse $g(z, \zeta)^{-1} \sigma_1(D)^*(z, \zeta)$; this is the princiapl symbol of Q. Using Leibniz' formula one concludes that for any polynomial r_l of degree j

$$\partial_{\zeta_i} \left(g(z, \zeta)^{-2l+1} r_j(z, \zeta) \right) = g(z, \zeta)^{-2l} r'_{j+1}(z, \zeta)$$

where r_{j+1} has degree (at most) $j+1$. Using this result repeatedly, and proceeding by induction, we may suppose that $q = q'_k + q''_{k+1}$ where q'_k has an expansion up to order k, and so may be taken to be such a sum, and q''_{k+1} is of order at most $-k-1$. The composition formula for left-reduced symbols then shows that

$$\sigma_1(D) q''_{k+1} \equiv g^{-2k} q_{k+1} \mod S^{-k-1}$$

where q_{k+1} is a polynomial of degree at most $3k$. Inverting $\sigma_1(D)(\zeta)$ as at the initial step then shows that q''_{k+1} is of the desired form, $g^{-2k-1} r_{k+1}$ with r_{k+1} of degree $3k + 1 = 3(k+1) - 2$, modulo terms of lower order. This completes the proof of the lemma. □

With this form for the symbol of Q we proceed to the proof of Proposition 8.7. That is, we consider (8.43). Since we only need to consider each term, we shall drop the indicies. A term of low order in the amplitude q_N gives an operator with kernel in $\mathcal{C}^{N-d}$. Such a kernel gives an operator

$$\mathcal{C}^\infty(H; V_2) \longrightarrow \mathcal{C}^{N-d}(M; V_1)$$

with kernel in $\mathcal{C}^{N-d}$. The result we want will therefore follow if we show that each term in the expansion of the symbol q gives an operator as in (8.42).

To be more precise, we can assume that the amplitude q is of the form

$$q = (1 - \phi) g^{-2l} q'$$

where q' is a polynomial of degree $3l-2$ and $\phi = \phi(\xi, \eta)$ is a function of compact support which is identically one near the origin. The cutoff function is to remove the singularity at $\zeta = (\xi, \eta) = 0$. Using continuity in the symbol topology the integrals in x' and y' can be carried out. By assumption $w \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1})$, so the resulting integral is absolutely convergent in η. If $l > 1$ it is absolutely convergent in ξ as well, so becomes

$$Q(w(y) \cdot \delta(x)) = (2\pi)^{-n} \int e^{ix\xi + iy\cdot\eta} q(x, y, \xi, \eta) \hat{w}(\eta) d\xi d\eta.$$

In $|\xi| > 1$ the amplitude is a rational function of ξ, decaying quadratically as $\xi \to \infty$. If we assume that $x > 0$ then the exponential factor is bounded in the half plane $\Im\xi \geq 0$. This means that the limit as $R \to \infty$ over the integral in $\Im\xi \geq 0$ over the semicircle $|\xi| = R$ tends to zero, and does so with uniform rapid decrease in η. Cauchy's theorem shows that, for $R > 1$ the real integral in ξ can be replaced by the contour integral over $\gamma(R)$, which is, for$R >> |\eta|$ given by the real interval $[-R, R]$ together with the semicircle of radius R in the upper half plane. If $|\eta| > 1$ the integrand is meromorphic in the upper half plane with a possible pole at the singular point $g(x, y, \xi, \eta) = 0$; this is at the point $\xi = ir^{\frac{1}{2}}(x, y, \eta)$ where $r(x, y, \eta)$ is a positive-definite quadratic form in η. Again applying Cauchy's theorem

$$Q(w(y)\delta(x) = (2\pi)^{-n+1} i \int e^{xr^{\frac{1}{2}}(x,y,\eta) + iy\cdot\eta} q'(x, y, \eta)\hat{w}(\eta) d\eta$$

where q' is a symbol of order $-k+1$ in η.

The product $e^{xr^{\frac{1}{2}}(x,y,\eta)} q'(x, y, \eta)$ is uniformly a symbol of order $-k+1$ in $x > 1$, with x derivatives of order p being uniformly symbols of order $-k+1+p$. It follows from the properties of pseudodifferential operators that $Q(w \cdot \delta(x))$ is a smooth function in $x > 0$ with all derivatives locally uniformly bounded as $x \downarrow 0$.

8.8. Proofs of Proposition 8.7 and Theorem 8.1

This completes the proof of (8.42), since a similar argument applies in $x < 0$, with contour deformation into the lower half plane. Thus we have shown that (8.40) is an isomorphism which is the first half of the statement of Proposition (8.7). Furthermore we see that the limiting value from above is a pseudodifferential operator on H :

$$Q_0 w = \lim_{x \downarrow 0} D^{-1}(w \cdot \delta(x)), \; Q_0 \in \Psi^0(H; V_2, V_1). \tag{8.45}$$

This in turn implies that Π_C, defined by (8.41) is an element of $\Psi^0(H; V_1)$, since it is $Q_0 \circ \frac{1}{i}\sigma(D)(dx)$.

Next we check that Π_C is a projection, i.e. that $\Pi_C^2 = \Pi_C$. If $w = \Pi_C v$, $v \in \mathcal{C}^\infty(H; V_1)$, then $w = b_H u$, $u = I_D \frac{1}{i}\sigma(D)(dx)v|_{M_+}$, so $u \in \mathcal{C}^\infty(M_+; V_1)$ satisfies $Du = 0$ in M_+°. In particular, by (8.39), $Pu_c = \frac{1}{i}\sigma_1(D)(dx)w \cdot \delta(x)$, which means that $w = \Pi_C w$ so $\Pi_C^2 = \Pi_C$. This also shows that the range of Π_C is precisely the range of b_H as stated in (8.37). The same argument shows that this choice of the projection gives (8.38). □

8.9. Inverses

Still for the case of a generalized Dirac operator on a compact manifold with dividing hypersurface, consider what we have shown. The operator D defines a map in (8.39) with inverse

$$I_D : \{v \in \mathcal{C}^\infty(H; V_1); \Pi_C v = v\} \longrightarrow \{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0 \text{ in } M_+\}. \tag{8.46}$$

This operator is the 'Poisson' operator for the canonical boundary condition given by the Calderòn operator, that is $u = I_D v$ is the unique solution of

$$Du = 0 \text{ in } M_+,\ u \in \mathcal{C}^\infty(M_+; V_1),\ \Pi_C b_H u = v. \tag{8.47}$$

We could discuss the regularity properties of I_D but we shall postpone this until after we have treated the 'one-sided' case of a genuine boundary problem.

As well as I_D we have a natural right inverse for the operator D as a map from $\mathcal{C}^\infty(M_+; V_1)$ to $\mathcal{C}^\infty(M_-; V_2)$. Namely

Lemma 8.6. *If $f \in \mathcal{C}^\infty(M_+; V_2)$ then $u = D^{-1}(f_c)|_{M_+} \in \mathcal{C}^\infty(M_+; V_1)$ and the map $R_D : f \longmapsto u$ is a right inverse for D, i.e.*
$D \circ R_D = \mathrm{Id}$.

Proof. Certainly $D(D^{-1}(f_c) = f_c$, so $u = D^{-1}(f_c)|_{M_+} \in \mathcal{C}^{-\infty}(M_+; V_1)$ satisfies $Du = f$ in the sense of extendible distributions. Since $f \in \mathcal{C}^\infty(M_+; V_2)$ we can solve the problem $Du \equiv f$ in the sense of Taylor series at H, with the constant term freely prescibable. Using Borel's lemma, let $u' \in \mathcal{C}^\infty(M_+; V_1)$ have the appropriate Taylor series, with $b_H u' = 0$.. Then $D(u'_c) - f_c = g \in \dot{\mathcal{C}}^\infty(M +_; V_2)$. Thus $u'' = D^{-1} g \in \mathcal{C}^\infty(M; V_1)$. Since $D(u' - u'') = f_c$, the uniqueness of solutions implies that $u = (u' - u'')|_{M_+} \in \mathcal{C}^\infty(M_+; V_1)$. □

Of course R_D cannot be a two-sided inverse to D since it has a large null space, described by I_D.

Problem 8.2. Show that, for D as in Theorem 8.1 if $f \in \mathcal{C}^\infty(M_+; V_2)$ and $v \in \mathcal{C}^\infty(H; V_1)$ there exists a unique $u \in \mathcal{C}^\infty(M_+; V_2)$ such that $Du = f$ in $\mathcal{C}^\infty(M_+; V_2)$ and $b_H u = \Pi_C v$.

8.10. Smoothing operators

The properties of smoothing operators on a compact manifold with boundary are essentially the same as in the boundaryless case. Rather than simply point to the earlier discussion we briefly repeat it here, but in an abstract setting.

Let $\mathcal{H}$ be a separable Hilbert space. In the present case this would be $L^2(X)$ or $L^2(X; E)$ for some vector bundle over X, or some space $H^m(X; E)$ of Sobolev sections. Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the ideal of compact operators. Where necessary the norm on $\mathcal{B}$ will be written $\|\ \|_{\mathcal{B}}$; $\mathcal{K}$ is a closed subspace of $\mathcal{B}$ which is the closure of the ideal $\mathcal{F} = \mathcal{F}(\mathcal{H})$ of finite rank bounded operators.

We will consider a subspace $\mathcal{J} = \mathcal{J}(\mathcal{H}) \subset \mathcal{B}$ with a stronger topology. Thus we suppose that $\mathcal{J}$ is a Fréchet algebra. That is, it is a Fréchet space with countably many norms $\|\ \|_k$ such that for each k there exists k' and C_k with

$$\|AB\|_k \le C_k \|A\|_{k'} \|B\|_{k'} \quad \forall\ A, B \in \mathcal{J}. \tag{8.48}$$

In particular of course we are supposing that $\mathcal{J}$ is a subalgebra (but *not* an ideal) in $\mathcal{B}$. To make it a topological *-subalgebra we suppose that

$$\|A\|_{\mathcal{B}} \le C \|A\|_k \quad \forall\ A \in \mathcal{J},\ * : \mathcal{J} \longrightarrow \mathcal{J}. \tag{8.49}$$

In fact we may suppose that $k = 0$ by renumbering the norms. The third condition we impose on $\mathcal{J}$ implies that it is a subalgebra of $\mathcal{K}$, namely we insist that

$$\mathcal{F} \cap \mathcal{J} \text{ is dense in } \mathcal{J}, \tag{8.50}$$

in the Fréchet topology. Finally, we demand, in place of the ideal property, that $\mathcal{J}$ be a bi-ideal in $\mathcal{B}$ (also called a 'corner') that is,

$$\begin{gathered} A_1, A_2 \in \mathcal{J}, B \in \mathcal{B} \Longrightarrow A_1 B A_2 \in \mathcal{J}, \\ \forall\ k\ \exists\ k' \text{ such that } \|A_1 B A_2\|_k \le C \|A_1\|_{k'} \|B\|_{\mathcal{B}} \|A_2\|_{k'}. \end{gathered} \tag{8.51}$$

Proposition 8.8. *The space of operators with smooth kernels acting on sections of a vector bundle over a compact manifold satisfies* (8.48)–(8.52) *with* $\mathcal{H} = H^m(X; E)$ *for any vector bundle* E.

Proof. The smoothing operators on sections of a bundle E can be written as integral operators

$$Au(x) = \int_E A(x, y) u(y), A(x, y) \in \mathcal{C}^\infty(X^2; \operatorname{Hom}(E) \otimes \Omega_R). \tag{8.52}$$

Thus $\mathcal{J} = \mathcal{C}^\infty(X^2; \operatorname{Hom}(E) \otimes \Omega_R)$ and we make this identification topological. The norms are the C^k norms. If $P_1, \dots, p_{N^{(m)}}$ is a basis, on $\mathcal{C}^\infty(X^2)$, for the differential operators of order m on $\operatorname{Hom}(E) \otimes \Omega_L$ then we may take

$$\|A\|_m = \sup_j \|P_j A\|_{L^\infty} \tag{8.53}$$

for some inner products on the bundles. In fact $\operatorname{Hom}(E) = \pi_L^* E \otimes \pi_R^* E^*$ from it which follows easily that this is a basis $P_j = P_{j,k} \otimes P_{j,R}$ decomposing as products. From this (8.48) follows easily since

$$\|AB\|_m = \sup_j \|(P_{jL} A) \cdot (P_{j,R} B)\|_\infty \|AB\|_{L^\infty} \le C \|A\|_{L^\infty} \|B\|_{L^\infty} \tag{8.54}$$

by the compactnes of X. From this (8.53) follows with $k = 0$.

The density (8.50) is the density of the finite tensor product $\mathcal{C}^\infty(X; E) \otimes \mathcal{C}^\infty (X; E^* \otimes \Omega_L)$ in $\mathcal{C}^\infty(X^2; \operatorname{Hom}(E) \otimes \Omega_L)$. This follows from the boundaryless case by doubling (or directly). Similarly the bi-ideal condition (8.52) can be seen from the regularity of the kernel. A more satisfying argument using distribution theory follows from the next result.

□

Proposition 8.9. *An operator* $A : \dot{\mathcal{C}}^\infty(X; E) \to \mathcal{C}^{-\infty}(X; F)$ *is a smoothing operator if and only if it extends by continuity to* $\dot{\mathcal{C}}^{-\infty}(X; E)$ *and then has range in* $\mathcal{C}^\infty(X; F) \hookrightarrow \mathcal{C}^{-\infty}(X; F)$.

Proof. If A has the stated mapping property then compose with a Seeley extension operator, then $EA = \tilde{A}$ is a continuous linear map

$$\tilde{A} : \dot{\mathcal{C}}^{-\infty}(X; E) \to \mathcal{C}^\infty(\tilde{X}; \tilde{F}),$$

for an extension of F to $\tilde{F}$ over the double $\tilde{X}$. Localizing in the domain to trivialize E and testing with a moving delta function we recover the kernel of $\tilde{A}$ as

$$\tilde{A}(x, y) = \tilde{A} \cdot \delta_y \in \mathcal{C}^\infty(\tilde{X}; \tilde{F}).$$

Thus it follows that $\tilde{A} \in \mathcal{C}^\infty(\tilde{X} \times X; \operatorname{Hom}(E, \tilde{F}) \otimes \Omega_R)$. The converse is more obvious.

□

Returning to the general case of a bi-ideal as in (8.48)–(8.52) we may consider the invertibility of $\mathrm{Id}+A$, $A \in \mathcal{J}$.

PROPOSITION 8.10. *If* $A \in \mathcal{J}$*, satisfying* (8.48)–(8.52)*, then* $\mathrm{Id}+A$ *has a generalized inverse of the form* $\mathrm{Id}+B$, $B \in \mathcal{J}$*, with*

$$AB = \mathrm{Id} - \pi_R, BA = \mathrm{Id} - \pi_L \in \mathcal{J} \cap \mathcal{F}$$

both finite rank self-adjoint projections.

PROOF. Suppose first that $A \in \mathcal{J}$ and $\|A\|_{\mathcal{B}} < 1$. Then $\mathrm{Id}+A$ is invertible in $\mathcal{B}$ with inverse $Id + B \in \mathcal{B}$,

$$B = \sum_{j \geq 1} (-1)^j A^j. \tag{8.55}$$

Not only does this Neumann series converge in $\mathcal{B}$ but also in $\mathcal{J}$ since for each k

$$\|A^j\|_k \leq C_k \|A\|_{k'} \|A^{j-2}\|_{\mathcal{B}} \|A\|_{k'} \leq C'_k \|A\|_{\mathcal{B}}^{j-2}, j \geq 2. \tag{8.56}$$

Thus $B \in \mathcal{J}$, since by assumption $\mathcal{J}$ is complete (being a Fréchet space). In this case $\mathrm{Id}+B \in \mathcal{B}$ is the unique two-sided inverse.

For general $A \in \mathcal{J}$ we use the assumed approximability in (8.50). Then $A = A' + A''$ when $A' \in \mathcal{F} \cap \mathcal{J}$ and $\|A''\|_{\mathcal{B}} \leq C\|A''\|_k < 1$ by appropriate choice. It follows that $\mathrm{Id}+B'' = (\mathrm{Id}+A'')^{-1}$ is the inverse for $\mathrm{Id}+A''$ and hence a parameterix for $\mathrm{Id}+A$:

$$\begin{aligned} (\mathrm{Id}+B'')(\mathrm{Id}+A) &= \mathrm{Id}+A' + B''A' \\ (\mathrm{Id}+A)(\mathrm{Id}+B'') &= \mathrm{Id}+A' + A'B'' \end{aligned} \tag{8.57}$$

Unfinished. Lemma on subprojections.

with both 'error' terms in $\mathcal{F} \cap \mathcal{J}$. □

8.11. Left and right parametrices

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a bounded linear operator between them. Let $\mathcal{J}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{J}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be bi-ideals as in the previous section. A left parametrix for A, modulo $\mathcal{J}_1$, is a bounded linear map $B_L : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

$$B_L \circ A = \mathrm{Id} + J_L, \ J_L \in \mathcal{J}_1. \tag{8.58}$$

Similarly a right parametrix for A, modulo $\mathcal{J}_2$ is a bounded linear map $B_R : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

$$A \circ B_R = \mathrm{Id} + J_R, \ J_R \in \mathcal{J}_2. \tag{8.59}$$

PROPOSITION 8.11. *If a bounded linear operator* $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ *has a left parametrix* B_L *modulo a bi-ideal* $\mathcal{J}_1$*, satisfying* (8.48)–(8.52)*, then* A *has closed range, null space of finite dimension and there is a generalized left inverse, differing from the original left parametrix by a term in* $\mathcal{J}_1$*, such that*

$$B_L \circ A = \mathrm{Id} - \pi_L, \ \pi_L \in \mathcal{J}_1 \cap \mathcal{F}, \tag{8.60}$$

with π_L *the self-adjoint projection onto the null space of* A*.*

Proof. Applying Proposition 8.10, $\mathrm{Id}+J_L$ has a generalized inverse $\mathrm{Id}+J$, $J \in \mathcal{J}_1$, such that $(\mathrm{Id}+J)(\mathrm{Id}+J_L) = (\mathrm{Id}-\pi'_L)$, $\pi'_L \in \mathcal{J}_1 \cap \mathcal{F}$. Replacing B_L by $\tilde{B}_L = (\mathrm{Id}+J)B_L$ gives a new left parametrix with error term $\pi'_L \in \mathcal{J}_1 \cap \mathcal{F}$. The null space of A is contained in the null space of $B'_L \circ A$ and hence in the range of F_L; thus it is finite dimensional. Furthermore the self-dajoint projection π_L onto the null space is a subprojection of π'_L, so is also an element of $\mathcal{J}_1 \cap \mathcal{F}$. The range of A is closed since it has finite codimension in $\mathrm{Ran}(A(\mathrm{Id}-\pi_L))$ and if $f_n \in \mathrm{Ran}(A(\mathrm{Id}-\pi_L)) = Au_n$, $u_n = (\mathrm{Id}-\pi_L)u_n$, converges to $f \in \mathcal{H}_2$, then $u_n = B_L f_n$ converges to $u \in \mathcal{H}_1$ with $A(\mathrm{Id}-\pi_L)u = f$. □

Proposition 8.12. *If a bounded linear operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ has a right parametrix B_R modulo a bi-ideal $\mathcal{J}_2$, satisfying* (8.48)–(8.52), *then it has closed range of finite codimension and there is a generalized right inverse, differing from the original right parametrix by a term in $\mathcal{J}_2$, such that*

$$A \circ B_R = \mathrm{Id}-\pi_R,\ \pi_R \in \mathcal{J}_2 \cap \mathcal{F}, \tag{8.61}$$

with $\mathrm{Id}-\pi_R$ *the self-adjoint projection onto the range space of A.*

Proof. The operator $\mathrm{Id}+J_R$ has, by Proposition 8.10, a generalized inverse $\mathrm{Id}+J$ with $J \in \mathcal{J}_1$. Thus $B'_R = B_R \circ (\mathrm{Id}+J)$ is a right parametrix with error term $\mathrm{Id}-\pi'_R$, $\pi'_R \in \mathcal{J}_1 \cap \mathcal{F}$ being a self-adjoint projection. Thus the range of A contains the range of $\mathrm{Id}-\pi'_R$ and is therefore closed with a finite-dimensional complement. Furthemore the self-adjoint projection onto the range of A is of the form $\mathrm{Id}-\pi_R$ where π_R is a subprojection of π'_R, so also in $\mathcal{J}_1 \cap \mathcal{F}$. □

The two cases, of an operator with a right or a left parametrix are sometimes combined in the term ‘semi-Fredholm.’ Thus an operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is semi-Fredholm if it has closed range and either the null space or the orthocomplement to the range is finite dimensional. The existence of a right or left parametrix, modulo the ideal of compact operators, is a necessary and sufficient condition for an operator to be semi-Fredholm.

8.12. Right inverse

In treating the ‘general’ case of an elliptic operator on compact manifold with boundary we shall start by constructing an analogue of the right inverse in Lemma 8.6. So now we assume that $D \in \mathrm{Diff}^1(X; V_1, V_2)$ is an operator of Dirac type on a compact manifold with boundary.

To construct a right inverse for D we follow the procedure in the boundaryless case. That is we use the construction of a pseudodifferential parametrix. In order to make this possible we need to extend M and D ‘across the boundary.’ This is certainly possible for X, since we may double it to a compact manifold without boundary, $2X$. Then there is not obstruction to extending D ‘a little way’ across the boundary. We shall denote by M an open extension of X (of the same dimension) so $X \subset M$ is a compact subset and by $\tilde{D}$ an extension of Dirac type to M.

The extension of D to $\tilde{D}$, being elliptic, has a parametrix $\tilde{Q}$. Consider the map

$$\tilde{Q}' : L^2(X; V_2) \longrightarrow H^1(X; V_1),\ \tilde{Q}'f = \tilde{Q}f_c|X \tag{8.62}$$

where f_c is the extension of f to be zero outside X. Then $\tilde{Q}'$ is a right parametrix, $D\tilde{Q}' = \mathrm{Id}+E$ where E is an operator on $L^2(X; V_2)$ with smooth kernel on X^2.

Following Proposition 8.12, D has a generalized right inverse $\tilde{Q}'' = \tilde{Q}'(\mathrm{Id} + E')$ up to finite rank smoothing and

$$D : H^1(X; V_1) \longleftrightarrow L^2(X; V_2) \tag{8.63}$$

has closed range with a finite dimensional complement in $\mathcal{C}^\infty(X; V_2)$.

PROPOSITION 8.13. *The map* (8.63) *maps* $\mathcal{C}^\infty(X; V_2)$ *to* $\mathcal{C}^\infty(X; V_1)$, *it is surjective if and only if the only solution of* $D^* u = 0$, $u \in \dot{\mathcal{C}}^\infty(X; V_2)$ *is the trivial solution.*

PROOF. The regularity statement, that $Q'\mathcal{C}^\infty(X; V) \subset \mathcal{C}^\infty(X; V_1)$ follows as in the proof of Lemma 8.6. Thus Q' maps $\mathcal{C}^\infty(X; V_1)$ to $\mathcal{C}^\infty(X; V_2)$ if and only if any paramatrix $\tilde{Q}'$ does so. Given $f \in \mathcal{C}^\infty(X; V_2)$ we may solve $Du' \equiv f$ in Taylor series at the boundary, with $u' \in \mathcal{C}^\infty(X; V_1)$ satisfying $b_H u' = 0$. Then $D(u')_c - f \in \dot{\mathcal{C}}^\infty(X; V_2)$ so it follows that $Q'(f_c)|_X \in \mathcal{C}^\infty(X; V_1)$.

Certainly any solution of $D^* u = 0$ with $u \in \dot{\mathcal{C}}^\infty(X; V_2)$ is orthogonal to the range of (8.63) so the condition is necessary. So, suppose that (8.63) is not surjective. Let $f \in L^2(X; V_2)$ be in the orthocomplement to the range. Then Green's formula gives the pairing with any smooth section

$$(Dv, f)_X = (D\tilde{v}, f_c)_{\tilde{X}} = (\tilde{v}, D^* f_c)_{\tilde{X}} = 0.$$

This means that $D^* f_c = 0$ in $\tilde{X}$, that is as a supported distribution. Thus, $f \in \dot{\mathcal{C}}^\infty(X; V_2)$ satisfies $D^* f = 0$. □

As noted above we will proceed under the assumption that $D^* f$ has no such non-trivial solutions in $\dot{\mathcal{C}}^\infty(X; V_2)$. This condition is discussed in the next section.

THEOREM 8.2. *If unique continuation holds for* D^* *then* D *has a right inverse*

$$Q : \mathcal{C}^\infty(X : V_2) \longrightarrow \mathcal{C}^\infty(X; V_1), \; DQ = \mathrm{Id} \tag{8.64}$$

where $Q = \tilde{Q}' + E$, $\tilde{Q}' f = \tilde{Q} f | X$ *where* $\tilde{Q}$ *is a parametrix for an extension of* D *across the boundary and* E *is a smoothing operator on* X.

PROOF. As just noted, unique continuation for D^* implies that D in (8.63) is surjective. Since the parametrix maps $\mathcal{C}^\infty(X; V_2)$ to $\mathcal{C}^\infty(X; V_1)$, D must be surjective as a map from $\mathcal{C}^\infty(X; V_1)$ to $\mathcal{C}^\infty(X; V_2)$. The parametrix modulo finite rank operators can therefore be corrected to a right inverse for D by the addition of a smoothing operator of finite rank. □

8.13. Boundary map

The map b from $\mathcal{C}^\infty(X; E)$ to $\mathcal{C}^\infty(\partial X; E)$ is well defined, and hence is well defined on the space of smooth solutions of D. We wish to show that it has closed range. To do so we shall extend the defintion to the space of square-integrable solutions. For any $s \in \mathbb{R}$ set

$$\mathcal{N}^s(D) = \{u \in H^s(X; E); Du = 0\}. \tag{8.65}$$

Of course the equation $Du = 0$ is to hold in the sense of extendible distributions, which just means in the interior of X. Thus $\mathcal{N}^\infty(D)$ is the space of solutions of D smooth up to the boundary.

LEMMA 8.7. *If $u \in \mathcal{N}^0(D)$ then*

$$\dot{D}u_c = v \cdot \delta(x),\ v \in H^{-\frac{1}{2}}(\partial X; E) \tag{8.66}$$

defines an injective bounded map $\tilde{b} : \mathcal{N}^0(D) \longrightarrow H^{-\frac{1}{2}}(\partial X; E)$ by $\tilde{b}(u) = i\sigma(D)(dx)v$ which is an extension of $b : \mathcal{N}^\infty(D) \longrightarrow \mathcal{C}^\infty(\partial X; E)$ defined by restriction to the boundary.

PROOF. Certainly $\dot{D}u_c \subset \dot{\mathcal{C}}^{\infty}_{\partial X}(X; E)$ has support in the boundary, so is a sum of products in any product decomposition of X near ∂X,

$$D(u_c) = \sum_j v_j \cdot \delta^{(j)}(x).$$

Since D is a first order operator and $u_c \in L^2(\tilde{X}; E)$, for any local extension, $\dot{D}u_c \in \dot{H}^{-1}(X; E)$. Localizing so that E is trivial and the localized v_j have compact supports this means that

$$(1 + |\eta|^2 + |\xi|^2)^{-\frac{1}{2}} \widehat{v_j}(\eta)\xi^j \in L^2(\mathbb{R}^n). \tag{8.67}$$

If $v_j \neq 0$ for some $j > 0$ this is not true even in some region $|\eta| < C$. Thus $v_j \equiv 0$ for $j > 0$ and (8.66) must hold. Furthermore integration in ξ gives

$$\begin{gathered}\int_{\mathbb{R}} (1 + |\eta|^2 + |\xi|^2)^{-1} d\xi = c(1 + |\eta|^2)^{-\frac{1}{2}},\ c > 0, \quad \text{so} \\ \int_{\mathbb{R}^{n-1}} (1 + |\eta|^2)^{-\frac{1}{2}} |\hat{v}(\eta)|^2 d\eta < 0.\end{gathered} \tag{8.68}$$

Thus $v \in H^{-\frac{1}{2}}(\partial X; E)$ and $\tilde{b}$ is well defined. The jumps formula shows it to be an extension of b. The injectivity of $\tilde{b}$ follows from the assumed uniqueness of solutions to $\dot{D}u = 0$ in X. □

Notice that (8.68) is actually reversible. That is if $v \in H^{-\frac{1}{2}}(\partial X; E)$ then $v \cdot \delta(x) \in H^{-1}(X; E)$. This is the basis of the construction of a left parametrix for $\tilde{b}$, which then shows its range to be closed.

LEMMA 8.8. *The boundary map $\tilde{b}$ in Lemma 8.7 has a continuous left parametrix $\widetilde{I_D} : H^{-\frac{1}{2}}(\partial X; E) \longrightarrow \mathcal{N}^0(D)$, $I_D \circ \tilde{b} = \mathrm{Id} + G$, where G has smooth kernel on $X \times \partial X$, and the range of $\tilde{b}$ is therefore a closed subspace of $H^{-\frac{1}{2}}(\partial X; E)$.*

PROOF. The parametrix $\widetilde{I_D}$ is given directly by the parametrix $\tilde{Q}$ for $\tilde{D}$, and extension to $\tilde{X}$. Applying $\tilde{Q}$ to (8.66) gives

$$u = \widetilde{I_D}v + Ru,\ \widetilde{I_D} = R_X \circ \tilde{Q} \circ \frac{1}{i}\sigma(D)(dx) \tag{8.69}$$

with R having smooth kernel. Since $\widetilde{I_D}$ is bounded from $H^{-\frac{1}{2}}(\partial X; E)$ to $L^2(X; E)$ and R is smoothing it follows from Proposition 8.11 that the range of $\tilde{b}$ is closed. □

8.14. Calderòn projector

Having shown that the range of $\tilde{b}$ in Lemma 8.7 is closed in $H^{-\frac{1}{2}}(\partial X; E)$ we now deduce that there is a pseudodifferential projection onto it. The discussion above of the boundary values of the $\tilde{Q}(w \cdot \delta(x))$ is local, and so applies just as well in the present more general case. Since this is just the definition of the map $\widetilde{I_D}$ in Lemma 8.8, we conclude directly that

$$Pv = \lim_{X^\circ} \widetilde{I_D} v,\ v \in \mathcal{C}^\infty(\partial X; E) \tag{8.70}$$

defines $P \in \Psi^0(\partial X; E)$.

Lemma 8.9. *If P is defined by* (8.70) *then $P^2 - P \in \Psi^{-\infty}(\partial X; E)$ and there exist A, $B \in \Psi^{-\infty}(\partial X; E)$ such that $P - \mathrm{Id} = A$ on $\mathrm{Ran}(\tilde{b})$ and $\mathrm{Ran}(P + B) \subset \mathrm{Ran}(\tilde{b})$.*

Proof needs clarification.

Proof. That $P^2 - P \in \Psi^{-\infty}(\partial X; E)$ follows, as above, from the fact that $\tilde{Q}$ is a two-sided parametrix on distributions supported in X. Similarly we may use the right inverse of D to construct B. If $v \in H^{-\frac{1}{2}}(\partial X; E)$ then by construction,

$$D\widetilde{I_D} v = R' v$$

where R' has a smooth kernel on $X \times \partial X$. Applying the right inverse Q it follows that $u' = \widetilde{I_D} v - (Q \circ R')v \in \mathcal{N}^0(D)$, where $Q \circ R'$ also has smooth kernel on $X \times \partial X$. Thus $\tilde{b}(u') = (P + B)v \in \mathrm{Ran}(\tilde{b})$ where B has kernel arising from the restriction of the kernel of $A \circ R'$ to $\partial X \times \partial X$, so $B \in \Psi^{-\infty}(\partial X; E)$. □

Now we may apply Proposition 6.11 with $F = \mathrm{Ran}(\tilde{b})$ and $s = -\frac{1}{2}$ to show the existence of a Calderòn projector.

Proposition 8.14. *If D is a generalized Dirac operator on X then there is an element $\Pi_C \in \Psi^0(\partial X; E)$ such that $\Pi_C^2 = \Pi_C$, $\mathrm{Ran}(\Pi_C) = \mathrm{Ran}(\tilde{b})$ on $H^{-\frac{1}{2}}(\partial X; E)$, $\Pi_C - P \in \Psi^{-\infty}(\partial X; E)$ where P is defined by* (8.70) *and $\mathrm{Ran}(\Pi_C) = \mathrm{Ran}(b)$ on $\mathcal{C}^\infty(\partial X; E)$.*

Proof. The existence of psuedodifferential projection, Π_C, differing from P by a smoothing operator and with range $\mathrm{Ran}(\tilde{b})$ is a direct consequence of the application of Proposition 6.11. It follows that $\mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ is dense in $\mathrm{Ran}(\tilde{b})$ in the topology of $H^{-\frac{1}{2}}(\partial X; E)$. Furthermore, if follows that if $v \in \mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ then $u \in \mathcal{N}^0(D)$ such that $\tilde{b}u = v$ is actually in $\mathcal{C}^\infty(X; E)$, i.e. it is in $\mathcal{N}^\infty(D)$. Thus the range of b is just $\mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ so $\mathrm{Ran}(b)$ is the range of Π_C acting on $\mathcal{C}^\infty(\partial X; E)$. □

In particular $\tilde{b}$ is just the continuous extension of b from $\mathcal{N}^\infty(D)$ to $\mathcal{N}^0(D)$, of which it is a dense subset. Thus we no longer distinguish between these two maps and set $\tilde{b} = b$.

8.15. Poisson operator

8.16. Unique continuation

8.17. Boundary regularity

8.18. Pseudodifferential boundary conditions

The discussion above shows that for any operator of Dirac type the 'Calderòn realization' of D,

$$D_{\mathcal{C}} : \{u \in H^s(X; E_1); \Pi_{\mathcal{C}} bu = 0\} \longrightarrow H^{s-1}(X; E_2),\ s > \frac{1}{2} \tag{8.71}$$

is an isomorphism.

We may replace the Calderòn projector in (8.71) by a more general projection Π, acting on $\mathcal{C}^\infty(\partial X, V_1)$, and consider the map

$$D_\Pi : \{u \in \mathcal{C}^\infty(X; V_1); \Pi bu = 0\} \longrightarrow \mathcal{C}^\infty(X; V_2). \tag{8.72}$$

In general this map will not be particularly well-behaved. We will be interested in the case that $\Pi \in \Psi^0(\partial X; V_1)$ is a pseudodifferential projection. Then a condition for the map D_Π to be Fredholm can be given purely in terms of the relationship between Π and the (any) Calderòn projector $\Pi_{\mathcal{C}}$.

THEOREM 8.3. *If $D \in \operatorname{Diff}^1(X; E_1, E_2)$ is of Dirac type and $Pi \in \Psi^0(\partial X; E_1)$ is a projection then the map*

$$D_\Pi : \{u \in \mathcal{C}^\infty(X; E_1); \Pi(u_{\partial X}) = 0\} \xrightarrow{D} \mathcal{C}^\infty(X; E_2) \tag{8.73}$$

is Fredholm if and only if

$$\Pi \circ \Pi_C : \operatorname{Ran}(\Pi_C) \cap \mathcal{C}^\infty(\partial V_1) \longrightarrow \operatorname{Ran}(\Pi) \cap \mathcal{C}^\infty(\partial E_1) \text{ is Fredholm} \tag{8.74}$$

and then the index of D_Π is equal to the relative index of $\Pi_{\mathcal{C}}$ and Π, that is the index of (8.74).

Below we give a symbolic condition equivalent which implies the Fredholm condition. If enough regularity conditions are imposed on the generalized inverse to (8.71) then this symbolic is also necessary.

PROOF. The null space of D_Π is easily analysed. Indeed $Du = 0$ implies that $u \in \mathcal{N}$, so the null space is isomorphic to its image under the boundary map:

$$\{u \in \mathcal{N}; \Pi bu = 0\} \xrightarrow{b} \{v \in \mathcal{C}; \Pi v = 0\}.$$

Since $\mathcal{C}$ is the range of $\Pi_{\mathcal{C}}$ this gives the isomorphism

$$\operatorname{Nul}(D_\Pi) \simeq \operatorname{Nul}(\Pi \circ \Pi_{\mathcal{C}} : \mathcal{C} \longrightarrow \operatorname{Ran}(\Pi)). \tag{8.75}$$

In particular, the null space is finite dimensional if and only if the null space of $\Pi \circ \Pi_{\mathcal{C}}$ is finite dimensional.

Similarly, consider the range of D_Π. We construct a map

$$\tau : \mathcal{C}^\infty(\partial X; V_1) \longrightarrow \mathcal{C}^\infty(X; V_2)/\operatorname{Ran}(D_\Pi). \tag{8.76}$$

Indeed each $v \in \mathcal{C}^\infty(\partial X; V_1)$ is the boundary value of some $u \in \mathcal{C}^\infty(X : V_1)$, let $\tau(v)$ be he class of DU. This is well-defined since any other extension u' is such that $b(u - u') = 0$, so $D(u - u') \in \operatorname{Ran}(D_\Pi)$. Furthermore, τ is surjective, since $D_{\mathcal{C}}$ is surjective. Consider the null space of τ. This certainly contains the null space of Π. Thus consider the quotient map

$$\tilde{\tau} : \operatorname{Ran}(\Pi) \longrightarrow \mathcal{C}^\infty(X : V_2)/\operatorname{Ran}(D_\Pi),$$

which is still surjective. Then $\tilde{\tau}(v) = 0$ if and only if there exists $v' \in \mathcal{C}$ such that $\Pi(v - v') = 0$. That is, $\tilde{\tau}(v) = 0$ if and only if $\Pi(v) = \Pi \circ \Pi_{\mathcal{C}}$. This shows that the finer quotient map

$$\tau' : \operatorname{Ran}(\Pi)/\operatorname{Ran}(\Pi \circ \Pi_{\mathcal{C}}) \longleftrightarrow \mathcal{C}^\infty(X; V_2)/\operatorname{Ran}(D_\Pi) \tag{8.77}$$

is an isomorphism. This shows that the range is closed and of finite codimension if $\Pi \circ \Pi_{\mathcal{C}}$ is Fredholm.

The converse follows by reversing these arguments. $\square$

8.19. Gluing

Returning to the case of a compact manifold without boundary, M, with a dividing hypersurface H we can now give a gluing result for the index.

THEOREM 8.4. *If* $D \in \mathrm{Diff}^1(M; E_1, E_2)$ *is of Dirac type and* $M = M_1 \cap M_2$ *is the union of two manifolds with boundary intersecting in their common boundary* $\partial M_1 \cap \partial M_2 = H$ *then*

$$\mathrm{Ind}(D) = \mathrm{Ind}(\Pi_{1,\mathcal{C}}, \mathrm{Id} - \Pi_{2,\mathcal{C}}) = \mathrm{Ind}(\Pi_{2,\mathcal{C}}, \mathrm{Id} - \Pi_{1,\mathcal{C}}) \tag{8.78}$$

where $\Pi_{i,\mathcal{C}}$, $i = 1, 2$, *are the Calderòn projections for* D *acting over* M_i.

8.20. Local boundary conditions

8.21. Absolute and relative Hodge cohomology

8.22. Transmission condition

CHAPTER 9

The wave kernel

Let us return to the subject of "good distributions" as exemplified by Dirac delta 'functions' and the Schwartz kernels of pseudodifferential operators. In fact we shall associate a space of "conormal distributions" with any submanifold of a manifold.

9.1. Conormal distributions

Thus let X be a $\mathcal{C}^\infty$ manifold and $Y \subset X$ a closed embedded submanifold – we can easily drop the assumption that Y is closed and even replace embedded by immersed, but let's treat the simplest case first! To say that Y is embedded means that each $\bar{y} \in Y$ has a coordinate neighbourhood U, in X, with coordinate $x_1, \dots, x_n$ in terms of which $\bar{y} = 0$ and

$$Y \cap U = \{x, = \cdots = x_k = 0\}. \tag{9.1}$$

We want to define

$$I^*(X, Y; \Omega^{\frac{1}{2}}) \subset \mathcal{C}^{-\infty}(X; \Omega^{\frac{1}{2}}) \tag{9.2}$$

to consist of distributions which are singular only at Y and small "along Y."

So if $u \in \mathcal{C}_c^{-\infty}(U)$ then in local coordinates (9.1) we can identify u with $u' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ so $u' \in H_c^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. To say that u is 'smooth along Y' means we want to have

$$D_{x_{k+1}}^{l_1} \dots D_{x_n}^{l_{n-k}} u' \in H_c^{s'}(\mathbb{R}^n) \quad \forall\ l_1, \dots, l_{n-k} \tag{9.3}$$

and a fixed s', independent of l (but just possibly different from the initial s); of course we can take $s = s'$. Now conditions like (9.3) do *not* limit the singular support of u' at all! However we can add a requirement that multiplication by a function which *vanishes* on Y makes u' smooth, by one degree, i.e.

$$x_1^{p_1} \dots x_k^{p_k} u' \in H^{s+|p|}(\mathbb{R}^n), |p| = p_1 + \cdots + p_k. \tag{9.4}$$

This last condition implies

$$D_1^{q_1} \dots D_k^{q_k} x_1^{p_1} \dots x_k^{p_k} u' \in H^s(\mathbb{R}^n) \text{ if } |q| \le |p|. \tag{9.5}$$

Consider what happens if we rearrange the order of differentiation and multiplication in (9.5). Since we demand (9.5) for *all* p, q with $|q| \le |p|$ we can show in tial that

$$\forall\ |q| \le |p| \le L \tag{9.6}$$

$$\Longrightarrow \tag{9.7}$$

$$\prod_{i=1}^{L} (x_{j_i} D_{\ell_i}) u \in H^s(\mathbb{R}^n) \quad \forall\ \text{ pairs, } (j_{i,\ell_i}) \in (1, \dots, k)^2. \tag{9.8}$$

Of course we can combine (9.3) and (9.8) and demand

$$\prod_{i=1}^{L_2} D_{p_i} \prod_{i=1}^{L_1}(x_{j_i}D_{\ell_i})u' \in H^s_c(\mathbb{R}^n)(j_j,\ell_i) \in (1,\dots,k)^2 \\ \forall\, L_1, L_2\; p_i \in (k+1,\dots u). \tag{9.9}$$

PROBLEM 9.1. Show that (9.9) implies (9.3) and (9.4)

The point about (9.9) is that it is easy to interpret in a coordinate independent way. Notice that putting $\mathcal{C}^\infty$ coefficients in front of all the terms makes no difference.

LEMMA 9.1. *The space of all $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^n$ tangent to the submanifold $\{x_1 = \cdots = x_k = 0\}$ is spanning over $\mathcal{C}^\infty(\mathbb{R}^n)$ by*

$$x_iD_j, D_p\;\; i,j \le k, p > k. \tag{9.10}$$

PROOF. A $\mathcal{C}^\infty$ vector field is just a sum

$$V = \sum_{j\le k} a_jD_j + \sum_{p>k} b_pD_p. \tag{9.11}$$

Notice that the D_p, for $p > k$, *are* tangent to $\{x_1 = \cdots = x_k = 0\}$, so we can assume $b_p = 0$. Tangency is then given by the condition

$$Vx)i = 0 \text{ and } \{x_1 = \cdots = x_k = 0\}, i = 1,\dots,h \tag{9.12}$$

i.e. $a_j = \sum\limits_{\ell=1} a_{j\ell}x_\ell, 1 \le j \le h$. Thus

$$V = \sum_{\ell=1} a_{j\ell}x_\ell D_j \tag{9.13}$$

which proves (9.10). □

This allows us to write (9.9) in the compact form

$$\mathcal{V}(\mathbb{R}^n, Y_k)^p u' \subset H^s_c(\mathbb{R}^n)\;\forall\, p \tag{9.14}$$

where $\mathcal{V}(\mathbb{R}^n, Y_k)$ is just the space of all $\mathcal{C}^\infty$ vector fields tangent to $Y_k = \{x_1 = \cdots = x_k = 0\}$. Of course the local coordinate just reduce vector fields tangent to Y to vector fields tangent to Y_k so the *invariant* version of (9.14) is

$$\mathcal{V}(X,Y)^p u \subset H^s(X;\Omega^{\frac12})\;\forall\, p. \tag{9.15}$$

To interpret (9.15) we only need recall the (Lie) action of vector fields on half-densities. First for densities: The *formal* transpose of V is $-V$, so set

$$^L V\phi(\psi) = \phi(-V\psi) \tag{9.16}$$

if $\phi \in \mathcal{C}^\infty(X;\Omega), \psi \in \mathcal{C}^\infty(X)$. On $\mathbb{R}^n$ then becomes

$$\begin{aligned}\int {}^L V\phi\cdot\psi &= -\int \phi\cdot V\psi \\ &= -\int \phi(x)V\psi\cdot dx \\ &= \int (V\phi(x) + \delta_V\phi)\,\psi\; dx \\ \delta_V = \sum_{i=1}^n D_ia_i &\quad \text{if } V = \Sigma a_iD_i.\end{aligned} \tag{9.17}$$

i.e.

$$L_V\left(\phi|dx|\right) = (V\phi)|dx| + \delta_V\phi. \tag{9.18}$$

Given the tensorial properties of density, set

$$L_V(\phi|dx|^t) = V\phi|dx|^t + t\delta_V\phi. \tag{9.19}$$

This corresponds to the *natural* trivialization in local coordinates.

DEFINITION 9.1. *If $Y \subset X$ is a closed embedded submanifold then*

$$\begin{aligned} IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right) &= \left\{u \in H^s\left(X; \Omega^{\frac{1}{2}}\right) \text{ satisfying (11)}\right\} \\ I^*\left(X, Y; \Omega^{\frac{1}{2}}\right) &= \bigcup_s IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right). \end{aligned} \tag{9.20}$$

Clearly

$$u \in I^*(X, Y; \Omega^{\frac{1}{2}}) \Longrightarrow u \restriction X\backslash Y \in \mathcal{C}^\infty\left(X\backslash Y; \Omega^{\frac{1}{2}}\right) \tag{9.21}$$

and

$$\bigcap_s IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right) = \mathcal{C}^\infty\left(X; \Omega^{\frac{1}{2}}\right). \tag{9.22}$$

Let us try to understand these distributions *in some detail!* To do so we start with a very simple case, namely $Y = \{p\}$ is a point; so we only have one coordinate system. So construct $p = 0 \in \mathbb{R}^n$.

$$\begin{gathered} u \in I_c^*\left(\mathbb{R}^n, \{0\}; \Omega^{\frac{1}{2}}\right) \Longrightarrow u = u'|dx|^{\frac{1}{2}} \text{ when} \\ x^\alpha D_x^\beta u' \in H_c^s(\mathbb{R}^n), \quad s \text{ fixed } \; \forall\, |\alpha| \ge |\beta|. \end{gathered} \tag{9.23}$$

Again by a simple commutative argument this is equivalent to

$$D_x^\beta x^\alpha u' \in H_c^s(\mathbb{R}^n) \;\; \forall\, |\alpha| \ge |\beta|. \tag{9.24}$$

We can take the Fourier transform of (9.24) and get

$$\xi^\beta D_\xi^\alpha \hat{u}' \in \langle\xi\rangle^{-s} L^2(\mathbb{R}^n) \;\forall\, |\alpha| \ge |\beta|. \tag{9.25}$$

In this form we can just replace ξ^β by $\langle\xi\rangle^{|\beta|}$, i.e. (9.25) just says

$$D_\xi^\alpha \hat{u}'(\xi) \in \langle\xi\rangle^{-s-|\beta|} L^2(\mathbb{R}^n) \;\forall\; \alpha. \tag{9.26}$$

Notice that this is *very* similar to a symbol estimate, which would say

$$D_\xi^\alpha \hat{u}'(\xi) \in \langle\xi\rangle^{m-|\alpha|} L^\infty(\mathbb{R}^n) \;\forall\; \alpha. \tag{9.27}$$

LEMMA 9.2. *The estimate* (9.26) *implies* (9.27) *for any* $m > -s - \frac{n}{2}$; *conversely* (9.27) *implies* (9.26) *for any* $s < -m - \frac{n}{2}$.

PROOF. Let's start with the simple derivative, (9.27) implies (9.26). This really reduces to the case $\alpha = 0$. Thus

$$\langle\xi\rangle^M L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \Longrightarrow M < -\frac{n}{2} \tag{9.28}$$

is the inequality

$$\left(\int |u|^2 d\xi\right)^{\frac{1}{2}} \le \sup\langle\xi\rangle^{-M}|u| \left(\int \langle\xi\rangle^{2M} d\xi\right)^{\frac{1}{2}} \tag{9.29}$$

and

$$\int \langle\xi\rangle^{2M} d\xi = \int \left(1+|\xi|^2\right)^M d\xi < \infty \text{ iff } M < -\frac{n}{2}. \tag{9.30}$$

To get (9.27) we just show that (9.27) implies

$$\langle\xi\rangle^{s+|\alpha|} D_\xi^\alpha \hat{u}' \in \langle\xi\rangle^{m+s} L^\infty \subset L^2 \text{ if } m+s < -\frac{n}{2}. \tag{9.31}$$

The converse is a little trickier. To really see what is going on we can reduce (9.26) to a one dimensional version. Of course, near $\xi = 0$, (9.26) just says $\hat{u}'$ is $\mathcal{C}^\infty$, so we can assume that $|\xi| > 1$ on $\operatorname{supp} \hat{u}'$ and introduce polar coordinates:

$$\xi = tw,\ w \in S^{n-1} t > 1. \tag{9.32}$$

Then

Exercise 2. Show that (9.26) (or maybe better, (9.25)) implies that

$$D_t^k P \hat{u}'(tw) \in t^{-s-k} L^2(\mathbb{R}^+ \times S^{n-1}; t^{n-1} dt dw)\ \forall\ k \tag{9.33}$$

for *any* $\mathcal{C}^\infty$ differential operator on S^{n-1}. □

In particular we can take P to be elliptic of any order, so (9.33) actually implies

$$\sup_w D_t^k P \hat{u}(t,w) \in t^{-s-k} L^2(\mathbb{R}^+; t^{n-1} dt) \tag{9.34}$$

or, changing the meaning to dt,

$$\sup_{w \in S^{n-1}} \left| D_t^k P \hat{u}(t,w) \right| \in t^{-s-k-\frac{n-1}{2}} L^2\left(\mathbb{R}^+, dt\right). \tag{9.35}$$

So we are in the one dimensional case, with s replaced by $s + \frac{n-1}{2}$. Now we can rewrite (9.35) as

$$D_t t^q D_t^k P \hat{u} \in t^r L^2,\ \forall\ k, r-q = -s-k-\frac{n-1}{2} - 1. \tag{9.36}$$

Now, observe the simple case:

$$f = 0 t < 1, D_t f \in t^r L^2 \Longrightarrow f \in L^\infty \text{ if } r < -\frac{1}{2} \tag{9.37}$$

since

$$\sup |f| = \int_{-\infty}^{t} t^r g \leq \left(\int |g|^2\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{t} t^{2r}\right)^{\frac{1}{2}}. \tag{9.38}$$

Thus from (9.36) we deduce $\leq (\int |g|^2)^{\frac{1}{2}}$

$$D_t^k P \hat{u} \in t^{-q} L^\infty \text{ if } r < -\frac{1}{2}, \text{ i.e. } -q > -s-k-\frac{n}{2}. \tag{9.39}$$

Finally this gives (9.27) when we go back from polar coordinates, to prove the lemma.

Definition 9.2. *Set, for $m \in \mathbb{R}$,*

$$I_c^m(\mathbb{R}^n, |[0\}) = \{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); \hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)\} \tag{9.40}$$

with this definition,

$$IH^s(\mathbb{R}^n, \{0\}) \subset I_c^m(\mathbb{R}^n, \{0\}) \subset I_c^{s'}(\mathbb{R}^n, \{0\}) \tag{9.41}$$

provided

$$s > -m - \frac{n}{4} > s'. \tag{9.42}$$

Exercise 3. Using Lemma 24, prove (9.41) carefully.

So now what we want to do is to *define* $I_c^m(X, \{p\}; \Omega^{\frac{1}{2}})$ for any $p \in X$ by

$$\begin{gathered} u \in I_c^m(X, \{p\}; \Omega^{\frac{1}{2}}) \Longleftrightarrow F^*(\phi u) \in I_c^m(\mathbb{R}^n, \{0\}), \\ u \restriction X \backslash \{p\} \in \mathcal{C}^\infty(X \backslash \{p\}). \end{gathered} \tag{9.43}$$

Here we have a little problem, namely we have to check that $I^m(\mathbb{R}^n, \{0\})$ is invariant under coordinate changes. Fortunately we can do this using (9.41).

Lemma 9.3. *If $F : \Omega \longrightarrow \mathbb{R}^n$ is a diffeomorphism of a neighbourhood of* 0 *onto its range, with $F(0) = 0$, then*

$$F^*\{u \in I_c^m(\mathbb{R}^n, \{0\}; \operatorname{supp}(u) \subset F(\Omega)\} \subset I_c^m(\mathbb{R}^n, \{0\}). \tag{9.44}$$

Proof. Start with a simple case, that F is *linear*. Then

$$u = (2\pi)^{-n} \int e^{ix\xi} a(\xi) d\xi, a \in S^{m-\frac{n}{4}}(\mathbb{R}^n). \tag{9.45}$$

so

$$\begin{aligned} F^*u &= (2\pi)^{-n} \int e^{iAx\cdot\xi} a(\xi) d\xi \ Fx = Ax \\ &= (2\pi)^{-n} \int i^{ix\cdot A^t\xi} a(\xi) d\xi \\ &= (2\pi)^{-n} \int e^{ix\cdot\eta} a((A^t)^{-1}\eta) |\det A|^{-1} d\eta. \end{aligned} \tag{9.46}$$

Since $a((A^t)^{-1}\eta)|\det A|^{-1} \in S^{m-\frac{n}{4}}\mathbb{R}^n)$ we have proved the result for linear transformations. We can always factorize F is

$$F = G \cdot A, \qquad A = (F_*) \tag{9.47}$$

so that the differential of G at 0 is the identity, i.e.

$$G(x) = x + O(|x|^2). \tag{9.48}$$

Now (9.48) allows us to use an homotopy method, i.e. set

$$G_s(x) = x + s(G(x) - x) \qquad s \in [0, 1) \tag{9.49}$$

so that $G_0 = \mathrm{Id}$, $G_s = G$. Such a 1-parameter family is given by integration of a vector field:

$$
\begin{aligned}
G_s^*\phi &= \int_0^s \frac{d}{ds} G_s^* \phi dx \\
&= \int_0 s \frac{d}{ds}\phi(G_x(x))ds \\
&= \sum_1 \int_0^s \frac{d}{G}{}^{\xi}_{s,i} ds\,(\partial x_j \phi)\,(G_\delta(x))\,ds \\
&= \int_0^s G_s^*\,(V_s\phi)\,ds
\end{aligned}
\tag{9.50}
$$

when the coefficients of V_s are

$$
G_s^* V_{s,j} = \frac{d}{ds} G_{s,i}. \tag{9.51}
$$

Now by (9.49) $\frac{d}{ds}G_{s,i} = \Sigma x_i x_j a_{ij}^s(x)$, so the same is true of the $V_{s,i}$, again using (9.49).

We can apply (9.50) to compute

$$
G^* u = \int_0^{\prime} G_s^*\,(V_s u)\,ds \tag{9.52}
$$

when $u \in I_c^m\,(\mathbb{R}^n, \{0\})$ has support near 0. Namely, by (9.41), $u \in IH_c^s\,(\mathbb{R}^n, \{0\})\,,$ with $s < -m - \frac{n}{4}$, but then

$$
V_s u \in IH_c^{s+1}(\mathbb{R}^n, \{0\}) \tag{9.53}
$$

since $V = \sum\limits_{i,j=1}^{n} b_{ij}^s(x) x_i x_j D_j$. Applying (9.41) again gives

$$
G_s^*(V_s u) \in I^{m'}\,(\mathbb{R}^n, \{0\})\,,\ \forall\ m' > m - 1. \tag{9.54}
$$

This proves the coordinates invariance. □

Last time we defined the space of conormal distributions associated to a closed embedded submanifold $Y \subset X$:

$$
\begin{aligned}
IH^s(X,Y) &= \{u \in H^s(X); \mathcal{V}(X,Y)^k u \subset H^s(X)\ \forall\ k\} \\
IH^*(X,Y) &= I^*(X,Y) = \bigcup s IH^s(X,Y).
\end{aligned}
\tag{9.55}
$$

Here $\mathcal{V}(X,Y)$ is the space of $\mathcal{C}^\infty$ vector fields on X tangent to Y. In the special case of a point in $\mathbb{R}^n$, say 0, we showed that

$$
u \in I_c^*(\mathbb{R}^n), \{0\}) \Longleftrightarrow u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \text{ and } \hat{u} \in S^M(\mathbb{R}^n), M = M(u). \tag{9.56}
$$

In fact we then defined the "standard order filtration" by

$$
u \in I_c^m(\mathbb{R}^n, \{0\}) = \left\{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); \hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)\right\}, \tag{9.57}
$$

and found that

$$
IH_c^s(\mathbb{R}^n, \{0\}) \subset I_c^{-s-\frac{n}{4}}(\mathbb{R}^n, \{0\}) \subset IH_c^{s'}(\mathbb{R}^n, \{0\})\ \forall\ s' < s. \tag{9.58}
$$

Our next important task is to show that $I_c^m(\mathbb{R}^n,\{0\})$ is invariant under coordinate changes. That is, if $F: U_1 \longrightarrow \mathbb{R}^n$ is a diffeomorphism of a neighbourhood of 0 to its range, with $F(0)=0$, then we want to show that

$$F^*u \in I_c^m(\mathbb{R}^n,\{0\})\ \forall\ u \in I_c^m(\mathbb{R}^n,\{0\}), \operatorname{supp}(u) \subset F(U_1). \tag{9.59}$$

Notice that we already know the coordinate independence of the Sobolev-based space, so using (9.58), we deduce that

$$F^*u \in I_c^{m'}(\mathbb{R}^n,\{0\})\ \forall\ u \in I_c^m(\mathbb{R}^n,\{0\}), n' > m, \operatorname{supp}(u) \subset F(U_1). \tag{9.60}$$

In fact we get quite a lot more for our efforts:

Lemma 9.4. *There is a coordinate-independent symbol map:*

$$I^m(X,\{p\};\Omega^{\frac12})@>\sigma_Y^m>>S^{m+\frac{n}{4}-[J]}\left(T_p^*\mathbb{R}^n;\Omega^{\frac12}\right) \tag{9.61}$$

given by the local prescription

$$\sigma_Y^m(u) = \hat{u}(\xi)|d\xi|^{\frac12} \tag{9.62}$$

where $u = v|dx|^{\frac12}$ is local coordinate based at 0, with ξ the dual coordinate in T_p^*X.

Proof. Our definition of $I^m(X,\{p\};\Omega^{\frac12})$ is just that in any local coordinate based at p

$$u \in I^m(X,\{p\};\Omega^{\frac12}) \Longrightarrow \phi u = v|dx|^{\frac12}, v \in I_c^m(\mathbb{R}^n,\{0\}) \tag{9.63}$$

and $u \in \mathcal{C}^\infty(X\backslash\{p\};\Omega^{\frac12})$. So the symbol map is clearly supposed to be

$$\sigma^m(u)^{(\zeta)} \equiv_\downarrow \hat{v}(\xi)|d\xi|^{\frac12} \in S^{m+\frac{n}{4}-[1]}(\mathbb{R}^n;\Omega^{\frac12}) \tag{9.64}$$

where $\zeta \in T_p^*X$ is the 1-form $\zeta = \xi\cdot dx$ in the local coordinates. Of course we have to show that (9.64) is independent of the choice of coordinates. We already know that a change of coordinates changes $\hat{v}$ by a term of order $m-\frac{n}{4}-1$, which disappears in (9.64) so the residue class is determined by the Jacobian of the change of variables. From (9.46) we see exactly how $\hat{v}$ transforms under the Jacobian, namely as a density on

$$T_0^*\mathbb{R}^n : A \in GL(n,\mathbb{R}) \Longrightarrow \widehat{A^*v}(\eta)|d\eta|^{\frac12}$$
$$= \hat{v}((A^t)^{-1}\eta)|\det A|^{-1}|dy|$$

so $\eta = A^t\xi \Longrightarrow$

$$\widehat{A^*v}(\eta)|dy| = \hat{v}(\xi)|d\xi|. \tag{9.65}$$

However recall from (9.63) that u is a half-density, so actually in the new coordinates $v' = A^*v\cdot|\det A|^{\frac12}$. This shows that (9.64) is well-defined.

Before going on to consider the general case let us note a few properties of $I^m(X,\{p\},\Omega^{\frac12})$: $\square$

Exercise: Prove that

$$\begin{gathered}\text{If } P \in \operatorname{Diff}^m(X;\Omega^{\frac12}) \text{ then}\\ P: I^m(X,\{p\};\Omega^{\frac12}) \longrightarrow I^{m+M}(X,\{p\};\Omega^{\frac12})\ \forall\ m\\ \sigma^{m+M}(Pu) = \sigma^M(P)\cdot\sigma^m(u).\end{gathered} \tag{9.66}$$

To pass to the general case of $Y \subset X$ we shall proceed in two steps. First let's consider a rather 'linear' case of $X = V$ a vector bundle over Y. Then Y can be identified with the zero section of V. In fact V is locally trivial, i.e. each $p \in y$ has a neighbourhood U s.t.

$$\pi^{-1}(U) \simeq \mathbb{R}^n_x \times U'_{y'} U' \subset \mathbb{R}^p \tag{9.67}$$

by a fibre-linear diffeomorphism projecting to a coordinate system on this base. So we want to define

$$I^m(V, Y; \Omega^{\frac{1}{2}}) = \{u \in I^*(V, Y; \Omega^{\frac{1}{2}}); \tag{9.68}$$

of $\phi \in \mathcal{C}^\infty_c(U)$ then under *any* trivialization (9.67)

$$\begin{gathered}\phi u(x, y) \equiv (2\pi)^{-n} \int e^{ix\cdot\xi} a(y, \xi) d\xi |dx|^{\frac{1}{2}}, \quad \bmod \mathcal{C}^\infty, \\ a \in S^{m - \frac{n}{2} - \frac{p}{4}}(\mathbb{R}^p_y, \mathbb{R}^n_\xi).\end{gathered} \tag{9.69}$$

Here $p = \dim Y, p+n = \dim V$. Of course we have to check that (9.69) is coordinate-independent. We can write the order of the symbol, corresponding to u having order m as

$$m - \frac{\dim V}{4} + \frac{\dim Y}{2} = m + \frac{\dim V}{4} - \frac{\operatorname{codim} Y}{2}. \tag{9.70}$$

These additional shifts in the order are only put there to confuse you! Well, actually they make life easier later.

Notice that we know that the space is invariant under *any* diffeomorphism of the fibres of V, varying smoothly with the base point, it is also obvious that (9.69) in independent the choice of coordinates is U', since that just transforms these variables. So a general change of variables preserving Y is

$$(y, x) \longmapsto (f(y, x), X(y, x)) \quad X(y, 0) = 0. \tag{9.71}$$

In particular f is a local diffeomorphism, which just changes the base variables in (9.69), so we can assume $f(y) \equiv y$. Then $X(y, x) = A(y) \cdot x + O(x^2)$. Since $x \longmapsto A(y) \cdot x$ is a fibre-by-fibre transformation it leaves the space invariant too, So we are reduced to considering

$$G : (y, x) \longmapsto (y, x + \Sigma a_{ij}(x, y) x_i x_j) y + \Sigma b_i(x, y) x_i. \tag{9.72}$$

To handle these transformations we can use the same homotopy method as before i.e.

$$G_s(x, y_= (y + s) \sum_i b_i(x, y) x_i, x + s \sum_{i,j} a_{ij}(x, y) x_i x_j) \tag{9.73}$$

is a 1-parameter family of diffeomorphisms. Moreover

$$\frac{d}{ds} G^*_s u = G^*_s V_s k \tag{9.74}$$

where

$$V_s = \sum_{i,\ell} \beta_{i,\ell}(s, x, y) x_i \partial_{y_\ell} + \sum_{i,j,k} \alpha_{i,j,k} + \sum_{i,j,k} \alpha_{ijk}(\alpha, y, s) \ell_i, \ell_j \frac{\partial}{\partial x_k}. \tag{9.75}$$

So all we really have to show is that

$$V_s : I^M(U' \times \mathbb{R}^n, U' \times \{0\}) \longrightarrow I^{M-1}(U' \times \mathbb{R}^n, U' \times \{0\}) \ \forall\ M. \tag{9.76}$$

Again the spaces are $\mathcal{C}^\infty$-modules so we only have to check the action of $x_i \partial_{y_\ell}$ and $x_i x + j \partial_{x_k}$. These change the symbol to

$$D_{\xi_i} \partial_{y_\ell} a \text{ and } i D_{\xi_i} D_{\xi_j} \cdot \xi_k a \tag{9.77}$$

respectively, all one order lower.

This shows that the definition (9.69) is actually a reasonable one, i.e. as usual it suffices to check it for any covering by coordinate partition.

Let us go back and see what the symbol showed before.

LEMMA 9.5. *If*

$$u \in I^m(V, Y; \Omega^{\frac{1}{2}})u = v|dx|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}} \tag{9.78}$$

defines an element

$$\sigma^m(u) \in S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(V^*; \Omega^{\frac{1}{2}}) \tag{9.79}$$

independent of choices.

Last time we discussed the invariant symbol for a conormal distribution associated to the zero section of a vector bundle. It turns out that the general case is not any more complicated thanks to the "tubular neighbourhood" or "normal fibration" theorem. This compares $Y \hookrightarrow X$, a closed embedded submanifold, to the zero section of a vector bundle.

Thus at each point $y \in Y$ consider the normal space:

$$N_yY = N_y\{X, Y\} = T_yx/T_yY. \tag{9.80}$$

That is, a normal vector is just *any* tangent vector to X modulo tangent vectors to Y. These spaces define a vector bundle over Y :

$$NY = N\{X; Y\} = \bigsqcup_{y \in Y} N_yY \tag{9.81}$$

where smoothness of a section is inherited from smoothness of a section of T_yX, i.e.

$$NY = T_yX/T_yY. \tag{9.82}$$

Suppose $Y_i \subset X_i$ are $\mathcal{C}^\infty$ submanifolds for $i = 1, 2$ and that $F : X_1 \longrightarrow X_2$ is a $\mathcal{C}^\infty$ map such that

$$F(Y_1) \subset Y_2. \tag{9.83}$$

Then $F_* : T_yX_1 \longrightarrow T_{F(y)}X_2$, must have the property

$$F_* : T_yY_1 \longrightarrow T_{F(y)}Y_2 \;\forall\; y \in Y_1. \tag{9.84}$$

This means that F_* defines a map of the normal bundles

$$\begin{tikzcd} F_* : NY_1 \arrow[r] \arrow[d] & NY_2 \arrow[d] \\ Y_1 \arrow[r, "F"'] & Y_2. \end{tikzcd} \tag{9.85}$$

Notice the very special case that $W \longrightarrow Y$ is a vector bundle, and we consider $Y \hookrightarrow W$ as the zero section. Then

$$N_y\{W; Y\} \longleftrightarrow W_y \qquad \forall\; y \in Y \tag{9.86}$$

since

$$T_yW = T_yY \oplus T_y(W_y) \qquad \forall\; y \in W. \tag{9.87}$$

That is, the normal bundle to the zero section is *naturally* identified with the vector bundle itself.

So, suppose we consider $\mathcal{C}^\infty$ maps

$$f : B \longrightarrow N\{X;Y\} = NY \tag{9.88}$$

where $B \subset X$ is an open neighbourhood of the submanifold Y. We can demand that

$$f(y) = (y,0) \in N_yY \quad \forall\ y \in Y \tag{9.89}$$

which is to say that f induces the natural identification of Y with the zero section of NY and moreover we can demand

$$f_* : NY \longrightarrow NY \text{ is the identity.} \tag{9.90}$$

Here f_* is the map (9.85), so maps NY to the normal bundle to the zero section of NY, which we have just observed is naturally just NY again.

THEOREM 9.1. *For any closed embedded submanifold $Y \subset X$ there exists a normal fibration, i.e. a diffeomorphism (onto its range)* (9.88) *satisfing* (9.89) *and* (9.90)*; two such maps f_1, f_2 are such that $g = f_2 \circ f_1^{-1}$ is a diffeomorphism near the zero section of NY, inducing the identity on Y and inducing the identity* (9.90).

PROOF. Not bad, but since it uses a little Riemannian geometry I will *not* prove it, see [ ], [ ]. (For those who know a little Riemannian geometry, f^{-1} can be taken as the exponential map near the zero section of NY, identified as a subbundle of T_YX using the metric.) Of course the uniqueness part is obvious. □

Actually we do *not* really need the global aspects of this theorem. Locally it is immediate by using local coordinates in which $Y = \{x_1 = \cdots = x_k = 0\}$.

Anyway using such a normal fibration of X near Y (or working locally) we can simply *define*

$$\begin{aligned} I^m(X,Y;\Omega^{\frac12}) = \{u \in \mathcal{C}^{-\infty}(X;\Omega^{\frac12}); u \text{ is } \mathcal{C}^\infty \text{ in } X\backslash Y \text{ and}\\ (f^{-1})^*(\phi u) \in I^m(NY,Y;\Omega^{\frac12}) \text{ if } \phi \in \mathcal{C}^\infty(X), \operatorname{supp}(\phi) \subset B\}. \end{aligned} \tag{9.91}$$

Naturally we should check that the definition doesn't depend on the choice of f. This means knowing that $I^m(NY,Y;\Omega^{\frac12})$ is invariant under g, as in the theorem, but we have already checked this. In fact notice that g is exactly of the type of (9.72). Thus we actually know that

$$\sigma^m(g^*u) = \sigma^m(u) \text{ in } S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y;\Omega^{\frac12}). \tag{9.92}$$

So we have shown that there is a coordinate invariance symbol map

$$\sigma^m : I^m(X,Y;\Omega^{\frac12}) \longrightarrow S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y;\Omega^{\frac12}) \tag{9.93}$$

giving a short exact sequence

(9.94)

$$0 \hookrightarrow I^{m-1}(X,Y;\Omega^{\frac12}) \longrightarrow I^m(X,Y;\Omega^{\frac12}) @>\sigma^m>> S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y;\Omega^{\frac12}) \longrightarrow 0$$

$$\text{where } n = \dim X - \dim Y, p = \dim Y. \tag{9.95}$$

Asymptotic completeness carries over immediately. We also need to go back and check the extension of (9.66):

Proposition 9.1. *If $Y \hookrightarrow X$ is a closed embedded submanifold and $A \in \Psi_c^m(X;\Omega^{\frac12})$ then*

$$A : I^M(X,Y;\Omega^{\frac12}) \longrightarrow I^{M+m}(X,Y;\Omega^{\frac12})\ \forall\ M \tag{9.96}$$

and

$$\sigma^{m+M}(Au) = \sigma^m(A)\sigma^m(A) \restriction N^*Y\sigma^M(u). \tag{9.97}$$

Notice that $\sigma^m(A) \in S^{m-[1]}(T^*X)$ so the product here makes perfectly good sense.

Proof. Since everything in sight is coordinate-independent we can simply work in local coordinates where

$$X \sim \mathbb{R}^p_y \times \mathbb{R}^n_x, Y = \{x = 0\}. \tag{9.98}$$

Then $u \in I^m_c(X,Y;\Omega^{\frac12})$ means just

$$u = (2\pi)^{-n}\int e^{ix\cdot\xi}a(y,\xi)d\xi\cdot|dx|^{\frac12}, a \in S^{m-\frac{n}{4}+\frac{p}{4}}(\mathbb{R}^p,\mathbb{R}^n). \tag{9.99}$$

Similarly A can be written in the form

$$A = (2\pi)^{-n-p}\int e^{i(x-x')\cdot\xi+i(y-y')\cdot\eta}b(x,y,\xi,\eta)d\xi d\eta. \tag{9.100}$$

Using the invariance properties of the Sobolev based space if we write

$$A = A_0 + \Sigma x_jB_j, A_0 = q_L(b(0,y,\xi,\eta)) \tag{9.101}$$

we see that $Au \in I^{m+M}(X,Y;\Omega^{\frac12})$ is equivalent to $A_0u \in I^{m+M}(X,Y;\Omega^{\frac12})$. Then

$$A_0u = (2\pi)^{-n-p}\int e^{ix\cdot\xi+i(y-y')\cdot\eta}b(0,y',\xi,\eta)b(y',\xi)dy'd\eta d\xi, \tag{9.102}$$

where we have put A_0 in right-reduced form. This means

$$A_0u = (2\pi)^{-n}\int e^{ix\cdot\xi}c(y,\xi)d\xi \tag{9.103}$$

where

$$c(y,\xi) = (2\pi)^{-p}\int e^{i(y-y')\cdot\eta}b(0,y',\xi,\eta)a(y',\xi)dy'd\eta. \tag{9.104}$$

Regarding ξ as a parameter, this is, before y' integration, the kernel of a pseudo-differential operator is y. It can therefore be written in left-reduced form, i.e.

$$c(y,\xi) = (2\pi)^{-p}\int e^{i(y-y')\eta}e(y,\xi,\eta)d\eta dy' = e(y,\xi,0) \tag{9.105}$$

where $e(y,\xi,\eta) = b(0,y,\xi,\eta)a(y,\xi)$ plus terms of order at most $m+M-\frac{n}{4}+\frac{p}{4}-1$. This proves the formula (9.97). □

Notice that if A is elliptic then $Au \in \mathcal{C}^\infty$ implies $u \in \mathcal{C}^\infty$, i.e. there are *no* singular solutions. Suppose that P is say a *differential* operator which is not elliptic and we look for solutions of

$$Pu \in \mathcal{C}^\infty(X\Omega^{\frac12}). \tag{9.106}$$

How can we find them? Well suppose we try

$$u \in I^M(X,Y;\Omega^{\frac12}) \tag{9.107}$$

for some submanifold Y. To know that u is singular we will want to have

$$\sigma(u) \text{ is elliptic on } N^*Y \tag{9.108}$$

(which certainly implies that $u \notin \mathcal{C}^\infty$).

The simplest case would be Y a hypersurface. In any case from (9.97) and (9.106) we deduce

$$\sigma^m(P) \cdot \sigma^M(u) \equiv 0. \tag{9.109}$$

So if we assume (9.108) then we *must* have

$$\sigma^m(P) \restriction N^*Y = 0. \tag{9.110}$$

Definition 9.3. *A submanifold is said to be* characteristic *for a given operator* $P \in \operatorname{Diff}^m(X; \Omega^{\frac12})$ *if* (9.110) *holds.*

Of course even if P *is* characteristic for y, and so (9.109) holds we do *not* recover (9.106), just

$$Pu \in I^{m+M-1}(X, Y; \Omega^{\frac12}) \tag{9.111}$$

i.e. one order smoother than it "should be". The task might seem hopeless, but let us note that these are examples, and important ones at that!!

Consider the (flat) wave operator

$$P = P_t^2 - \sum_{i=1}^n D_i^2 = D_t^2 - \Delta \text{ on } \mathbb{R}^{n+1}. \tag{9.112}$$

A hypersurface in $\mathbb{R}^{n+1}$ looks like

$$H = \big\{h(t,x) = 0\big\}, (dh \neq 0 \text{ on } H)\,. \tag{9.113}$$

The symbol of P is

$$\sigma^2(P) = \tau^2 - |\xi|^2 = \tau^2 - \xi_1^2 - \cdots - \xi_n^2, \tag{9.114}$$

where τ, ξ are the dual variables to t, x. So consider (9.110),

$$N^*Y = \big\{(t, x; \lambda dh(t,y)); h(t,x) = 0\big\}. \tag{9.115}$$

Inserting this into (9.114) we find:

$$\left(\lambda\frac{\partial h}{\partial t}\right)^2 - \left(\lambda\frac{\partial h}{\partial x_1}\right)^2 - \cdots - \left(\lambda\frac{\partial h}{\partial x_n}\right)^2 = 0 \text{ on } h = 0 \tag{9.116}$$

i.e. simply:

$$\left(\frac{\partial h}{\partial t}\right)^2 = |d_x h|^2 \text{ on } h(t,x) = 0. \tag{9.117}$$

This is the "eikonal equation" for h (and hence H).

Solutions to (9.117) are easy to find – we shall actually find all of them (locally) next time. Examples are given by taking h to be linear:

$$H = \big\{h = at + b \cdot x = 0\big\} \text{ is characteristic for } P \Longleftrightarrow a^2 = |b|^2. \tag{9.118}$$

Since h/a defines the same surface, all the linear solutions correspond to planes

$$t = \omega \cdot x, \omega \in \mathbb{S}^{n-1}. \tag{9.119}$$

So, do solutions of $Pu \in \mathcal{C}^\infty$ which are conormal with respect to such hypersurfaces exist? Simply take

$$u = v(t - \omega \cdot x) \quad v \in I^*(\mathbb{R}, \{0\}; \Omega^{\frac{1}{2}}). \tag{9.120}$$

Then

$$Pu = 0, u \in I^*(\mathbb{R}^{n+1}, H; \Omega^{\frac{1}{2}}). \tag{9.121}$$

For example $v(s) = \delta(s), u = \delta(t - \omega \cdot x)$ is a "travelling wave".

9.2. Lagrangian parameterization

We will consider below the push-forward of conormal distributions under a fibration and how this gives rise to the more general notion of a Lagrangian distribution. So we first consider the local model for a fibration, which is projection, π, off a Euclidean factor

$$\pi : \mathbb{R}^n_y \times \mathbb{R}^k_z \to \mathbb{R}^n_y.$$

The most important case of conormal distributions associated to a submanifold here is that of a hyperspace $H \subset \mathbb{R}^n_y \times bbR^k_z$ with global defining function $h \in \mathcal{C}^\infty(\mathbb{R}^{n+k})$, $H = \{h = 0\}$, $dh \neq 0$ on H.

Recall from the general properties of conormal distributions that if u is conormal to H then $\mathrm{WF}(u) \subset N^*H = \{\lambda \cdot dh(y,z); h(y,z) = 0\}$. From the properties of wavefront set under push-forward, if u has compact support then

$$\begin{aligned}\mathrm{WF}(\pi_* u) \subset \{(y,\eta);\ \exists\ z \text{ s.t. } (y,z) \in H,\\ \eta = \lambda dh(y,z),\ \frac{\partial h}{\partial z}(y,z) = 0\}.\end{aligned}$$

That is, the singularities of u are (co-)normal to H and any singularities not (co-)normal to the fibres are wiped out by integration.

So, we are interested in the set

$$C_H = \{(y,z) \in H; \frac{\partial h}{\partial z}(y,z) = 0\} \tag{9.122}$$

the 'fibre critical' set; a point is in this set if the fibre through it is tangent to H at that point. In general this can be quite singular but by the implicit function theorem

$$dh(\bar{y},\bar{z}),\ d\frac{\partial h}{\partial z_j}(\bar{y},\bar{z}) \text{ linearly independent} \Rightarrow C_H \text{ is smooth near } (\bar{y},\bar{z}). \tag{9.123}$$

Observe that the set (9.122) only depends on H, not on the chosen defining function, h. Indeed any other defining functions is just $h' = ah$ with $a \neq 0$. Of course this defines the same hypersurface H and since

$$\frac{\partial h'}{\partial z_j} = a\frac{\partial h}{\partial z_j} + \frac{\partial a}{\partial z_j} h \tag{9.124}$$

leads to the same fibre critical set C_H, justifying the notation.

A fibre-preserving map in local coordinates is just one of the form

$$(y,z) = \tilde{F}(y',z'),\ z = F(y',z'),\ y = G(y') \tag{9.125}$$

so under a diffeomorphism of this form the fibre above y pulls back to the fibre above y'. The definition of C_H is also invariant under fibre-preserving diffeomorphisms. Namely, if H' is the pull back of H then C_H also pulls back to $C_{H'}$ since

$$h' = \tilde{F}^* h, \text{ i.e. } h'(y', z') = h(y, z) \Longrightarrow d_{z'} h'(y', z') = \frac{\partial F}{\partial z'} \cdot d_z h(y, z). \tag{9.126}$$

PROPOSITION 9.2. *Under the non-degeneracy assumption* (9.123) *on* $H \subset \mathbb{R}^n \times \mathbb{R}^k$, *the map*

$$N^* H \setminus 0\big|_{C_H} \ni (y, z; \lambda dh) \mapsto (y, \lambda dh) \in T^* \mathbb{R}^n \setminus 0 \tag{9.127}$$

is locally an embedding with range a conic Lagrangian submanifold Λ_H, *i.e., a homogeneous submanifold of dimension* n *such that*

$$\alpha = \sum_j \eta_j dy_j \text{ vanishes as a 1-form on } \Lambda_H. \tag{9.128}$$

PROOF. In local coordinates the map (9.127) is the projection

$$\tilde{\pi} : (y, z, \eta, \zeta) \mapsto (y, \eta)$$

restricted to the submanifold

$$\begin{aligned} N^* H \setminus 0\big|_{C_H} = \{(y, z; \eta, \zeta); h(y, z) = 0 \\ \zeta = \frac{\partial h}{\partial z_j}(y, z) = 0, \ \eta = \lambda d_y h(y, z)\}, \end{aligned} \tag{9.129}$$

By the implicit function theorem it suffices to show that the differential is injective when restricted to the tangent space of (9.129), i.e., that no element of the null space of $\tilde{\pi}_*$ is tangent to $M = N^* H \setminus 0|_{C_H}$ (other than zero of course). The null space of $\tilde{\pi}_*$ is spanned by ∂_{z_j} and ∂_{ζ_i}. Since $\zeta = 0$ in $N^* H$ over C_H, only $a \cdot \partial_z$ could be tangent to it. However, $\eta_j = \lambda \frac{\partial h}{\partial z_j y_j}$ on M and also $\frac{\partial h}{\partial z_h} = 0$ on M so

$$\sum_k \frac{\partial^2 h}{\partial y_j \partial z_k} a_k = 0 = \sum_k \frac{\partial^2 h}{\partial z_j \partial z_k} a_k$$

which implies $a = 0$ because of (9.123).

Thus (9.127) is locally an embedding, i.e., is an immersion as long as (9.123) holds, with the image denoted Λ_H. To see (9.128), i.e. that $\alpha = 0$ when restricted to Λ_H it is enough to show that $\tilde{\pi}^* \alpha = \sum_j \eta_j \lambda y_j = 0$ on $M = (N^* H \setminus 0)|_{C_H}$. Since $\eta_j = \lambda \frac{\partial h}{\partial y_j}$ on M,

$$\alpha = \lambda \sum_j \frac{\partial h}{\partial_{y_j}} dy_j = \lambda dh = 0$$

on M, since $h = 0$. $\square$

Notice that under a coordinate transformation in the variables y, say $y = G(y')$, the hypersurface H is transformed to H' defined by $h'(y', z) = h(G(y'), z)$ and Λ_H is replaced by

$$\Lambda_{H'} = \{(y', \eta'), \ y = G(y'), \ \eta' \cdot dy' = \eta \cdot G^* dy, \ (y, \eta) \in \Lambda_H\}. \tag{9.130}$$

That is, Λ_H is a well-defined submanifold of $T^* \mathbb{R}^n \setminus 0$ with $\mathbb{R}^n$ treated as a manifold.

We shall say a hypersurface $H \subset \mathbb{R}^n \times \mathbb{R}^k$ such that (9.123) holds near $\bar{p} \in H$ is a *parameterization* of Λ_H near $(\bar{y}, dh(\bar{y}, \bar{z}))$, $\bar{p} = (\bar{y}, \bar{z})$ given by (9.127). Proposition 9.2 has a converse, namely any $\Lambda \subset T^*\mathbb{R}^n \setminus 0$ which is homogeneous and Lagrangian arises this way locally, that is provided

$$\begin{aligned} &\Lambda \subset T^*\mathbb{R}^n \setminus 0 \text{ is smooth of dimension } n, \\ &t \cdot \Lambda = \Lambda, t > 0 \text{ is Lagrangian} \\ &\omega = \sum_j d\eta_j dy_j \text{ vanishes on } \Lambda. \end{aligned} \tag{9.131}$$

Note that as a consequence of the assumed homogeneity of Λ, this last condition is equivalent to

$$\alpha = \sum_j \eta_j dy_j \text{ vanishes on } \Lambda. \tag{9.132}$$

Certainly (9.132) implies that $\omega = d\alpha$ vanishes on Λ. Conversely, the homogeneity means exactly that $R = \eta \cdot \partial_\eta$ is everywhere tangent to Λ. Then for any $v \in T_p\Lambda$,

$$\alpha(v) = \omega(R, v) = 0. \tag{9.133}$$

Proposition 9.3. *Any homogeneous Lagrangian submanifold has a parameterization near each point $(\bar{y}, \bar{\eta}) \in \Lambda$ and H can be chosen to be minimal in the sense that if $\bar{p}$ is the base point of the parameterization*

$$\frac{\partial^2 h}{\partial z_i \partial z_j}(\bar{p}) = 0. \tag{9.134}$$

Proof. Fix $(\bar{y}, \bar{\eta}) \in \Lambda$, $\bar{\eta} \notin 0$ by assumption in (9.131). Let $S \subset \mathbb{R}^n$ by the projection of $T_{(\bar{y},\bar{\eta})}\Lambda$ onto the first factor. Thus

$$\dim S = n - k - 1 \leq n - 1 \tag{9.135}$$

by homogeneity (which implies $\eta \cdot \partial_\eta$ is tangent to Λ) so $k \geq 0$. By an affine change of variables in $\mathbb{R}^n$ we may assume $\bar{y} = 0$ and that $S = \operatorname{sp}\{\partial_{z_{k+2}}, \dots, \partial_{y_n}\}$. Thus on Λ, near $(\bar{y}, \bar{\eta})$, the variables y_j, $j \geq k+2$, have independent differentials and $dy_j = 0$ at $(\bar{y}, \bar{\eta})$ for $j = 1, \dots, k+1$. The vanishing of α, and $d\alpha$, on Λ, and in particular on the tangent space $T_{(\bar{y},\bar{\eta})}\Lambda$ implies that

$$\begin{aligned} \eta_{k+2} = \cdots = \eta_n = 0 \text{ at } (\bar{y}, \bar{\eta}), \\ d\eta_{k+2} = \cdots = d\eta_n = 0 \text{ at } (\bar{y}, \bar{\eta}). \end{aligned}$$

Thus the variables η_j, $j = 1, \dots, k+1$ and y_l, $l \geq k+2$ together give local coordinates on Λ near $(\bar{y}, \bar{\eta})$. By a further linear transformation among only the first $k+1$ variables we can assume that $\bar{\eta} = (1, 0, \dots, 0)$.

Write

$$\alpha = \eta_1 dy_1 - \sum_{2 \leq j \leq k+1} y_j d\eta_j + \sum_{l \geq k+2} \eta_l dy_l + d(\sum_{2 \leq j \leq k+1} \eta_j y_j).$$

By assumption in (9.131) this 1-form vanishes identically on Λ. Next restrict to $\Gamma = \Lambda \cap \{\eta_1 = 1\}$, which involves no essential loss of information due to the assumed homogeneity of Λ. Then $z_j = \eta_{j+1}$, $1 \leq j \leq k$ and $y'' = (y_{k+2}, \dots, y_n)$ are local coordinates on Γ near the base point and the other variables can therefore be expressed in terms of them and so we may define a function $g(z, y'')$ by

$$g(z, y'') = y_1 + \sum_{2 \le j \le k+1} \eta_j y_j \text{ on } \Gamma \tag{9.136}$$

Thus on Γ,

$$\begin{gathered} \alpha = dg - \sum_{1 \le j \le k} y_{j+1} dz_j + \sum_{l \ge k+2} \eta_l dy_l = 0 \text{ on } \Gamma \Longrightarrow \\ \eta_{j+1} = z_j,\ y_{j+1} = \frac{\partial g}{\partial z_j},\ j = 1, \dots, k,\ \eta_l = -\frac{\partial g}{\partial y_l},\ l \ge k+2. \end{gathered} \tag{9.137}$$

We shall show that the zero set of the function

$$h(y, z) = y_1 + \sum_{j=1}^{k} z_j y_{j+1} - g(z, y'') \tag{9.138}$$

parameterizes Λ near $(\bar{y}, \bar{\eta})$. Certainly (9.123) holds so it suffices to check that the Lagrangian it parameterizes is indeed Λ. Differentiating h,

$$C_H = \{y_{j+1} = \frac{\partial g}{\partial z_j},\ j = 1, \dots, k,\ h = y_1 + \sum_{j=1}^{k} z_j y_{j+1} - g(z, y'') = 0\}$$

shows that the z_j and y_l, $l \ge k+2$ are coordinates on C_H and from (9.137)

$$d_y h = dy_1 + \sum_{j=1}^{k} z_j dy_{j+1} - d_{y''} g(z, y'') \Longrightarrow (y, d_y h) \in \Gamma \tag{9.139}$$

so H does parameterize Λ.

This completes the proof of Proposition 9.3 since h is minimal, in that $\partial^2 h/\partial z_i \partial z_j = 0$ at the chosen base point. □

As we shall see below, it is important to observe that two minimal paramerizations of a conic Lagrangian near a given point are *equivalent* in the sense that there is is a fibre-preserving diffeomorphism mapping base point to base point and taking one hypersurface to the other.

LEMMA 9.6 (Minimal parameterizations). *If $H' \subset \mathbb{R}^n \times \mathbb{R}^{k'}$ is a hypersurface satisfying* (9.123) *at $\bar{p} = (\bar{y}, \bar{z})$ which is minimal in the sense that* (9.134) *holds and which locally parameterizes a conic Lagrangian Λ then $k' = k$, the integer in* (9.135) *for that Lagrangian and there is a local fibre-preserving diffemorphism reducing H' to the hypersurface H constructed in Proposition 9.3.*

PROOF. We may work in the local coordinates introduced in the proof of Proposition 9.3. Thus, in addition to assuming that $H' = \{h'(y, z) = 0\}$ parameterizes Λ near $(\bar{y}, \bar{\eta})$ we may suppose that (9.135) holds and also that $\bar{p} = (\bar{y}, \bar{z})$ is the base point of both the given parameterization and that constructed in Proposition 9.3.

Thus y_j, for $j \ge n - k + 2$ η_l, $l \le k+1$ are coordinates on Λ, $\bar{y} = 0$, $\bar{\eta} = (1, 0, \dots, 0)$ and $T_{(\bar{y},\bar{\eta})}\Lambda$ is reduced to normal form. First we arrange that, locally, $C_{H'} = C_H$ by a fibre-preserving diffeomorphism. Of necessity $dh' = dy_1$ at the base point, so $h' = a(y_1+g(y_2,\dots,y_n,z))$. So may assume that $h' = y_1+g(y_2,\dots,y_n,z)$. From the arranged form of the tangent space to Λ at the base point we know that

$$d_y\frac{\partial h'}{\partial z_j}(\bar p) \text{ define } \{dy_j = 0,\ 2 \le j \le k+1\}. \tag{9.140}$$

Thus, after a linear change of fibre coordinates, we may suppose that

$$d_y\frac{\partial h'}{\partial z_j} = dy_j \text{ at } \bar p. \tag{9.141}$$

Now the assumption that H' and H parameterize the same Lagrangian means that

$$\begin{tikzcd} C_H \ni (y,z) \arrow[d, mapsto] & & \\ (y, d_y h(y,z)) \arrow[r, equal] & (y, d_y h'(y,z')) & \text{in } \Lambda \cap \{\eta_1 = 1\} \\ & C_{H'} \ni (y,z') \arrow[u, mapsto] & \end{tikzcd} \tag{9.142}$$

induces a diffeomorphism from C_H to $C_{H'}$. We need to check that this can be extended to a fibre preserving diffeomorphism, but this is clear since z and the $y'' = y_{k+2},\dots,y_n$ give coordinates on C_H and similarly on $C_{H'}$ and in terms of these (9.142) is the restriction of the identity in y and

$$z_j = \frac{\partial h'}{\partial y_{j+1}}(y,z') \tag{9.143}$$

which is fibre-preserving.

Thus we have arranged that $C_H = C_{H'}$ and that $d_y h = d_y h'$ there, which means that

$$h' = h + O((h, d_z h)^2) \tag{9.144}$$

i.e. the difference vanishes quadratically on $C_H = C_{H'}$.

So, we need to make a further fibre-preserving transformation which removes these quadratic terms, leaving C_H fixed of course. This can be done using the Morse lemma. Since a proof is not included here, it seem appropriate to prove it directly – this amounts to Moser's proof of the Morse Lemma.

Since we have arranged that h' and h are equal up to quadratic terms on C_H it follows that $h_t = (1-t)h + th'$ is, for $t \in [0,1]$, a 1-parameter family of parameterizations of the same Lagrangian Λ with C_H fixed (and of course dh_t constant on C_H.) So, Moser's idea applied to this case, is to look for a 1-parameter family of fibre-preserving diffeomorphisms,

$$F_t(y,z) = (y, Z(t,z)),\ F_0(y,z) = (y,z), \tag{9.145}$$

starting at the identity and such that

$$F_t^* h_t = h_t(y, Z(t,z)) \equiv h(y,z) = h_0(y,z). \tag{9.146}$$

The nice feature of this is that the condition can be expressed differentially and written in the form

$$0 = \frac{d}{dt}F_t^* h_t = F_t^*(V_t h_t(y,z) + h_t') \Longrightarrow V_t h_t(y,z) + h_t' = 0 \tag{9.147}$$

where V_t is the 1-parameter family of vector fields defining F_t. That is, F_t can be recovered from V_t and the intial condition $F_0 = \mathrm{Id}$, and will be fibre-preserving if and only if

$$V_t = \sum_{j=1}^{k} v_j(t,z)\partial_{z_j} \tag{9.148}$$

is tangent to the fibres. The remarkable property of (9.147) is that 'F_t has disappeared' and we only need to find V_t.

By construction

$$h_t = h + \sum_{i,j=1}^{k} G_{ij}(t,y',z)\frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}$$
$$\Longrightarrow \frac{\partial h_t}{\partial z_i} = \sum_j A_{ij}(t,y,y')\frac{\partial h}{\partial z_j}$$

where the G_{ij} are smooth and A_{ij} is invertible near C_H. Thus

$$\frac{dh}{dt} = \sum_{i,j=1}^{k} \frac{dG_{ij}(t,y',z)}{dt}\frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}(A^{-1}\frac{\partial h_t}{\partial z})_j \tag{9.149}$$

constructs V_t. □

It also follows from Proposition 9.3 that there is a parameterization of Λ, near a given point, with any number of fibre variables z, greater than or equal to k. Namely, if $z' \in \mathbb{R}^q$ and $p(z')$ is a non-degenerate quadratic form in z' then

$$H' = \{(y,z,z') \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^q;\ h' = h(y,z) + p(z')\}$$

also parameterizes Λ and has $k+q$ fibre variables, simply because $\frac{\partial h'}{\partial z'} = 0 \Leftrightarrow z' = 0$.[1]

Conversely we may remove 'unnecessary' fibre variables.

Lemma 9.7. *If $H \subset \{\mathbb{R}^{k+l}_{(z,Z)} \times \mathbb{R}^n\}$ is defined by h where at the base point $\partial^2 h_{\partial Z^2}$ is invertible and $Z = S(z,y)$ is the local sollution of $\partial h/\partial Z(z,S,y) = 0$ then $H' = \{h' = h(z,S,y)$ locally parameterizes the same Lagrangian as H.*

Proof. The invertibility of $\partial^2 h/\partial Z^2$ at the base point $\bar{y}, \bar{z}, \bar{Z})$ implies that the local solution of $\partial h/\partial Z(z,S,y)$ is of the indicated form and then h' exists. Then

$$\frac{\partial h'}{\partial z} = \frac{\partial h}{\partial z} + \frac{\partial S}{\partial z}\cdot\frac{\partial h}{\partial Z} \tag{9.150}$$

from which it follows that $C_H\partial(y,z,Z) \longmapsto (y,z) \in C_{H'}$ is an isomorphism and $d_y h' = d_y h$ at the points so identified. Thus h' parameterizes the same Lagrangian as h. □

The most familiar case of a conic Lagrangian submanifold of $T^*\mathbb{R}^n$ is the conormal bundle of a submanifold. If the manifold is of codimension $k+1$ then

$$G = \{y \in \mathbb{R}^n; g_1(y) = \cdots = g_{k+1}(y) = 0,\ dg_j \text{ independent}\}$$
$$N^*G = \{(y,\eta) \in T^*\mathbb{R}^n; \eta = \sum_{i=1}^{k+1} \eta_i dg_i(y)\}.$$

[1] See Problem N

Clearly G is parameterized near $\eta = (1, 0, \dots, 0)$ by

$$h(y, z) = g_1(y) + \sum_{j=1}^{k} z_j g_{j+1}(h).$$

Then $\partial^2 h/\partial z_i \partial z_j \equiv 0$. Conversely,

PROPOSITION 9.4. *If $H \subset \mathbb{R}^n \times \mathbb{R}^k$ is such that $\partial^2 h/\partial z_i \partial z_j \equiv 0$ on C_H then H parameterizes the conormal bundle of a submanifold locally.*

PROOF. See Problem MM. □

9.3. Lagrangian distributions

Now we are in a position to associate a space of distributions with a conic Lagrangian, $\Lambda \subset T^*X \setminus 0$, in a way that generalizes the conormal distributions discussed earlier.

DEFINITION 9.4. *If $\Lambda \subset T^*X \setminus 0$ is a smooth conic Lagrangian submanifold then*

$$I^*(X, \Lambda) \subset \mathcal{C}^{-\infty}(X) \tag{9.151}$$

is defined to consist of those distributions satisfying

$$\mathrm{WF}(u) \subset \Lambda \tag{9.152}$$

and such that for each $p \in \Lambda$ there is a local parameterization $H \subset X \times \mathbb{R}^k$ of Λ near p and $v \in I^*(X \times \mathbb{R}^k, H)$ with compact support such that

$$p \notin \mathrm{WF}(u(\cdot) - \int_{\mathbb{R}^k} v(\cdot, z) dz). \tag{9.153}$$

Thus by definition a distribution is Lagrangian if it is 'smooth away from the Lagrangian' and microlocally given by push-forward of a conormal distribution on a parameterizing hypersurface near each point of the Lagrangian.

As usual this definition only really makes good sense because the same class of singularities near a given point of Λ arises by pushing forward, independent of which parameterization of the Lagrangian is used. So, we check this first

One thing to check is that this does indeed reduce to the conormal distributions discussed earler.

PROPOSITION 9.5. *If $\Lambda = N^*G \setminus 0$ is the conormal bundle of an embedded submanifold then*

$$I^*(X, N^*G) = I^*(X, G). \tag{9.154}$$

PROOF. Stationary phase to minimal parameterization. □

LEMMA 9.8. *If $H_i \subset X \times \mathbb{R}^{k_i}$, $i = 1, 2$, near $p_i \in C_{H_i}$ are two parameterizations of a conic Lagrangian Λ near $p \in \Lambda$ and $\chi \in \mathcal{C}^\infty_c(X \times \mathbb{R}^{k_i})$ then for each $v \in I^*(X \times \mathbb{R}^{k_1}; H_1)$ there exists $w \in I^*(X \times \mathbb{R}^{k_1}; H_1)$ such that*

$$p \notin \mathrm{WF}(\int_{\mathbb{R}^{k_1}} v(\cdot, z) dz - \int_{\mathbb{R}^{k_2}} w(\cdot, z') dz'. \tag{9.155}$$

Nothing is said about the orders of v and w, but we will work this out as we go along.

Proof. Suppose first that both the H_i are minimal parameterizations at p. Then we know from Proposition 9.6 that the two parameterizations are related by a fibre-preserving diffeomorphism. This means that the resulting spaces of conormal distributions are mapped onto each other by the diffeomorphism and its inverse locally and then w is the pull-back of v with a Jacobian factor inserted to ensure that the integrals are the same.

So, to prove the general case it suffices to work with an arbitrary parameterization H_1 and we may suppose that H_2 is any convenient minimal parameterization. At the base point,

$$\frac{\partial^2 h_1}{\partial z_i \partial z_j} \text{ has rank p} \tag{9.156}$$

where minimality corresponds to $p = 0$. After a linear change of variables, we may take this matrix to be the identity in the last $p \times p$ block. Then by the implicit function theorem,

$$\frac{\partial h_1}{\partial z_i} = 0,\ k_1 - p + 1 \le i \le k_i \Longrightarrow z_j = Z_j(y, z'),\ k_1 - p + 1 \le k_1,\ z' = (z_1, \dots, z_k),\ k = k_1 - p.$$

Thus,

$$h_1(y,z) = h(y,z') + \sum_{i,j=k_1-p+1}^{k_1} H_{ij}(z_i - Z_i(y,z'))(z_j - Z_j(y,z')),\ h(z',y) = h_1(y,z',Z(y,z')) \tag{9.157}$$

with H_{ij} symmetric and invertible. □

9.4. Keller's example of a caustic

Keller was the first to effectively compute with Lagrangian distributions in a context, that of a caustic, where what is now called the Keller-Maslov line bundle cannot be avoided. This example will be discussed here and should help to motivate the general, invariant, definition of the symbol of a Lagrangian distribution in the next section.

Consider the wave operator in $2 + 1$ dimensions

$$P = D_t^2 - D_x^2 - D_y^2. \tag{9.158}$$

The forward forcing problem for P is uniquely solvable. That is, if $f \in \mathcal{C}^{-\infty}(\mathbb{R}^3)$ has support in $t \ge 0$ then there is a unique distribution

$$u \in \mathcal{C}^{-\infty}(\mathbb{R}^3),\ Pu = f,\ \operatorname{supp}(u) \subset \{t \ge 0\}. \tag{9.159}$$

It is also the case that if in addition $f \in \mathcal{C}^{\infty}(\mathbb{R}^3)$ then $u \in \mathcal{C}^{\infty}(\mathbb{R}^3)$. In particular this means that if u is a solution of $Pu = 0$ in $t < 0$ then u can be extended uniquely to a solution in the whole of $\mathbb{R}^3$[2] and the singularities in the future only depend on the singularities in the past.

So, suppose that we have arrange that $u \in \mathcal{C}^{-\infty}(\mathbb{R}^3)$ is conormal to some hypersurface in $t < 0$ and satisfies the wave equation, or at least has Pu smooth there. It is possible to find such solutions, $u \in I^m(\mathbb{R}^3, G)$ which are elliptic (so in particular not smooth) if G is characteristic for the wave equation, meaning that

$$N^*G \subset \Sigma(P) = \{(x, y, t, \tau, \xi, \eta); \tau^2 = \xi^2 + \eta^2\}. \tag{9.160}$$

[2]Problem NN

The most obvious example of this is a characteristic plane $G = \{t = \omega \cdot (x, y)\}$ where $|\omega|^2 = 1$. Then for instance

$$P(\delta(t - \omega \cdot (x, y)) = 0. \tag{9.161}$$

As we shall see below, it is possible to continue any smooth curve $C \in \mathbb{R}^2$ as a characteristic hypersurface (in two ways in fact) for $|t| < \epsilon$ where $\epsilon > 0$ depends on C, and especially on its curvature. As opposed to the case of the line $\omega \cdot (x, y) = 0$ which leads to the global surface above, in general this characteristic hypersurface will develop singularities. Again as we shall see below, the conormal bundle of the curve defines a global smooth conic Lagrangian and the singularities correspond to the places where the projection of this to the base is not locally smooth. The particular example we consider here, following the idea of Keller, is where G is a parabola. The general construction is carried out below but for the parabola

$$C = \{y = \frac{x^2}{2}\},\ N^*C = \{(x, \frac{x^2}{2}, -x\eta, \eta),\ x,\ \eta \in \mathbb{R}\} \tag{9.162}$$

we can find the global Lagrangian – it is 'the union of the light rays through the points of $N^*C \setminus 0$'. Here, by a light ray, we mean a straight line in $\Sigma(P) \subset T^*\mathbb{R}^3$ on which τ, ξ and η are constant (with $\tau^2 = \xi^2 + \eta^2$) and $t = t_0 + s$, $x = x_0 - \xi s/\tau$ and $y = y_0 - \eta s/\tau$. Here $(t_0, x_0, y_0, \tau, \xi, \eta)$ is the initial point, so we can take $t_0 = 0$ and $s = t$ and so initially $\tau = \pm(x_0^2 + 1)^{\frac{1}{2}}\eta$ and

(9.163)

$$\Lambda_C = \{(t, x_0 \pm \frac{x_0 t}{(x_0^2 + 1)^{\frac{1}{2}}}, \frac{x_0^2}{2} - \mp \frac{t}{(x_0^2 + 1)^{\frac{1}{2}}}, \pm(x_0^2+1)^{\frac{1}{2}}\eta, -x_0\eta, \eta); x_0,\ \eta,\ t \in \mathbb{R}, \eta \neq 0\}.$$

If we take τ to have the oppsosite sigh to η, meaning the negative sign in (9.163) then y increases with t from its initial (non-negative value) snd x increases if negative and decreases if positive. It is straightforward to check[3] that

$$\Lambda_C^- = N^*G \text{ in } t < 1, G \text{ smooth}. \tag{9.164}$$

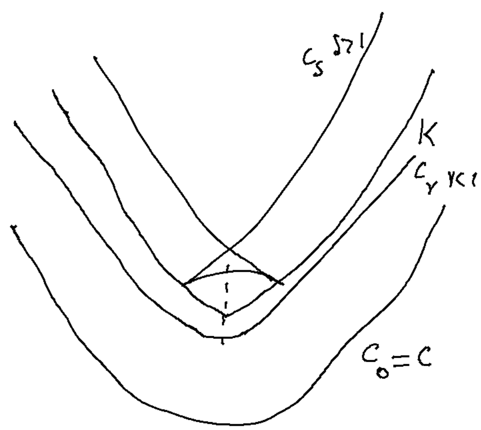
In fact the first singularity which occurs, meaning the first point at which the intersection of the tangent space to Λ_C^- and the fibre of $T^*\mathbb{R}^3$ has dimension greater than 1 is at $(1, 0, 1, -1, 0, 1)$ at which it has dimension 2 – it always has dimension 1 in $t < 1$. In fact we can easily see exactly where the tangent space to Λ_C^- meets the fibre with dimension greater than one since this is exactly where

$$\begin{gathered}
\frac{d}{dx_0}\left(x_0(1 - \frac{t}{(x_0^2 + 1)^{\frac{1}{2}}})\right) = 1 - \frac{t}{(x_0^2 + 1)^{\frac{3}{2}}} = 0 \text{ and} \\
\frac{d}{dx_0}\left(\frac{x_0^2}{2} + \frac{t}{(x_0^2 + 1)^{\frac{1}{2}}}\right) = x_0 - \frac{tx_0}{(1 + x_0^2)^{\frac{3}{2}}} = 0 \\
\Longleftrightarrow t = (1 + x_0^2)^{\frac{3}{2}},\ x = x_0^3,\ y = 1 + \frac{3}{2}x_0^2.
\end{gathered} \tag{9.165}$$

This curve is the full caustic, K. As a curve, it is smooth as a curve except for a singular point at $(1, 0, 1)$, which is the point we are most interested in. Notice that if we think of Λ_C^- as projecting to the conormal bundle to a family C_t of curves in $\mathbb{R}^2$ starting at $C_0 = C$ and parameterized by t, then for $t < 1$, C_t is smooth, for $t = 1$ it has a single singular point at $(1, 0, 1)$ and for $t > 1$ these curves each have

[3] Problem ***

two singular points, on K. If you check[4] what is happening to the curvature of C_t you will see that it is positive in the 'upwards direction' (i.e. for y increasing) and this remains true for $t > 1$ for the part of the curve outside K; however the part of the curve above K has the opposite curvature. This is reflected in the behaviour of the symbols as we shall see.



Using the construction in the previous sections we can find an explicit parameterization for Λ_C^- near $(1,0,1)$ and show that there are solutions of $Pu = 0$ nearby which are Lagrangian with respect to Λ_C^-. Thus following the proof of Propositon 9.3 we first make an affine change of coordinates setting

$$S = y - t,\ R = \frac{y+t}{2} - 1,\ x = x. \tag{9.166}$$

The dual variables are then

$$\eta = \sigma + \frac{\rho}{2},\ \tau = -\sigma + \frac{\rho}{2}, \xi = \xi \text{ i.e. } \sigma = \frac{\eta - \tau}{2},\ \rho = \eta + \tau. \tag{9.167}$$

Thus in the canonically dual coordinates to these coordinates is

$$\begin{aligned}(9.168)\quad \Lambda_C^- = \{(S, x, R, \sigma, \xi, \rho) = \\ (\frac{1}{2}x_0^2 + t((x_0^2+1)^{-\frac{1}{2}} - 1), x_0 - x_0 t(x_0^2+1)^{-\frac{1}{2}}, \frac{x_0^2}{4} + \frac{1}{2}t(1 + (x_0^2+1)^{-\frac{1}{2}}) - 1, \\ \frac{1}{2}(1 + (x_0^2+1)^{\frac{1}{2}})\eta, -x_0\eta, (1 - (x_0^2+1)^{\frac{1}{2}})\eta); x_0,\ \eta,\ t \in \mathbb{R}, \eta \neq 0\}\end{aligned}$$

where we use the same parameterization. Now the base point has been moved to the point $(1,0,0)$ above the origin in (S, x, R) and the projection to the base of the tangent space is fixed by

$$dx = 0,\ dS = 0. \tag{9.169}$$

[4]Problem ***

Then a parameterizing hypersurface is given by

(9.170) $$h = S + zx - F(R, z),\ F = x\xi$$

where $z = \xi$ and R are taken as coordinates on $\Lambda_C^- \cap \{\sigma = 1\}$. Thus F is given implicitly by:-

(9.171)
$$F(R,z) = (x_0 - x_0 t(x_0^2+1)^{-\frac12})z\eta,\ \frac12(1+(x_0^2+1)^{\frac12})\eta = 1,\ z = -x_0\eta,\ R = \frac{x_0^2}{4} + \frac12 t(1+(x_0^2+1)^{-\frac12}) - 1,$$

where we eliminate η using the second equation and x_0 and t in terms of z and R using the last two. Now, we know that $\partial_z^2 h = 0$ at the base point and we can easily check that

(9.172) $$\partial_z^3 h = ?\text{ at }\bar{p}.$$

This of course means that $\partial_z^2 h < 0$ above $x = 0$ for $t = 1 - \delta$ and $\partial_z^2 h > 0$ above $x = 0$ for $t = 1 + \delta$, $\delta > 0$ small. The effect of this is Keller's observation

(9.173) As a conormal distribution the symbol of u is multiplied by i across the swallowtail tip.

This shows that we must expect 'factors of i' to appear in the definition of the symbol of a Lagrangian distributions when we generalize from the conormal case. These factors are what constitutes the Keller-Maslov line bundle over a conic Lagrangian.

9.5. Oscillatory testing and symbols

The symbol of a conormal distribution is defined by taking the Fourier transform across the submanifold. To extend this to the Lagrangian case requires some care. We shall show that if $u \in I^*(X, \Lambda; \Omega^{\frac12})$ is a half-density then we can define its symbol as an object on Λ (but not quite a function) by pairing with oscillating functions. Thus consider

(9.174) $$A(s, f, \nu) = u(e^{-isf}\nu),\ f \in \mathcal{C}^\infty(X),\ u \in I^*(X, \Lambda; \Omega^{\frac12}).$$

The argument is really local, so it is enough to take $X = \mathbb{R}^n$, but we do want to ensure coordinate invariance. In order for (9.174) to make sense, ν should be a half-density. Obviously to find (i.e. define) the symbol at some point $\lambda \in \Lambda$, or really the ray through that point, we will suppose that ν has support near the projection of that point. The main question is then, what we should demand of f. It is clearly natural to expect to take

(9.175) $$df(\pi(\lambda)) = \lambda \in T^*X \setminus 0.$$

Lemma 9.9. *For any $\lambda \in \Lambda$, the phase $f \in \mathcal{C}^\infty(X)$ can be chosen so that*

(9.176) $$f(\pi(\lambda)) = 0 \text{ and } \operatorname{graph}(df) \pitchfork \Lambda = \{\lambda\}$$

and then, if ν has sufficiently small support near $\pi(\lambda)$, $A(s, f, \nu)$ in (9.174) is a classical symbol for any $u \in I^*(X, \Lambda; \Omega^{\frac12})$.

NB: Cutoffs need to be done better and argument cleaned up!

Proof. As shown in the proof of Proposition 9.3 coordinates can be introduced near the projection of λ in terms of which the base point and the tangent space to Λ at the base point takes the form

(9.177) $$\lambda = dy_1,\ T_\lambda\Lambda = \operatorname{sp}\{\partial_{\eta_j},\ 1 \le j \le k+1,\ \partial_{y_l},\ l \ge k+2\}.$$

Since, for any choice of a real-valued function $f \in \mathcal{C}^\infty(X)$, and any coordinates

$$\operatorname{graph}(df) = \{(y, d_y f)\} \subset T^*X \tag{9.178}$$

is a smooth Lagrangian submanifold (since it clearly has dimension n, being a graph, and $\alpha = \eta \cdot dy = df$ is closed on Λ, and hence $\omega = d\alpha$ vanishes there). Thus, if $df(\pi(\lambda)) = \lambda$ then (9.176) is just the condition that the pairing between $T^*\lambda \operatorname{graph}(df)$ and $T_\lambda\Lambda$ be non-degenerate. Thus the condition on f is just

$$\det\left(\frac{\partial^2 f}{\partial_i \partial_j}\Big|_{i,j\ge k+2}\right) \neq 0. \tag{9.179}$$

Put more invariantly, this condition can be stated in terms of any submanifold S through $\pi(\lambda) = \bar{y}$ which is conormal bundle tangent to Λ at λ, i.e. $\lambda \in N^*S$ and $T_\lambda N^*S = T_\lambda\Lambda$ as

$$df(\bar{y}) = \lambda,\ f\big|_A \text{ has a non-degenerate critical point at } \bar{y}. \tag{9.180}$$

Of course such a submanifold S exists, for example that given locally by $y_j = \bar{y}_j$ for $j \ge k+2$ and (9.180) only depends on $T_{\bar{y}}S$.[5]

Under this assumption of transversality we need to examine (9.174). We know from the properties of the wave front set that only the points where $df \in \operatorname{WF}(u)$ can make asymptotic contributions to $A(s, f, \nu)$. Thus, if ν has small enough support then only the point $\lambda \in \operatorname{WF}(u)$ is relevant and we may suppose that, in local coordinates,

$$u(y) = \int e^{i\tau h(y,z)} a(y,z,\tau) dz d\tau |dy|^{\frac12},\ a \in S^m_{\mathrm{phg}} \tag{9.181}$$

for some m and some parameterizing hypersurface for Λ at λ. Then

$$A(s, f, \nu) = \int e^{i(\tau h(y,z) - s f(y))} a'(y,z,\tau) dz d\tau dy,\ a|dy|^{\frac12}\nu = a'|dy|. \tag{9.182}$$

Consider the inverse Fourier transform in s

$$u(t) = \int e^{i(\tau h(y,z) - s f(y) + st)} a'(y,z,\tau) dz d\tau dy ds. \tag{9.183}$$

A cutoff keeping $\tau > 1$ and $r = s/\tau$ bounded from above an below can be inserted here making on a $\mathcal{C}^\infty$ change to u. It then follows that

$$\tilde{h}(t, y, z) = h(y, z) - r f(y) + rt \tag{9.184}$$

defines a hypersurface parameterizing $N^*\{t = 0\}$ which means that u is conormal to 0 and its Fourier transform $A(s, f, \nu)$ is equivalently a classical symbol. □

[5] Note that this shows something rather less than obvious, which is worth checking by hand. Namely if one takes a $\mathcal{C}^\infty$ perturbation of f to $f + \epsilon g$ then if ϵ is small enough and $df + \epsilon df = \lambda' \in \Lambda$ then the condition (9.180) must hold at the new point λ' – even though the dimension of the tangent space may well be different (it can only be larger). This is just the stability of transversality.

9.6. Hamilton-Jacobi theory

Let X be a $\mathcal{C}^\infty$ manifold and suppose $p \in \mathcal{C}^\infty(T^*X\backslash 0)$ is homogeneous of degree m. We want to find characteristic hypersurfaces for p, namely hypersurfaces (locally) through $\bar{x} \in X$

$$H = \{f(x) = 0\} \quad h \in \mathcal{C}^\infty(x) h(\bar{x}) = 0, dh(\bar{x}) \neq 0 \tag{9.185}$$

such that

$$p(x, dh(x)) = 0. \tag{9.186}$$

Here we demand that (9.186) hold near $\bar{x}$, not just on H itself. To solve (9.186) we need to impose some additional conditions, we shall demand

$$p \text{ is real-valued} \tag{9.187}$$

and

$$d_{\text{fibre}} p \neq 0 \text{ or } \Sigma(p) = \{p = 0\} \subset T^*X\backslash 0. \tag{9.188}$$

This second condition is actually stronger than really needed (as we shall see) but in any case it implies that

$$\Sigma(P) \subset T^*X\backslash 0 \text{ is a } \mathcal{C}^\infty \text{ conic hypersurface} \tag{9.189}$$

by the implicit function theorem.

The strategy for solving (9.186) is a geometric one. Notice that

$$\Lambda_h = \{(x, dh(x)) \in T^*X\backslash 0\} \tag{9.190}$$

actually determines h up to an additive constant. The first question we ask is – precisely which submanifold $\Lambda \subset T^*X\backslash 0$ corresponds to graphs of differentials of $\mathcal{C}^\infty$ functions? The answer to this involves the tautologous *contact* form.

$$\begin{gathered} \alpha : \ T^*X \longrightarrow T^*(T^*X) \not\subset \tilde{\pi} \circ \alpha = \text{Id} \\ \alpha(x, \xi) = \tilde{\pi}^* \xi \in T^*_{(x,\xi)}(T^*X). \end{gathered} \tag{9.191}$$

Here $\tilde{\pi} : T^*(T^*X) \longrightarrow T^*X$ is the projection. Notice that if $x_1, \dots, x_n$ are local coordinates in X then $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are local coordinates T^*X, where $\xi \in T^*_x X$ is written

$$\xi = \sum_{i=1}^n \xi_i dx_i. \tag{9.192}$$

Since $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are local coordinates in T^*X they together with the dual coordinates $\Xi_1, \dots, \Xi_n$, $X_1, \dots, X_n$ are local coordinates in $T^*(T^*X)$ where

$$\zeta \in T^*_{(x,\xi)}(T^*X) \Longrightarrow \zeta = \sum_{j=1}^n \Xi_j dx_j + \sum_{j=1}^n X_j d\xi_j. \tag{9.193}$$

In these local coordinates

$$\alpha = \sum_{j=1}^n \xi_j dx_j! \tag{9.194}$$

The first point is that α is independent of the original choice of coordinates, as is evident from (9.191).

LEMMA 9.10. *A submanifold* $\Lambda \subset T^*X\backslash 0$ *is, near* $(\bar{x},\bar{\xi}) \in \Lambda$, *of the form* (9.190) *for some* $h \in \mathcal{C}^\infty(X)$, *if*

$$\pi : \Lambda \longrightarrow X \text{ is a local diffeomorphism} \tag{9.195}$$

and

$$\alpha \text{ restricted to } \Lambda \text{ is exact.} \tag{9.196}$$

PROOF. The first condition, (9.195), means that Λ is locally the image of a section of T^*X :

$$\Lambda = \{(x,\zeta(x)), \zeta \in \mathcal{C}^\infty(X;T^*X)\}. \tag{9.197}$$

Thus the section ζ gives an inverse Z to π in (9.195). It follows from (9.191) that

$$Z^*\alpha = \zeta. \tag{9.198}$$

Thus if α is exact on Λ then ζ is exact on X, $\zeta = dh$ as required. □

Of course if we are only working locally near some point $(\bar{x},\bar{\xi}) \in \Lambda$ then (9.196) can be replaced by the condition

$$\omega = d\alpha = 0 \text{ on } X. \tag{9.199}$$

Here $\omega = d\alpha$ is the symplectic form on T^*X :

$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j. \tag{9.200}$$

DEFINITION 9.5. *A submanifold* $\Lambda \subset T^*X$ *of dimension equal to that of* X *is said to be* Lagrangian *if the fundamental* 2*-form,* ω*, vanishes when pulled back to* Λ.

By definition a symplectic manifold is a $\mathcal{C}^\infty$ manifold S with a $\mathcal{C}^\infty$ 2-form $\omega \in \mathcal{C}^\infty(S;\Lambda^2)$ fixed satisfying two constraints

$$d\omega = 0 \tag{9.201}$$

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}} \neq 0 \qquad \dim S = 2n. \tag{9.202}$$

A particularly simple example of a symplectic manifold is a real vector space, necessarily of even dimension, with a non-degenerate antisymmetric 2-form:

$$\begin{cases} \omega : E \times E \longrightarrow \mathbb{R} \\ \tilde{\omega} : E \longleftrightarrow E^*. \end{cases} \tag{9.203}$$

Here $\tilde{\omega}(v)(w) = \omega(v,w)\ \forall\ w \in E$. Now (9.201) is trivially true if we think of ω as a translation-invariant 2-form on E, thought of as a manifold.

Then a subspace $V \subset E$ is Lagrangian if

$$\begin{gathered} \omega(v,w) = 0\ \forall\ v,w \in V \\ 2\dim V = \dim E. \end{gathered} \tag{9.204}$$

Of course the point of looking at symplectic vector spaces and Lagrangian subspaces is:

LEMMA 9.11. *If* S *is a symplectic manifold then* T_zS *is a symplectic vector space for each* $z \in S$. *A submanifold* $\Lambda \subset S$ *is Lagrangian iff* $T_z\Lambda \subset T_zS$ *is a Lagrangian subspace* $\forall\ z \in \Lambda$.

We can treat ω, the antisymmetric 2-form on E, as though it were a Euclidean inner product, at least in some regards! Thus if $W \subset E$ is any subspace set

$$W^\omega = \{v \in E; \omega(v, w) = 0 \ \forall \ w \in W\}. \tag{9.205}$$

LEMMA 9.12. *If $W \subset E$ is a linear subspace of a symplectic vector space then* $\dim W^\omega + \dim W = \dim E$; W *is Lagrangian if and only if*

$$W^\omega = W. \tag{9.206}$$

PROOF. Let $W^0 \subset E^*$ be the usual annihilator:

$$W^0 = \{\alpha \in E^*; \alpha(v) = 0 \ \forall \ v \in W\}. \tag{9.207}$$

Then $\dim W^0 = \dim E - \dim W$. Observe that

$$\tilde{\omega} : W^\omega \longleftrightarrow W^0. \tag{9.208}$$

Indeed if $\alpha \in W^0$ and $\tilde{\omega}(v) = \alpha$ then

$$\alpha(w) = \tilde{\omega}(v)(w) = \omega(v, w) = 0 \ \forall \ w \in W \tag{9.209}$$

implies that $v \in W^\omega$. Conversely if $v \in W^\omega$ then $\alpha = \tilde{\omega}(v) \in W^0$. Thus $\dim W^\omega + \dim W = \dim E$.

Now if W is Lagrangian then $\alpha = \tilde{\omega}(w), w \in W$ implies

$$\alpha(v) = \tilde{\omega}(w)(v) = \omega(w, v) = 0 \ \forall \ v \in w. \tag{9.210}$$

Thus $\tilde{\omega}(W) \subset W^0 \Longrightarrow W \subset W^\omega$, by (9.208), and since $\dim W = \dim W^\omega$, (9.206) holds. The converse follows similarly. □

The "lifting" isomorphism $\tilde{\omega} : E \longleftrightarrow E^*$ for a symplectic vector space is like the Euclidean identification of vectors and covectors, but "twisted". It is of fundamental importance, so we give it several names! Suppose that S is a symplectic manifold. Then

$$\tilde{\omega}_z : T_z S \longleftrightarrow T_z^* S \ \forall \ z \in S. \tag{9.211}$$

This means that we can associate (by the inverse of (9.211)) a vector field with each 1-form. We write this relation as

$$\begin{gathered} H_\gamma \in \mathcal{C}^\infty(S; TS) \text{ if } \gamma \in \mathcal{C}^\infty(S; T^*S) \text{ and} \\ \tilde{\omega}_z(H_\gamma) = \gamma \ \forall \ z \in S. \end{gathered} \tag{9.212}$$

Of particular importance is the case $\gamma = df, f \in \mathcal{C}^\infty(S)$. Then H_{df} is written H_f and called the Hamilton vector field of f. From (9.212)

$$\omega(H_f, v) = df(v) = vf \ \forall \ v \in T_z S, \ \forall \ z \in S. \tag{9.213}$$

The identity (9.213) implies one important thing immediately:

$$H_f f \equiv 0 \ \forall \ f \in \mathcal{C}^\infty(S) \tag{9.214}$$

since

$$H_f f = df(H_f) = \omega(H_f, H_f) = 0 \tag{9.215}$$

by the antisymmetry of ω. We need a generalization of this:

LEMMA 9.13. *Suppose $L \subset S$ is a Lagrangian submanifold of a symplectic manifold then for each $f \in \mathcal{I}(S) = \{f \in \mathcal{C}^\infty(X); f \restriction \{s = 0\}$, H_f is tangent to Λ.*

PROOF. H_f tangent to Λ means $H_f(z) \in T_z\Lambda \ \forall \ z \in \Lambda$. If $f = 0$ on Λ then $df = 0$ on $T_z\Lambda$, i.e. $df(z) \in (T_z\Lambda)^0 \subset (T_zS) \ \forall \ z \in \Lambda$. By (9.206) the assumption that Λ is Lagrangian means $\tilde{\omega}_z(df(z)) \in T_z\Lambda$, i.e. $H_f(z) \in T_\zeta\Lambda$ as desired. $\square$

This lemma gives us a necessary condition for our construction of a Lagrangian submanifold

$$\Lambda \subset \Sigma(P). \tag{9.216}$$

Namely H_p must be tangent to Λ! We use this to construct Λ as a union of integral curves of H_p. Before thinking about this seriously, let's look for a moment at the conditions we imposed on p, (9.187) and (9.188). If p is real then H_p is real (since ω is real). Notice that

$$\text{If } S = T^*X \text{ then each fibre } T^*_xX \subset T^*X \text{ is Lagrangian }. \tag{9.217}$$

Remember that on $T^*X, \omega = d\alpha, \alpha = \xi \cdot dx$ the canonical 1-form. Thus T^*_xX is just $x =$ const, so $dx = 0$, so $\alpha = 0$ on T^*_xX and hence in particular $\omega = 0$, proving (9.217). This allows us to interpret (9.188) in terms of H_p as

$$(9.188) \longleftrightarrow H_p \text{ is everywhere transversal to the fibres } T^*_xX. \tag{9.218}$$

Now we want to construct a little piece of Lagrangian manifold satisfying (9.216). Suppose $z \in \Sigma(P) \subset T^*X\backslash 0$ and we want to construct a piece of Λ through z. Since $\pi_*(H_p(z)) \neq 0$ we can choose a local coordinate, $t \in \mathcal{C}^\infty(X)$, such that

$$\pi_*(H_p(z))t \neq 0, \text{ i.e. } H_p(\pi^*t)(z) \neq 0. \tag{9.219}$$

Consider the hypersurface through $\pi(z) \in X$,

$$H = \{t = t(z)\} \Longrightarrow \pi(z) \in H. \tag{9.220}$$

Suppose $f \in \mathcal{C}^\infty(H), df(\pi(z)) = 0$. In fact we can choose f so that

$$f = f' \restriction H, f' \in \mathcal{C}^\infty(X),\ df'(\pi(z)) = z \tag{9.221}$$

where $z \in \Xi(P)$ was our chosen base point.

THEOREM 9.2. *(Hamilton-Jacobi) Suppose $p \in \mathcal{C}^\infty(T^*X\backslash 0)$ satisfies (9.187) and (9.188) near $z \in T^*X\backslash 0, H$ is a hypersurface through $\pi(z)$ as in (9.219), (9.216) and $f \in \mathcal{C}^\infty(H)$ satisfies (9.221), then there exists $\tilde{f} \in \mathcal{C}^\infty(X)$ such that*

$$\begin{aligned} \Lambda = \ &\textit{graph } (d\tilde{f}) \subset \Sigma(P) \textit{ near } z \\ &\tilde{f} \restriction H = f \textit{ near } \pi(z) \\ &d\tilde{f}(\pi(z)) = z \end{aligned} \tag{9.222}$$

and any other such solution, $\tilde{f}'$, is equal to $\tilde{f}$ in a neighbourhood of $\pi(z)$.

PROOF. We need to do a bit more work to prove this important theorem, but let us start with the strategy. First notice that $\Lambda \cap \pi^{-1}(H)$ is already determined, near $\pi(z)$.

To see this we have to understand the relationship between $df(h) \in T^*H$ and $d\tilde{f}(h) \in T^*X$, $h \in H, \tilde{f} \restriction H = f$. Observe that $H = \{t = 0\}$ lifts to $T^*_HX \subset T^*X$ a *hypersurface*. By (9.214), H_t is tangent to T^*_HX and non-zero. In local coordinates $t, x, \ldots, x_{n-1}$, the x's in H,

$$H_t = -\frac{\partial}{\partial \tau} \tag{9.223}$$

where $\tau, \xi_1, \dots, \xi_n$ are the dual coordinates. Thus we see that

$$\pi_H : T^*_H X \longrightarrow T^*H \qquad \pi_H(\beta)(v) = \beta(v), v \in T_h H \subset T_h X, \tag{9.224}$$

is projection along ∂_τ. Now starting from $f \in \mathcal{C}^\infty(H)$ we have

$$\Lambda_f \subset T^*H. \tag{9.225}$$

Notice that if $\tilde{f} \in \mathcal{C}^\infty(X), \tilde{f}\big|H = f$ then

$$\Lambda_{\tilde{f}} \cap T^*_H X \text{ has dimension } n-1 \tag{9.226}$$

and

$$\pi_H(\Lambda_{\tilde{f}} \cap T^*_H X) = \Lambda_f. \tag{9.227}$$

The first follows from the fact that $\Lambda_{\tilde{f}}$ is a graph over X and the second from the definition, (9.224). So we find □

Lemma 9.14. *If $z \in \Sigma(P)$ and H is a hypersurface through $\pi(z)$ satisfying* (9.219) *and* (9.220) *then $\pi^P_H : (\Sigma(P) \cap T^*_H X) \longrightarrow T^*H$ is a local diffeomorphism in a neighbourhood z; if* (9.221) *is to hold then*

$$\Lambda_{\tilde{f}} \cap T^*_H X = (\pi^P_H)^{-1}(\Lambda_f) \textit{ near } z. \tag{9.228}$$

Proof. We know that H_p is tangent to $\Sigma(P)$ but, by assumption (9.221) is *not* tangent to $T^*_H X$ at z. Then $\Sigma(P) \cap T^*_H X$ does have dimension $2n-1-1 = 2(n-1)$. Moreover $\pi_H$is projection along ∂_τ which cannot be tangent to $\Sigma(P) \cap T^*_H X$ (since it would be tangent to $\Sigma(P)$). Thus π^P_H has injective differential, hence is a local isomorphism.

So this is our strategy:

Start with $f \in \mathcal{C}^\infty(H)$, look at $\Lambda_f \subset T^*H$, lift to $\Lambda \cap T^*_H X \subset \Sigma(P)$ by π^P_H. Now let

$$\Lambda = \bigcup \{H_p - \text{curves through } (\pi^P_H)^{-1}(\Lambda_f)\}. \tag{9.229}$$

This we will show to be Lagrangian and of the form $\Lambda_{\tilde{f}}$, it follows that

$$p(x, d\tilde{f}) = 0, \tilde{f} \restriction H = f. \tag{9.230}$$

□

9.7. Riemann metrics and quantization

Metrics, geodesic flow, Riemannian normal form, Riemann-Weyl quantization.

9.8. Transport equation

The first thing we need to do is to finish the construction of characteristic hypersurfaces using Hamilton-Jacobi theory, i.e. prove Theorem XIX.37. We have already defined the submanifold Λ as follows:

1) We choose $z \in \Sigma(P)$ and $t \in \mathcal{C}^\infty(X)$ s.t. $H_p \pi^*(t) \neq 0$ at dz, then selected $f \in \mathcal{C}^\infty(H), H = \{t = 0\} \cap \Omega, \Omega \ni \pi z$ s.t.

$$z(v) = df(v) \ \forall\ v \in T_{\pi z} H. \tag{9.231}$$

Then we consider

$$\Lambda_f = \text{ graph}\{df\} = \{(x, df(x)), x \in H\} \subset T^*H \tag{9.232}$$

as our "initial data" for Λ. To move it into $\Sigma(P)$ we noted that the map

$$\Sigma(P) \cap \underset{\{t=0 \text{ in } T^*X\}}{\underset{\|}{T^*_H X}} \longrightarrow T^*H \tag{9.233}$$

is a local diffeomorphism near z, $df(\pi(z))$ by (9.231). The inverse image of Λ_f in (9.233) is therefore a submanifold $\tilde{\Lambda}_f \subset \Sigma(p) \cap T^*_H X$ of dimension $\dim X - 1 = \dim H$. We define

$$\Lambda = \bigcup \big\{ H_p - \text{ curves of length } \epsilon \text{ starting on } \tilde{\Lambda}_f \big\}. \tag{9.234}$$

So we already know:

$$\Lambda \subset \Sigma(P) \text{ is a manifold of dimension } n, \tag{9.235}$$

and

$$\pi : \Lambda \longrightarrow X \text{ is a local diffeomorphism near } n, \tag{9.236}$$

What we need to know most of all is that

$$\Lambda \text{ is Lagrangian.} \tag{9.237}$$

That is, we need to show that the symplectic two form vanishes identically on $T_{z'}\Lambda$, $\forall\ z' \in \Lambda$ (at least near z). First we check this at z itself! Now

$$T_z\Lambda = T_z\tilde{\Lambda}_f + \text{ sp}(H_p). \tag{9.238}$$

Suppose $v \in T_z\tilde{\Lambda}_f$, then

$$\omega(v, H_p) = -dp(v) = 0 \text{ since } \tilde{\Lambda}_f \subset \Sigma(P). \tag{9.239}$$

Of course $\omega(H_p, H_p) = 0$ so it is enough to consider

$$\omega | (T_z\tilde{\Lambda}_f \times T_z\tilde{\Lambda}_f). \tag{9.240}$$

Recall from our discussion of the projection (9.233) that we can write it as projection along ∂_τ. Thus

$$\begin{gathered} \omega_X(v, w) = \omega_H(v', w') \text{ if } v, w \in T_z(T_H X), \\ (c^*_H)_* v = v' (c^*_H)_* w = w' \in T_z(T^*H) \end{gathered} \tag{9.241}$$

where $z = df(\pi(z))$. Thus the form (9.240) vanishes identically because Λ_f is Lagrangian.

In fact the same argument applies at every point of the initial surface $\tilde{\Lambda}_f \subset \Lambda$:

$$T_{z'}\Lambda \text{ is Lagrangian } \ \forall\ z' \in \tilde{\Lambda}_f. \tag{9.242}$$

To extend this result out into Λ we need to use a little more differential geometry. Consider the local diffeomorphisms obtained by exponentiating H_p :

$$\exp(\epsilon H_p)(\Lambda \cap \Omega) \subset \Lambda\ \forall\ \epsilon \text{ small, } \Omega \ni z \text{ small.} \tag{9.243}$$

This indeed is really the definition of Λ_j more precisely,

$$\Lambda = \bigcup_{\epsilon \text{ small}} \exp(\epsilon H_p)(\tilde{\Lambda}_f). \tag{9.244}$$

The main thing to observe is that, on T^*H, the local diffeomorphisms $\exp(\epsilon H_p)$ are *symplectic*:

$$\exp(\epsilon H_p)^* \omega_X = \omega_X. \tag{9.245}$$

Clearly (9.245), (9.243) and (9.242) prove (9.237). The most elegant wary to prove (9.245) is to use Cartan's identity (valid for H_p any vector field, ω any form)

$$\frac{d}{d\epsilon}\exp(\epsilon H_p)^*\omega = \exp(\epsilon H_p)^*(\mathcal{L}_{H_p}\omega) \tag{9.246}$$

where the Lie derivative is given explicitly by

$$\mathcal{L}_V = d\circ \iota_V + \iota_V \circ d, \tag{9.247}$$

c_V being contradiction with V (i.e. $\alpha(\cdot,\cdot,\dots)\longrightarrow \alpha(V,\cdot,\cdot,\dots)$). Thus

$$\mathcal{L}_{H_p}\omega = d(\omega(H_p,\cdot)) + \iota_V(\underset{\substack{\| \\ 0}}{d\omega}) = d(dp) = 0. \tag{9.248}$$

Thus from (9.235), (9.236) and (9.237) we know that

$$\Lambda = \text{ graph}(d\tilde{f}), \tilde{f}\in \mathcal{C}^\infty(X), \text{ near } \pi(z), \tag{9.249}$$

must satisfy the eikonal equation

$$p(x, d\tilde{f}(x)) = 0 \text{ near } \pi(z), H\tilde{f}\restriction H = f \tag{9.250}$$

where we may actually have to add a constant to $\tilde{f}$ to get the initial condition – since we only have $d\tilde{f} = df$ on TH.

So now we can return to the construction of travelling waves: We want to find

$$u \in I^*(X, G; \Omega^{\frac12}) \qquad G = \{f = 0\} \tag{9.251}$$

such that u is elliptic at $z\in\Sigma(p)$ and

$$Pu \in \mathcal{C}^\infty(X). \tag{9.252}$$

So far we have noticed that

$$\sigma_{m+M}(Pu) = \sigma_m(P)\restriction N^*G\cdot\sigma(u) \tag{9.253}$$

so that

$$N^*G\subset\Sigma(p) \Longleftrightarrow p(x, df) = 0 \text{ on } f = 0 \tag{9.254}$$

implies

$$Pu \in I^{m+M-1}(X, G; \Omega^{\frac12}) \text{ near } \pi(z) \tag{9.255}$$

which is one order smoother than without (9.254).

It is now clear, I hope, that we need to make the "next symbol" vanish as well, i.e. we want

$$\sigma_{m+M-1}(Pu) = 0. \tag{9.256}$$

Of course to arrange this it helps to know what the symbol is!

Proposition 9.6. *Suppose $P\in\Psi^m(X;\Omega^{\frac12})$ and $G\subset X$ is a $\mathcal{C}^\infty$ hypersurface characteristic for P (i.e. $N^*G\subset\Sigma(P)$) then $\forall\ u\in I^M(X, G; \Omega^{\frac12})$*

$$\sigma_{m+M-1}(Pu) = (-iH_p + a)\sigma_m(u) \tag{9.257}$$

where $a\in S^{m-1}(N^*G)$ and H_p is the Hamilton vector field of $p = \sigma_m(P)$.

Proof. Observe first that the formula makes sense since $\Lambda = N^*G$ is Lagrangian, $\Lambda \subset \Sigma(p)$ implies H_p is tangent to Λ and if p is homogeneous of degree m (which we are implicitly assuming) then

$$\mathcal{L}_{H_p} : S^r(\Lambda;\Omega^{\frac{1}{2}}) \longrightarrow S^{r+m-1}(\Lambda;\Omega^{\frac{1}{2}}) \ \forall\ m \tag{9.258}$$

where one can ignore the half-density terms. So suppose $G = \{x_1 = 0\}$ locally, which we can always arrange by choice of coordinates. Then

$$X = N^*G = \{(0, x', \xi_1, 0) \in T^*X\}. \tag{9.259}$$

To say $N^*G \subset \Sigma(p)$ means $p = 0$ on Λ, i.e.

$$p = x_1 q(x,\xi) + \sum_{j>1} \xi_j p_j(x,\xi) \text{ near } z \tag{9.260}$$

with q homogeneous of degree m and the p_j homogeneous of degree $m-1$. Working microlocally we can choose $Q \in \Psi^m(X,\Omega^{\frac{1}{2}}), P_j \in \Psi^{m-1}(X,\Omega^{\frac{1}{2}})$ with

$$\sigma_m(Q) = q, \sigma_{m-1}(P_j) = p_j \text{ near } z. \tag{9.261}$$

Then, from (9.260)

$$P = x_1 Q + D_{x_j} P_j + R + P', \quad R \in \Psi^{m-1}(X;\Omega^{\frac{1}{2}}) z \notin WF'(P'), P' \in \Psi^m(X,\Omega^{\frac{1}{2}}). \tag{9.262}$$

Of course P' does not affect the symbol near z so we only need observe that

$$\begin{aligned} &\sigma_{r-1}(x,u) = -d_{\xi_1}\sigma_r(u) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\ u \in I^r(X,G;\Omega^{\frac{1}{2}}) \\ &\sigma_r(D_{x_j}u) = D_{x_j}\sigma_r(u). \end{aligned} \tag{9.263}$$

This follows from the local expression

$$u(x) = (2\pi)^{-1} \int e^{ix_1\xi_1} a(x',\xi_1) d\xi_1. \tag{9.264}$$

Then from (9.262) we get

$$\begin{aligned} \sigma_{m+M-1}(Pu) &= -D_{\xi_1}(q\sigma_M(u)) + \sum_j D_{x_j}(p_j\sigma_M(u)) + r\cdot\sigma_m(u) \\ &= -i\left(\sum_{j>1} p_j \restriction \Lambda \frac{\partial}{\partial x_j} - q \restriction \Lambda \frac{\partial}{\partial \xi_i}\right)\sigma_M(u) + a'\sigma_M(u). \end{aligned} \tag{9.265}$$

Observe from (9.260) that the Hamilton vector field of p, at $x_1 = \xi' = 0$ is just the expression in parenthesis. This proves (9.257). $\square$

So, now we can solve (9.256). We just set

$$\sigma_M(u)(\exp(\epsilon H_p)z') = e^{i\epsilon A}\exp(\epsilon H_p)^*[b] \ \forall\ z' \in \tilde{\Lambda}_f = \Lambda \cap \{t = 0\}. \tag{9.266}$$

where A is the solution of

$$H_p A = a,\ A \restriction t = 0 = 0 \quad \text{ on } \Lambda_0 \tag{9.267}$$

and $b \in S^r(\Lambda_0)$ is a symbol defined on $\Lambda_0 = \Lambda \cap \{t = 0\}$ near z.

PROPOSITION 9.7. *Suppose $P \in \Psi^m(X;\Omega^{\frac12})$ has homogeneous principal symbol of degree m satisfying*

$$p = \sigma_m(P) \text{ is real} \tag{9.268}$$

$$d_{\text{fibre}}\, p \neq 0 \text{ on } p = 0 \tag{9.269}$$

and $z \in \Sigma(p)$ is fixed. Then if $H \ni \pi(z)$ is a hypersurface such that $\pi_*(H_p) \cap H$ and $G \subset H$ is an hypersurface in H s.t.

$$\bar{z} = c_H^*(z) \in H^*_{\pi z} G \tag{9.270}$$

there exist a characteristic hypersurface $\tilde{G} \subset X$ for P such that $\tilde{G} \cap H = G$ near $\pi(z)$, $z \in N^*_{\pi z}\tilde{G}$. For each

$$u_0 \in I^{m+\frac14}(H, G; \Omega^{\frac12}) \text{ with } WF(u_0) \subset \gamma, \tag{9.271}$$

γ a fixed small conic neighbourhood of $\bar{z}$ n T^*H there exists

$$u \in I(X, \tilde{G}; \Omega^{\frac12}) \text{ satisfying} \tag{9.272}$$

$$u \restriction G = u_0 \text{ near } \pi z \in H \tag{9.273}$$

$$Pu \in \mathcal{C}^\infty \text{ near } \pi z \in X. \tag{9.274}$$

PROOF. All the stuff about G and $\tilde{G}$ is just Hamilton-Jacobi theory. We can take the symbol of u_0 to be the b in (9.266), once we think a little about half-densities, and thereby expect (9.273) and (9.274) to hold, modulo certain singularities. Indeed, we would get

$$u_1 \restriction G - u_0 \in I^{r+\frac14-1}(H, G; \Omega^{\frac12}) \text{ near } \pi z \in H \tag{9.275}$$

$$Pu \in I^{r+m-2}(X, \tilde{G}; \Omega^{\frac12}) \text{ near } \pi z \in X. \tag{9.276}$$

So we have to work a little to remove lower order terms. Let me do this informally, without worrying too much about (9.273) for a moment. In fact I will put (9.275) into the exercises!

All we really have to observe to improve (9.276) to (9.274) is that

$$\begin{gathered} g \in I^r(X, \tilde{G}; \Omega^{\frac12}) \Longrightarrow \exists \quad u \in I^{r+m-1}(X; \tilde{G}; \Omega^{\frac12}) \\ \text{s.t.} \quad Pu - g \in I^{r-1}(X, \tilde{G}; \Omega^{\frac12}) \end{gathered} \tag{9.277}$$

which we can then iterate and asymptotically sum. In fact we can choose the solution so $u \restriction H \in \mathcal{C}^\infty$, near $\pi\bar{z}$. To solve (9.277) we just have to be able to solve

$$-i(H_p + a)\sigma(u) = \sigma(g) \tag{9.278}$$

which we can do by integration (duHamel's principle). □

The equation (9.278) for the symbol of the solution is the transport equation. We shall use this construction next time to produce a microlocal parametrix for P!

9.9. Problems

PROBLEM 9.2. Let X be a $\mathcal{C}^\infty$ manifold, $G \subset X$ on $\mathcal{C}^\infty$ hypersurface and $t \in \mathcal{C}^\infty(X)$ a real-valued function such that

$$dt \neq 0 \text{ on } T_pG \; \forall \; p \in L = G \cap \{t = 0\}. \tag{9.279}$$

Show that the transversality condition (9.279) ensures that $H = \{t = 0\}$ and $L = H \cap G$ are both $\mathcal{C}^\infty$ submanifolds.

PROBLEM 9.3. Assuming (9.279) show that dt gives an isomorphism of line bundles

$$\Omega^{\frac{1}{2}}(H) \equiv \Omega_H^{\frac{1}{2}}(X) \sim \Omega_H^{\frac{1}{2}}(X)/|dt|^{\frac{1}{2}} \tag{9.280}$$

and hence one can define a restriction map,

$$\mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(H; \Omega^{\frac{1}{2}}). \tag{9.281}$$

PROBLEM 9.4. Assuming 1 and 2, make sense of the restriction formula

$$\restriction H : I^m\left(X, G; \Omega^{\frac{1}{2}}\right) \longrightarrow I^{m+\frac{1}{4}}\left(H, L; \Omega^{\frac{1}{2}}\right) \tag{9.282}$$

and prove it, and the corresponding symbolic formula

$$\sigma_{m+\frac{1}{4}}\left(u \restriction H\right) = \left(\iota_H^*\right)^* \left(\sigma_m(u) \restriction N_L^* G\right) / |d\tau|^{\frac{1}{2}}. \tag{9.283}$$

NB. Start from local coordinates and try to understand restriction at that level before going after the symbol formula!

9.10. The wave equation

We shall use the construction of travelling wave solutions to produce a parametrix, and then a fundamental solution, for the wave equation. Suppose X is a Riemannian manifold, e.g. $\mathbb{R}^n$ with a 'scattering' metrice:

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j, g_{ij} = \delta_{ij}|x|R. \tag{9.284}$$

Then the associates Laplacian, on functions, i.e.

$$\Delta u = -\sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} (\delta g g^{ij}(x)) \frac{\partial}{\partial x_i} u \tag{9.285}$$

where $g^{ij}(x) = (g_{ij}(x))^{-1}$ and $g = \det g_{ij}$. We are interested in the wave equation

$$Pu = (D_t^2 - \Delta)u = f \qquad \text{on } \mathbb{R} \times X \tag{9.286}$$

For simplicity we assume X is either compact, or $X = \mathbb{R}^n$ with a metric of the form (9.284).

The cotangent bundle of $\mathbb{R} \times X$ is

$$T^*(\mathbb{R} \times X) \simeq T^*\mathbb{R} \times T^*X \tag{9.287}$$

with canonical coordinates (t, x, τ, ξ). In terms of this

$$\sigma(P) = \tau^2 - |\xi|^2 |\xi| = \sum_{i,j=1}^{n} g^{ij}(x) \xi_i \xi_j. \tag{9.288}$$

Thus we certainly have an operator satisfying the conditions of (9.286) and (9.288), since

$$d_{\text{ fibre }} p = \left(\frac{\partial p}{\partial \tau}, \frac{\partial p}{\partial \xi}\right) = 0 \Longrightarrow \tau = 0 \text{ and } g^{ij}(x)\xi_i = 0 \Longrightarrow \xi = 0. \tag{9.289}$$

As initial surface we consider the obvious hypersurface $\{t = 0\}$ (although it will be convenient to consider others). We are after the two theorems, one local and global, the other microlocal, although made to look global.

THEOREM 9.3. *If X is a Riemannian manifold, as above, then for every $f \in \mathcal{C}_c^{-\infty}(\mathbb{R} \times X)$ $\quad \exists!\ u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$ satisfying*

$$Pu = f, u = 0 \textit{ in } t < \inf\{\bar{t}; \quad \exists(\bar{t}, x) \in \operatorname{supp}(f)\}. \tag{9.290}$$

THEOREM 9.4. *If X is a Riemannian manifold, as above, then for every $u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$,*

$$WF(u)\backslash WF(Pu) \subset \Sigma(P)\backslash WF(Pu) \tag{9.291}$$

is a union of maximally extended H_o-curves in the open subset $\Sigma(P)\backslash WF(Pu)$ of $\Sigma(P)$.

Let us think about Theorem 9.3 first. Suppose $\bar{x}X$ is fixed on $\delta_{\bar{x}} \in \mathcal{C}^{-\infty}(X; \Omega)$ is the Dirac delta (g measure) at $\bar{x}$. Ignoring, for a moment, the fact that this is not quite a generalized function we can look for the "forward fundamental solution" of P with pole at $(0, \bar{x})$:

$$\begin{gathered} PE_{\bar{x}}(t, x) = \delta(t)\delta_{\bar{x}}(x) \\ E_{\bar{x}} = 0 \text{ in } t < 0. \end{gathered} \tag{9.292}$$

Theorem 9.3 asserts its existence and uniqueness. Conversely if we can construct $E_{\bar{x}}$ for each $\bar{x}$, and get reasonable dependence on $\bar{x}$ (continuity is almost certain once we prove uniqueness) then

$$K(t, x; \bar{t}, \bar{x}) = E_{\bar{x}}(t - \bar{t}, x) \tag{9.293}$$

is the kernel of the operator $f \mapsto u$ solving (9.290).

So, we want to solve (9.292). First we convert it (without worrying about rigour) to an initial value problem. Namely, suppose we can solve instead

$$\begin{gathered} PG_{\bar{x}}(t, x) = 0 \text{ in } \mathbb{R} \times X \\ G_{\bar{x}}(0, x) = 0,\ D_t G_{\bar{x}}(0, x) = \delta_{\bar{x}}(x) \text{ in } X. \end{gathered} \tag{9.294}$$

Note that

$$(g(t, x, \tau, 0) \notin \Sigma(P) \Longrightarrow (t, x; \tau, 0) \notin WF(G). \tag{9.295}$$

This means the restriction maps, to $t = 0$, in (9.294) are well-defined. In fact so is the product map:

$$E_{\bar{x}}(t, x) = H(t)G_{\bar{x}}(t, x). \tag{9.296}$$

Then if G satisfied (9.294) a simple computation shows that $E_{\bar{x}}$ satisfies (9.292). Thus we want to solve (9.294).

Now (9.294) seems very promising. The initial data, $\delta_{\bar{x}}$, is certainly conormal to the point $\{\bar{x}\}$, so we might try to use our construction of travelling wave solutions. However there is a serious problem. We already noted that, for the wave equation, there cannot be any smooth characteristic surface other than a hypersurface. The point is that if H has codimension k then

$$N^*_{\bar{x}}H \subset T^*_{\bar{x}}(\mathbb{R} \times X) \text{ has dimension } k. \tag{9.297}$$

To be characteristic we must have

$$N^*_{\bar{x}}H \subset \Sigma(P) \Longrightarrow k = 1 \tag{9.298}$$

Since the *only* linear space contained in a (proper) cone is a line.

However we can easily 'guess' what the characteristic surface corresponding to the point $(x, \bar{x})$ is – it is the *cone* through that point:

This certainly takes us *beyond* our conormal theory. Fortunately there is a way around the problem, namely the possibility of superposition of conormal solutions.

To see where this comes from consider the representation in terms of the Fourier transform:

$$\delta(x) = (2\pi)^{-n} \int e^{ix\xi} d\xi. \tag{9.299}$$

The integral of course is not quite a proper one! However introduce polar coordinates $\xi = r\omega$ to get, at least formally

$$\delta(x) = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{irx\cdot\omega} r^{n-1} dr \; d\omega. \tag{9.300}$$

In odd dimensions r^{n-1} is even so we can write

$$\delta(x) = \frac{1}{2(2\pi)^n} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} e^{irx\cdot\omega} r^{n-1} dr \; d\omega, n \text{ odd }. \tag{9.301}$$

Now we can interpret the r integral as a 1-dimensional inverse Fourier transform so that, always formally,

$$\begin{gathered} \delta(x) = \frac{1}{2(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} f_n(x \cdot \omega) d\omega \\ \qquad\qquad\qquad\qquad n \text{ odd} \\ f_n(s) = \frac{1}{(2\pi)} \int e^{irs} \gamma^{n-1} dr. \end{gathered} \tag{9.302}$$

In even dimensions we get the same formula with

$$f_n(s) = \frac{1}{2\pi} \int e^{irs} |r|^{n-1} dr. \tag{9.303}$$

These formulas show that

$$f_n(s) = |D_s|^{n-1} \delta(s). \tag{9.304}$$

Here $|S_s|^{n-1}$ is a pseudodifferential operator for n even or differential operator $(= D_s^{n-1})$ if n is odd. In any case

$$f_n \in I^{n-1+\frac{1}{4}} (\mathbb{R}, \{0\})! \tag{9.305}$$

Now consider the map

$$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x, \omega) \mapsto x \cdot \omega \in \mathbb{R}. \tag{9.306}$$

Thus $\mathcal{C}^\infty$ has different

$$\omega \cdot dx + x \cdot d\omega \neq 0 \text{ or } x \cdot \omega = 0 \tag{9.307}$$

So the inverse image of $\{0\}$ is a smooth hypersurface R.

LEMMA 9.15. *For each $n \geq 2$*

$$f_n(x,\omega) = \frac{1}{2\pi} \int e^{i(x\cdot\omega)r} |r|^{n-1} dr \in I^{\frac{n}{4}-\frac{1}{4}} \left(\mathbb{R} \times \mathbb{S}^{n-1}, R\right). \tag{9.308}$$

PROOF. Replacing $|r|^{n-1}$ by $\rho(r)|r|^{n-1} + (1-\rho(r))|r|^{n-1}$, where $\rho(r) = 0$ n $r < \frac{1}{2}, \rho(r) = 1$ in $r > 1$, expresses f_n as a sum of a $\mathcal{C}^\infty$ term and a conormal distribution. Check the order yourself! □

PROPOSITION 9.8. *(Radon inversion formula). Under pushforward corresponding to $\mathbb{R}^n \times \mathbb{S}^{n-1} @ > \pi_1 >> \mathbb{R}^n$*

$$\begin{gathered} (\pi_1)_* f_n' = 2(2\pi)^{n-1} \delta(x), \\ f_n' = f_n |d\omega||dx|. \end{gathered} \tag{9.309}$$

PROOF. Pair with a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$(\pi_1)_* f_n' = \iint f_n(x\cdot\omega)\phi(x) dx \ d\omega \tag{9.310}$$

by the Fourier inversion formula. □

So now we have a superposition formula expressing $\delta(x)$ as an integral:

$$\delta(x) = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} f_n(x\cdot\omega) d\omega \tag{9.311}$$

where for each fixed ω $f_n(x\cdot\omega)$ is conormal with respect to $x\cdot\omega = 0$. This gives us a strategy to solve (9.294).

PROPOSITION 9.9. *Each $\bar{x} \in X$ has a neighbourhood, $U_{\bar{x}}$, such that for $\bar{t} > 0$ (independent of $\bar{x}$) there are two characteristic hypersurfaces for each $\omega \in \mathbb{S}^{n-1}$*

$$H^{\pm}_{\bar{x},\omega)} \subset (-\bar{t},\bar{t}) \times U_{\bar{x}} \tag{9.312}$$

depending on $\bar{x}, \omega$, and there exists

$$u^{\pm}(t,x;\bar{x},\omega) \in I^*((-\bar{t}|\bar{t}| \times U_{\bar{x}}, H^{\pm}_{(\bar{x},\omega)}) \tag{9.313}$$

such that

$$Pu^{\pm} \in \mathcal{C}^\infty \tag{9.314}$$

$$\begin{cases} u^+ + \bar{u} \restriction t = 0 = \delta_{\bar{x}}(x\cdot\omega) & \text{in } U_{\bar{x}} \\ D_t(u^+ + u^-) \restriction \{t=0\} = 0 & \text{in } U_{\bar{x}}. \end{cases} \tag{9.315}$$

PROOF. The characteristic surfaces are constructed through Hamilton-Jacobi theory:

$$\begin{gathered} N^* H^{\pm} \subset \Sigma(P), \\ H_0 = H^{\pm} \cap \{t=0\} = \{x\cdot\omega = 0\}. \end{gathered} \tag{9.316}$$

There are two or three because the conormal direction to H_0 at 0; ωdx, has two $\Sigma(P)$:

$$\tau = \pm 1,\ (\tau, \omega) \in T_0^*(\mathbb{R} \times X). \tag{9.317}$$

With *each* of these two surfaces we can associate a microlocally unique conormal solution

$$\begin{gathered} Pu^{\pm} = 0, \quad u^{\pm} \restriction \{t = 0\} = u_0^{\pm} \\ u_0^{\pm} \in I^*(\mathbb{R}^n, \{x \cdot \omega = 0\}) \end{gathered} \tag{9.318}$$

Now, it is easy to see that there are unique choices

$$\begin{gathered} u_\delta^+ + u_0^- = \delta(x \cdot \omega) \\ D_t u^+ + D_t u^- \restriction \{t = 0\} = 0. \end{gathered} \tag{9.319}$$

(See exercise 2.) This solves (9.315) and proves the proposition (modulo a fair bit of hard work!).

□

So now we can use the superposition principle. Actually it is better to add the variables ω to the problem and see that

$$\begin{gathered} u^{\pm}(t, x; \omega, \bar{x}) \in I^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n; H^{\pm}) \\ H^{\pm} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n \end{gathered} \tag{9.320}$$

being fixed by the condition that

$$H^{\pm} \cap \mathbb{R} \times \mathbb{R}^n \times \{\omega\} \times \{\bar{x}\} = H^{\pm}_{\bar{x},\omega}. \tag{9.321}$$

Then we set

$$G'_{\bar{x}}(t, x) = \int\limits_{\mathbb{S}^{n-1}} (u^+ + u^-)(x, x; \omega, \bar{x}). \tag{9.322}$$

This satisfies (9.294) locally near $\bar{x}$ and modulo $\mathcal{C}^\infty$. i.e.

$$\begin{cases} PG'_{\bar{x}} \in \mathcal{C}^\infty((-\bar{t}(\bar{t})) \times U_{\bar{x}}) \\ G'_{\bar{x}} \restriction \{t = 0\} = xv, & \\ & v_i \in \mathcal{C}^\infty \\ D_t G'_{\bar{x}} = \delta_{\bar{x}}(x) + v_2 \end{cases} \tag{9.323}$$

Let us finish off by doing a calculation. We have (more or less) shown that $u^{\pm}$ are conormal with respect to the hypersurfaces $H^{\pm}$. A serious question then is, what is (a bound one) the wavefront set of $G'_{\bar{x}}$? This is fairly easy provided we understand the geometry. First, since $u^{\pm}$ are conormal,

$$WF(u^{\pm}) \subset N^* H^{\pm}. \tag{9.324}$$

Then the push-forward theorem says

$$\begin{gathered} WF(G^{\pm}) \subset \{(t, x, \tau, \xi);\ \exists \quad (t, x, \tau, \xi, \omega, w) \in WF(u^{\pm})\} \\ G^{\pm} = (\pi_1)_* u^{\pm} = \int\limits_{\mathbb{S}^{n-1}} u^{\pm}(t, s; \omega, \bar{x}) d\omega \end{gathered} \tag{9.325}$$

so here

$$(t, x, \tau, \xi, \omega, w) \in T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1}) = T^*(\mathbb{R} \times \mathbb{R}^n) \times T^*\mathbb{S}^{n-1}. \tag{9.326}$$

We claim that the singularities of $G'_{\bar{x}}$ lie on a cone:

$$WF(G'_{\bar{x}}) \subset \Lambda_{\bar{x}} \subset T^*(\mathbb{R} \times \mathbb{R}^n) \tag{9.327}$$

where $\Lambda_{\bar{x}}$ is the conormal bundle to a cone:

$$\begin{gathered} \Lambda_{\bar{x}} = \mathrm{cl}\{(t, x; \tau, \xi);\ t \neq 0, D(x, \bar{x}) = \pm t, \\ (\tau, \xi) = \tau(1, \mp d_x D(x, \bar{x})) \end{gathered} \tag{9.328}$$

where $D(x, \bar{x})$ is the Riemannian distance from x to $\bar{x}$.

9.11. Forward fundamental solution

Last time we constructed a local parametrix for the Cauchy problem:

$$\begin{cases} PG'_{\bar{x}} = f \in \mathcal{C}^\infty(\Omega) & (0, \bar{x}) \in \Omega \subset \mathbb{R} \times X \\ G'_{\bar{x}} \restriction t = 0 = u' & \\ D_t G'_{\bar{x}} \restriction \{t = 0\} = \delta_{\bar{x}}(x) + u'' & u', u'' \in \mathcal{C}^\infty(\Omega_0) \end{cases} \tag{9.329}$$

where $P = D_t^2 - \Delta$ is the wave operator for a Riemann metric on X. We also computed the wavefront set, and hence singular support of $G_{\bar{x}}$ and deduced that

$$\operatorname{sing}\cdot\operatorname{supp}.(G_{\bar{x}}) \subset \{(t, x); d(x, \bar{x}) = |t|\} \tag{9.330}$$

in terms of the Riemannian distance.

(9.331)

This allows us to improve (9.329) in a very significant way. First we can chop $G_{\bar{x}}$ off by replacing it by

$$\phi\left(\frac{t^2 - d^2(x, \bar{x})}{\epsilon^2}\right). \tag{9.332}$$

where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has support near 0 and is identically equal to 1 in some neighbourhood of 0. This gives (9.329) again, with $G'_{\bar{x}}$ now supported in say $d^2 < t^2 + \epsilon^2$.

(9.333)

Next we can improve (9.329) a little bit by arranging that

$$u' = u'' = 0,\ D_t^k f \Big|_{t=0} = 0\ \forall\ k. \tag{9.334}$$

This just requires adding to G' a $\mathcal{C}^\infty, v$, function, so that

$$v \Big|_{t=0} = u',\ D_t v \Big|_{t=0} = -u'', \quad D_t^k(Pu) \Big|_{t=0} = -D_t^k f \Big|_{t=0} \quad k > 0. \tag{9.335}$$

Once we have done this we consider

$$E'_{\bar{x}} = iH(t)G'_{\bar{x}} \tag{9.336}$$

which now satisfies

$$\begin{gathered} PE'_{\bar{x}} = \delta(t)\delta_{\bar{t}}(x) + F_{\bar{x}},\ F_{\bar{x}} \in \mathcal{C}^\infty(\Omega_{\bar{x}}) \\ \mathrm{supp}(E'_x) \subset \{d^2(x, \bar{x}) \leq t^2 + \epsilon^2\} \cap \{t \geq 0\}. \end{gathered} \tag{9.337}$$

Here F vanishes in $t < 0$, so vanishes to infinite order at $t = 0$.

Next we remark that we can actually do all this with smooth dependence of $\bar{x}$. This should really be examined properly, but I will not do so to save time. Thus we actually have

$$\begin{cases} E'(t,x,\bar{x}) \in \mathcal{C}^{-\infty}(P(-\infty,\epsilon)\times X\times X) \\ PE' = \delta(t)\sigma_{\bar{x}}(x) + F \\ \operatorname{supp} E' \subset \{d^2(x,\bar{x}) \geq t^2+\epsilon^2\} \cap \{t \geq 0\}. \end{cases} \tag{9.338}$$

We can, and later shall, estimate the wavefront set of E. In case $X = \mathbb{R}^n$ we can take E to be the *exact* forward fundamental solution where $|x|$ or $\bar{x} \geq R$, so

$$\begin{gathered} \operatorname{supp}(F) \subset \{t \geq 0\} \cap \{|x|, |\bar{x}| \leq R\} \cap \{d^2 \leq t^2+\epsilon^2\} \\ F \in \mathcal{C}^\infty((-\infty,\epsilon)\times X\times X). \end{gathered} \tag{9.339}$$

Of course we want to remove F, the error term. We can do this because it is a *Valterra operator*, very similar to an upper triangular metric. Observe first that the operators of the form (9.339) form an algebra under t-convolution:

$$F = F_1 \circ F_1,\ F(t,x,\bar{x}) = \int_0^t \int F_1(t,-t',x,x')F_2(t^1,x^1,\bar{x})dx'dt'. \tag{9.340}$$

In fact if one takes the iterates of a fixed operator

$$F^{(k)} = F^{(k-1)} \circ F \tag{9.341}$$

One finds exponential convergence:

$$\left|D_x^\alpha D_t^p F^{(k)}(t,x,\bar{x})\right| \leq \frac{C^{k+1}N,\delta}{k!}|t|^N \quad \text{ in } t < \epsilon-\delta\ \forall\ N. \tag{9.342}$$

Thus if F is as in (9.339) then $Id+F$ has inverse $Id+\tilde{F}$,

$$\tilde{F} = \sum_{j\geq 1}(-1)^j F^{(j)} \tag{9.343}$$

again of this form.

Next note that the composition of E' with $\tilde{F}$ is again of the form (9.339), with R increased. Thus

$$E = E' + E' \circ F \tag{9.344}$$

is a forward fundamental solution, satisfying (9.338) with $F \equiv 0$.

In fact E is also a left parametrix, in an appropriate sense:

Proposition 9.10. *Suppose $u \in \mathcal{C}^{-\infty}((-\infty,\epsilon)\times X)$ is such that*

$$\operatorname{supp}(u) \cap [-T,\tau]\times X \textit{ is compact } \forall\ T \textit{ and for } \tau < \epsilon \tag{9.345}$$

then $Pu = 0 \Longrightarrow u = 0$.

Proof. The trick is to make sense of the formula

$$0 = E \cdot Pu = u. \tag{9.346}$$

In fact the operators G with kernel $G(t,x,\bar{x})$, defined in $t < \epsilon$ and such that $G * \phi \subset \mathcal{C}^\infty\ \forall\ \phi \in \mathcal{C}^\infty$ and

$$\{t \geq 0\} \cap \{d(x,\bar{x}) \leq R\} \supset \operatorname{supp}(G) \tag{9.347}$$

act on the space (9.345) as t-convolution operators. For this algebra $E * P = \mathrm{Id}$ so (9.346) holds! $\square$

We can use this proposition to prove that E itself is unique. Actually we want to do more.

THEOREM 9.5. *If X is either a compact Riemann manifold or $\mathbb{R}^n$ with a scattering metric then P has a unique forward fundamental solution, ω.*

$$\operatorname{supp}(E) \subset \{t \geq 0\}, P^E = \mathrm{Id} \tag{9.348}$$

and

$$\operatorname{supp}(E) \subset \{(t, x, \bar{x}) \in \mathbb{R} \times X \times X; d(x, \bar{x}) \leq t\} \tag{9.349}$$

and further

$$WF'(E) \subset \mathrm{Id} \cup \mathcal{F}_+ \tag{9.350}$$

where $\mathcal{F}_+$ is the forward bicharacteristic relation on $T^*(\mathbb{R} \times X)$

$$\begin{gathered} \zeta = (t, x, \tau, \xi) \notin \Sigma(P) \Longrightarrow \mathcal{F}_+(\zeta) = \emptyset \\ \zeta = (t, x, \tau, \xi) \in \Sigma(P) \Longrightarrow \mathcal{F}_+(\zeta) = \{\zeta' = (t', x', \tau', \xi') \\ t' \geq t \times \zeta' = \exp(TH_p)\zeta \text{ for some } T\}. \end{gathered} \tag{9.351}$$

PROOF. (1) Use E_1 defined in $(-\infty, \epsilon \times X$ to continue E globally.
(2) Use the freedom of choice of $\{t = 0\}$ and uniqueness of E to show that (9.349)can be arranged for small, and hence all,
(3) Then get (9.351) by checking the wavefront set of G.

□

As corollary we get proofs of (9.333) and (9.334).

PROOF OF THEOREM XXI.5.

$$u(t, x) = \int E(t - t', x, x') f(t', x') dx' dt'. \tag{9.352}$$

□

PROOF OF THEOREM XXI.6. We have to show that if both $\mathrm{WF}(Pu) \not\ni z$ and $\mathrm{WF}(u) \not\ni z$ then $\exp(\delta H_p)z \notin WF(u)$ for small δ. The general case that follows from the (assumed) connectedness of H_p curves. This involves microlocal uniqueness of solutions of $Pu = f$. Thus if $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has support in $t > -\delta$, for $\delta > 0$ small enough, $\pi^* t(z) = \bar{t}$

$$P(\phi(t - \bar{t})u) = g \text{ has } z \notin WF(g), \tag{9.353}$$

and vanishes in $t < \delta$. Then

$$\begin{gathered} \phi(t - \bar{t})u = E \times g \\ \Longrightarrow \exp(\tau H_p)(z) \notin WF(u) \text{ for small } \tau. \end{gathered} \tag{9.354}$$

□

9.12. Operations on conormal distributions

I want to review and refine the push-forward theorem, in the general case, to give rather precise results in the conormal setting. Thus, suppose we have a projection

$$X \times Y @ > x >> X \tag{9.355}$$

where we can view $X \times Y$ as compact manifolds or Euclidean spaces as desired, since we actually work locally. Suppose

$$Q \subset X \times Y \text{ is an embeded submanifold.} \tag{9.356}$$

Then we know how to define and examine the *conormal* distribution associated to Q. If

$$u \in I^m(X \times Y, Q; \Omega) \tag{9.357}$$

when is $\pi_*(u) \in \mathcal{C}^{-\infty}(X; \Omega)$ conormal? The obvious thing we ned is a submanifold with respect to what it should be conormal! From our earlier theorem we know that

$$WF(\pi_*(u)) \subset \{(x,\xi); \quad \exists \quad (x,\xi,y,0) \in WF(u) \subset N^*Q\}. \tag{9.358}$$

So suppose $Q = \{q_j(x,y) = 0, j = 1, \dots, k\}$, $k = \operatorname{codim} Q$. Then we see that

$$(\bar{x}, \bar{\xi}, \bar{y}, 0) \in N^*Q \Longleftrightarrow (\bar{x}, \bar{y}) \in Q, \bar{\xi} = \sum_{j=1}^{k} \tau_j d_x q_j, \sum_{j=1}^{k} \tau_j dy q_j = 0. \tag{9.359}$$

Suppose for a moment that Q has a hypersurface, i.e. $k = 1$, and that

$$Q \longrightarrow \pi(Q) \text{ is a fibration} \tag{9.360}$$

then we expect

Theorem 9.6. $\pi_* : I^m(X \times Y, Q, \Omega) \longrightarrow I^{m'}(X, \pi(Q))$.

Proof. Choose local coordinates so that

$$Q = \{x_1 = 0\} \tag{9.361}$$

$$u = \frac{1}{2\pi} \int e^{ix_1\xi_1} a(x', y, \xi_1) d\xi_1 \tag{9.362}$$

$$\pi^* u = \frac{1}{2\pi} \int e^{ix_1\xi_1} b(x', \xi_1) d\xi_1 \tag{9.363}$$

$$b = \int a(x', y, \xi) dy. \tag{9.364}$$

□

Next consider the case of restriction to a submanifold. Again let us suppose $Q \subset X$ is a hypersurface and $Y \subset X$ is an embedded submanifold *transversal* to Q :

$$\begin{aligned} &Q \pitchfork Y = QY \\ &\text{i.e. } T_qQ + T_qY = T_qX \quad \forall\ q \in Qy \\ &\Longrightarrow Q_y \qquad \text{is a hypersurface in } X. \end{aligned} \tag{9.365}$$

Indeed locally we can take coordinates in which

$$Q = \{x_1 = 0\}, Y = \{x'' = 0\}, \quad x = (x_1, x', x''). \tag{9.366}$$

THEOREM 9.7.

$$C_Y^* : I^m(X,Q) \longrightarrow I^{m+\frac{k}{4}}(Y,Q_Y)k = \operatorname{codim} Y \ \textit{in } X. \tag{9.367}$$

PROOF. In local coordinates as in (9.366)

$$\begin{aligned} u &= \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x(x',x'',\xi_1))d\xi, \\ c^*u &= \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x',0,\xi_1)d\xi_1. \end{aligned} \tag{9.368}$$

Now let's apply this to the fundamental solution of the wave equation. Well rather consider the solution of the initial value problem

$$\begin{cases} PG(t,x,\bar{x}) = 0 \\ G(0,x,\bar{x}) = 0 \\ D_tG(0,x,\bar{x}) = \delta_{\bar{x}}(x). \end{cases} \tag{9.369}$$

We know that G exists for *all* time and that for short time it is

$$G - \int\limits_{\mathbb{S}^{n-1}} (u_+(t,x,\bar{x};\omega) + u_-(t,x,\bar{x};\omega))d\omega + \mathcal{C}^\infty \tag{9.370}$$

where $u_\pm$ are conormal for the term characteristic hypersurfaces H_p satisfying

$$\begin{aligned} N^*H_\pm &\subset \Sigma(P) \\ H_\pm \cap \{t=0\} &= \{(x-\bar{x})\cdot\omega = 0\} \end{aligned} \tag{9.371}$$

Consider the 2×2 matrix of distribution

$$U(t) = \begin{pmatrix} D_tG & G \\ D_t^2G & D_tG \end{pmatrix}. \tag{9.372}$$

Since $WFU \subset \Sigma(P)$, in polar $\tau \neq 0$ we can consider this as a smooth function of t, with values in distribution on $X \times X$. □

THEOREM 9.8. *For each $t \in \mathbb{R}$ $U(t)$ is a boundary operator on $L^2(X) \oplus H'(X)$ such that*

$$U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t) \\ D_tu(t) \end{pmatrix} \tag{9.373}$$

where $u(t,x)$ is the unique solution of

$$\begin{aligned} (D_t^2 - \Delta)u(t) &= 0 \\ u(0) &= u_0 \\ D_t + u(0) &= u_1. \end{aligned} \tag{9.374}$$

PROOF. Just check it! □

Consider again the formula (9.370). First notice that at $x = \bar{x}$, $t=0$, $dH^\pm = dt \pm d(x-\bar{x})\omega)$ (by construction). so

$$H_\pm \pitchfork \{x=\bar{x}\} = \{t=0\} \subset \mathbb{R}\times X \hookrightarrow \mathbb{R}\times X\times Y\times\mathbb{S}^{n-1}. \tag{9.375}$$

Moreover the projection

$$\mathbb{R}\times X\times\mathbb{S}^{n-1} \longrightarrow \mathbb{R} \tag{9.376}$$

clearly fibres $\{t = 0\}$ over $\{t = 0\} \in= \{0\} \subset \mathbb{R}$. Then we can apply the two theorems, on push-forward and pull-back, above to conclude that

$$T(t) = \int_X G(t, x, \bar{x}) \upharpoonright x = \bar{x} dx \in \mathcal{C}^{-\infty}(\mathbb{R}) \tag{9.377}$$

is conormal near $t = 0$ i.e. $\mathcal{C}^\infty$ in $(-\epsilon, \epsilon)\backslash\{0\}$ for some $\epsilon > 0$ and conormal at 0. Moreover, we can, at least in principle, work at the *symbol* of $T(t)$ at $t = 0$. We return to this point next time.

For the moment let us think of a more 'fundamental analytic' interpretation of (9.377). By this I mean

$$T(t) = trU(t). \tag{9.378}$$

REMARK 9.1. Trace class operators $\Delta\lambda$; Smoother operators are trace order, $tr = \int K(x, x)$

$$\int U(t)\phi(t) \text{ is smoothing} \tag{9.379}$$

$$\langle T(t), \phi(t)\rangle = tr\langle U(t), \phi(t)\rangle. \tag{9.380}$$

9.13. Weyl asymptotics

Let us summarize what we showed last time, and a little more, concerning the trace of the wave group

PROPOSITION 9.11. *Let X be a compact Riemann manifold and $U(t)$ the wave group, so*

$$U(t) : \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \ni (u_0, u_1) \mapsto (u, (t), D + tu(t)) \in \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \tag{9.381}$$

where u is the solution to

$$\begin{aligned} (D_t^2 - \Delta)u(t) &= 0 \\ u(0) &= u_0 \\ D_t u(0) &= u_1. \end{aligned} \tag{9.382}$$

The trace of the wave group, $T \in \mathcal{S}'(\mathbb{R})$, is well-defined by

$$T(\phi) = \operatorname{Tr} U(\phi), U(\phi) = \int U(t)\phi(t)dt \ \forall \ \phi \in \mathcal{S}(\mathbb{R}) \tag{9.383}$$

and satisfies

$$T = Y(\left(1 + \sum_{j=1}^{\infty} 2\cos(t\lambda_j)\right) \tag{9.384}$$

$$\textit{where } 0 = \lambda_0 < \lambda_1^2 \le \lambda_2^2 \dots \qquad \lambda_j \ge 0 \tag{9.385}$$

is the spectrum of the Laplacian repeated with multiplicity

$$\operatorname{sing.supp}(T) \subset \mathcal{L} \cup \{0\} \cup -\mathcal{L} \tag{9.386}$$

where $\mathcal{L}$ is the set of lengthes of closed geodesics of X and

$$\begin{gathered} \textit{if } \psi \in \mathcal{C}_c^\infty(\mathbb{R}), \psi(t) = 0 \textit{ if } |t| \ge \textit{ inf } \mathcal{L} - \epsilon, \epsilon > 0, \\ \psi T \in I(\mathbb{R}, \{0\}) \\ \sigma(\psi T) = \end{gathered} \tag{9.387}$$

Proof. We have already discussed (9.384) and the first part of (9.387) (given (9.386)). Thus we need to show (9.386), the *Poisson relation*, and compute the symbol of T as a cononormal distribution at 0 .

Let us recall that if G is the solution to

$$(D_t^2 = \Delta)G(t,x,\bar{x}) = 0 \\ G(0,x,\bar{x}) = 0 \\ D_tG(0,x,\bar{x}) = \delta_{\bar{x}}(x) \tag{9.388}$$

then

$$T = \pi_*(\iota_\Delta^* 2D_tG), \tag{9.389}$$

where

$$\iota_\Delta : \mathbb{R} \times X \hookrightarrow \mathbb{R} \times X \times X \tag{9.390}$$

is the embedding of the diagonal and

$$\pi : \mathbb{R} \times X \longrightarrow \mathbb{R} \tag{9.391}$$

is projective. We also know about the wavefront set of G. That is,

$$WF(G) \subset \{(t,x,\bar{x},\tau,\xi,\bar{\xi}); \tau^2 = |\xi|^2 = |\bar{\xi}|^2, \\ \exp(sH_p)(0,\bar{x},\tau,\bar{\xi}) = (t,x,\tau,\xi),\ \text{some } s\}. \tag{9.392}$$

Let us see what (9.392) says about the wavefront set of T. First under the restriction map to $\mathbb{R} \times \Delta$

$$WF(\iota_\Delta^* D_tG) \subset \{(t,y,\tau,\eta); \quad \exists \\ (t,x,y,\tau,\xi,\bar{\xi});\ \eta = \xi - \bar{\xi}\}. \tag{9.393}$$

Then integration gives

$$WF(T) \subset \{(t,\tau); \quad \exists \quad (t,y,\tau,0) \in WF(D_tG)\}. \tag{9.394}$$

Combining (9.393) and (9.394) we see

$$t \in \operatorname{sing.supp}(T) \Longrightarrow \quad \exists \quad (t,\tau) \in WF(T) \\ \Longrightarrow \quad \exists \quad (t,x,x,\tau,\xi,\xi) \in WF(D_tG) \\ \Longrightarrow \quad \exists \quad s \quad \text{s.t.} \quad \exp(sH_p)(0,x,\tau,\xi) = (t,x,\tau,\xi). \tag{9.395}$$

Now

$$p = \tau^2 - |\xi|^2,\ \text{so } H_p = 2\tau\partial_t - H_g,\ g = |\xi|^2, \tag{9.396}$$

H_g being a vector field on T^*X. Since WF is *conic* we can take $|\xi| = 1$ in the last condition in (9.395). Then it says

$$s = 2\tau t, \qquad \exp(tH_{\frac{1}{2}g})(x,\xi) = (x,\xi), \tag{9.397}$$

since $\tau^2 = 1$.

The curves in X with the property that their tangent vectors have unit length and the lift to T^*X is an integral curve of $H_{\frac{1}{2}g}$ are *by definition* geodesic, parameterized by arclength. Thus (9.397) is the statement that $|t|$ is the length of a closed geodesic. This proves (9.386).

So now we have to compute the symbol of T at 0. We use, of course, our local representation of G in terms of conormal distributions. Namely

$$G = \sum_j \phi_j G_j, \quad \phi_j \in \mathcal{C}^\infty(X), \tag{9.398}$$

where the ϕ_j has support in coordinate particles in which

$$\begin{aligned} G_j(t,x,\bar{x}) &= \int\limits_{\mathbb{S}^{n-1}} (u_+(t,x,\bar{x};\omega) + u_-(t,x,\bar{x};\omega))\,d\omega, \\ u_p m &= \frac{1}{2\pi}\int\limits_{\xi} e^{ih_\pm(t,x,\bar{x},\omega)\xi} a_\pm(x,\bar{x},\xi,\omega)d\xi. \end{aligned} \tag{9.399}$$

Here $h_\pm$ are solutions of the eikonal equation (i.e. are characteristic for P)

$$\begin{gathered} |\partial_t h_\pm|^2 = |h_\pm|^2 \\ h_\pm \Big|_{t=0} = (x-\bar{x})\cdot\omega \\ \pm\partial_t h_\pm > 0, \end{gathered} \tag{9.400}$$

which fixes them locally uniquely. The $a_\pm$ are chosen so that

$$(u_+ + u_\pm \Big|_{t=0} = 0, (D_t u_+ D_t u_-)\Big|_{t=0} \delta((x-\bar{x})\cdot\omega) P u_\pm \in \mathcal{C}^\infty. \tag{9.401}$$

Now, from (9.399)

$$u_+ + u_- \Big|_{t=0} = \frac{1}{2\pi}\int e^{((x-x\bar{x})\cdot\omega)\xi}(a_+ + a_-)(x,\bar{x},\xi,\omega)d\xi = 0 \tag{9.402}$$

so $a_+ - a_-$. Similarly

$$\begin{aligned} D_t u_+ + D_t u_- \Big|_{t=0} &= \frac{1}{2\pi}\int e^{i((x-\bar{x})\cdot\omega)\xi}\big[(D_t h_+)a_+ + (D_t h_-)a_-\big]d\xi \\ &= \frac{1}{2(2\pi)^{n-1}} f_n((x-\bar{x})\cdot\omega) \end{aligned} \tag{9.403}$$

From (9.400) we know that $D_t h_\pm = \mp i|d_x(x-\bar{x})\cdot\omega| = \mp i|\omega|$ where the length is with respect to the Riemann measure. We can compute the symbols or both sides in (9.403) and consider that

$$-2i|\omega|a_+ \equiv \frac{1}{2(2\pi)^{n-1}}|\xi|^{n-1} = D_t h_+ a_+ + D_t h_- a_- \Big|_{t=0} \tag{9.404}$$

is necessary to get (9.401). Then

$$\begin{aligned} T(t) &= 2\pi_*(\iota_\Delta^* D_t G) \\ &= \frac{1}{2\pi}\sum_{j,\pm} 2\int\limits_X \int\limits_{\mathbb{S}^{n-1}} e^{ih_\pm(t,x,x,\omega)\xi}(D_t h_\pm a_\pm)(x,\bar{x},\omega,\xi)d\xi d\omega dx. \end{aligned} \tag{9.405}$$

Here dx is really the Riemann measure on X. From (9.404) the leading part of this is

$$\frac{2}{2\pi}\sum_{j\pm}\int\limits_X \int\limits_{\mathbb{S}^{n-1}} e^{ih_\pm(t,x,x,\omega)\xi}\frac{1}{4(2\pi)^{n-1}}|\xi|^{n-1}d\xi d\omega dx \tag{9.406}$$

since any term vanishes at t contributes a weaker singularity. Now

$$h_\pm = \pm|\omega|t + (x-\bar{x})\cdot\omega + 0(t^2). \tag{9.407}$$

From which we deduce that

$$\begin{aligned}\psi(t)T(t) &= \frac{1}{2\pi}\int e^{it\tau}a(\tau)d\tau \\ a(\tau) &\sim C_n \operatorname{Vol}(X)|\tau|^{n-1}C_n = \end{aligned} \tag{9.408}$$

where C_n is a universal constant depending only on dimension. Notice that if n is odd this is a "little" function.

The final thing I want to do is to show how this can be used to describe the asymptotic behaviour of the eigenvalue of Δ : □

PROPOSITION 9.12. *("Weyl estimates with optimal remainder".) If $N(\lambda)$ is the number of eigenvalues at Δ satisfying $\lambda_1^2 \le \lambda$, counted with multiplicity, the*

$$N(\lambda) = C_n \operatorname{Vol}(X)\lambda^n + o(\lambda^{n-1}) \tag{9.409}$$

The estimate of the remainder terms is the here – weaker estimates are easier to get.

PROOF. (Tauberian theorem). Note that

$$T = \mathcal{F}(\mu) \text{ where } N(\lambda) = \int_0^\lambda \mu(\lambda), \tag{9.410}$$

$\mu(\lambda)$ being the measure

$$\mu(\lambda) = \sum_{\lambda_j^2 \in \operatorname{spec}(\Delta)} \delta(\lambda - \lambda_j). \tag{9.411}$$

Now suppose $\rho \in \mathcal{S}(\mathbb{R})$ is even and $\int \rho = 1$, $\rho \ge 0$. Then $N_\rho(\lambda) = \int(\lambda')\rho(\lambda - \lambda')$ is a $\mathcal{C}^\infty$ function. Moreover

$$\widehat{\frac{d}{d\lambda}N_\rho(\lambda)} = \hat{\mu}\cdot\hat{\rho}. \tag{9.412}$$

Suppose we can choose ρ so that

$$\rho \ge 0, \int \rho = 1, \rho \in \mathcal{S},\ \hat{\rho}(t) = 0,\ |t| > \epsilon \tag{9.413}$$

for a given $\epsilon > 0$. Then we *know* $\hat{\mu}\hat{\rho}$ is conormal and indeed

$$\begin{aligned}&\frac{d}{d\lambda}N\rho(\lambda) \sim C\operatorname{Vol}(X)\lambda^{n-1} + \dots \\ &\Longrightarrow N_\rho(\lambda) \sim C'\operatorname{Vol}(X)\lambda^n + \textit{ lots}.\end{aligned} \tag{9.414}$$

So what we need to do is look at the *difference*

$$N_\rho(\lambda) - N(\lambda) = \int N(\lambda - \lambda')\rho(\lambda') - N(\lambda)\rho(\lambda'). \tag{9.415}$$

It follows that a bound for N

$$|N(\lambda + \mu) - N(\lambda)| \le ((1 + |\lambda| + |\mu|)^{n-1}(1 + |\lambda|) \tag{9.416}$$

gives

$$N(\lambda) - N_\rho(\lambda) \le C\lambda^{n-1} \tag{9.417}$$

which is what we want. Now (9.418) follows if we have

$$N(\lambda + 1) - N(\lambda) \le C(1 + |\lambda|) \qquad t/\lambda. \tag{9.418}$$

This in turn follows from the *positivity* of ρ, since

$$\int \rho(\lambda - \lambda')\mu(\lambda') \leq C(1 + |\lambda|)^{n-1}. \tag{9.419}$$

Finally then we need to check the existence of ρ as in (9.413). If ϕ is real and even so is $\hat{\phi}$. Take ϕ with support in $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and construct $\phi * \phi$, real and even with ϕ. □

9.14. Problems

PROBLEM 9.5. Show that if E is a symplectic vector space, with non-degenerate bilinear form ω, then there is a basis $v_1, \dots, v_n, w_1, \dots, w_n$ of E such that in terms of the dual basis of E^*

$$\omega = \sum_j v_j^* \wedge w_j^*. \tag{9.420}$$

Hint: Construct the w_j, v_j successive. Choose $v_1 \neq 0$. Then choose w_1 so that $\omega(v_1, w_1) = 1$. Then choose v_2 so $\omega(v_1, v_2) = \omega(w_1, v_2) = 0$ (why is this possible?) and w_2 so $\omega(v_2, w_2) = 1$, $\omega(v_1, w_2) = \psi(w_1, w_2) = 0$. Then proceed and conclude that (9.420) must hold.

Deduce that there is a linear transformation $T : E \longrightarrow \mathbb{R}^{2n}$ so that $\omega = T^*\omega_D$, with ω_D given by (9.200).

PROBLEM 9.6. Extend problem 9.5 to show that T can be chosen to map a given Lagrangian plane $V \subset E$ to

$$\{x = 0\} \subset \mathbb{R}^{2n} \tag{9.421}$$

Hint: Construct the basis choosing $v_j \in V \ \forall \ j$!

PROBLEM 9.7. Suppose S is a symplectic manifold. Show that the *Poisson bracket*

$$\{f, g\} = H_f g \tag{9.422}$$

makes $\mathcal{C}^\infty(S)$ into a Lie algebra.

CHAPTER 10

K-theory

This is a brief treatment of K-theory, enough to discuss, and later to prove, the Atiyah-Singer index theorem. I am starting from the smoothing algebra discussed earlier in Chapter 4 in order to give a 'smooth' treatment of K-theory (this approach is in fact closely related to the currently-in-vogue subject of 'smooth K-theory'). In particular the K-groups $\mathrm{K}^1_{\mathrm{c}}(X)$ and $\mathrm{K}^0_{\mathrm{c}}(X)$ of any manifold X, corresponding to compactly-supported classes, are defined. The elementary properties are derived and important isomorphism between them are discussed. There is a plethora of such maps which are listed here to try to help keep them straight:-

(10.1) The clutching construction, Proposition 10.6
$$\mathrm{clu}: \mathrm{K}^1_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}\times X).$$

(10.2) The 1-dim isotropic index, Proposition 10.9
$$\mathrm{Ind}_{\mathrm{iso}}: \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.3) The 1-dim Toeplitz index, elliptics on the circle §10.7
$$\mathrm{Ind}_{\mathrm{To}}: \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X)$$

(10.4) The N-dim isotropic index, quantize elliptic symbols
$$\mathrm{Ind}_{\mathrm{iso}}: \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^{2N-1}\times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.5)
The N-dim odd semiclassical index, quantize invertible matrices, Proposition 10.14
$$\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso,sl}}: \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^{2N}\times X) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X).$$

(10.6) The N-dim even semiclassical index, quantize projections
$$\mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso,sl}}: \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N}\times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.7) N-dim isotropic index, quantize elliptic symbols
$$\mathrm{Ind}_{\mathrm{iso}}: \mathrm{K}^0_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.8) Odd semiclassical index quantize invertible matrices
$$\mathrm{Thom}^{\mathrm{odd}} = \mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso,sl}}: \mathrm{K}^1_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X).$$

(10.9) Even semiclassical index quantize projections
$$\mathrm{Thom} = \mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso,sl}}: \mathrm{K}^0_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.10) The Bott map, tensor with β_E
$$\mathrm{Bott} = \mathrm{Thom}^{-1}: \mathrm{K}^0_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(E).$$

The three maps before the last are for a real vector bundle E over X with symplectic structures on the fibres – they are the same as the preceeding three in the case of a trivial bundle except that the first of those then involves the inverse of the clutching construction.

10.1. What do I need for the index theorem?

Here is a summary of the parts of this chapter which are used in the proof of the index theorem to be found in Chapter 12.

(1) Odd K-theory ($\mathrm{K}^1_\mathrm{c}(X)$) defined as stable homotopy classes of maps into $\mathrm{GL}(N,\mathbb{C})$, or as homotopy classes of maps into $G^{-\infty}$.
(2) Even K-theory ($\mathrm{K}_\mathrm{c}(X)$) defined as stable isomorphism classes of $\mathbb{Z}_2$-graded bundles.
(3) The gluing identification of $\mathrm{K}^1_\mathrm{c}(X)$ and $\mathrm{K}_\mathrm{c}(\mathbb{R}\times X)$.
(4) The isotropic index map $\mathrm{K}^1_\mathrm{c}(\mathbb{R}\times X)\longrightarrow \mathrm{K}_\mathrm{c}(X)$ using the eigenprojections of the harmonic oscillator to stabilize the index.
(5) Bott periodicity – proof that this map is an isomorphism and hence that $\mathrm{K}_\mathrm{c}(X)\equiv \mathrm{K}_\mathrm{c}(\mathbb{R}^2\times X)$.
(6) Thom isomorphism $\mathrm{K}_\mathrm{c}(V)\longrightarrow \mathrm{K}_\mathrm{c}(X)$ for a complex (or symplectic) vector bundle over X. In particular the identification of the 'Bott element' $\beta_V\in \mathrm{K}_\mathrm{c}(V)$ which generates $\mathrm{K}_\mathrm{c}(V)$ as a module over $\mathrm{K}_\mathrm{c}(X)$.

With this in hand you should be able to proceed to the proof of the index theorem in K-theory in Chapter 12. If you want the 'index formula' which is a special case of the index theorem in cohomology you need a bit more, namely the discussion of the Chern character and Todd class below.

10.2. Odd K-theory

First recall the 'smoothing group'

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)=\left\{A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n);\ \exists\, B\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \mathrm{Id}+B=(\mathrm{Id}+A)^{-1}\right\}. \tag{10.11}$$

Note that the notation is potentially confusing here. Namely, I am thinking of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ as the subset consisting of those $A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that $\mathrm{Id}+A$ is invertible. The group product is then not the usual product on $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ since

$$(\mathrm{Id}+A_1)(\mathrm{Id}+A_2)=\mathrm{Id}+A_1+A_2+A_1A_2.$$

Just think of the operator as 'really' being $\mathrm{Id}+A$ but the identity is always there so it is dropped from the notation.

One consequence of the fact that $\mathrm{Id}+A$ is invertible if and only if $\det(\mathrm{Id}+A)\neq 0$ is that[1]

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\subset\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)=\mathcal{S}(\mathbb{R}^{2n})=\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^{2n}})\text{ is open and dense.} \tag{10.12}$$

In view of this there is no problem in understanding what a smooth map into $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is. Namely, it is a map into $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ which has range in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and the following statment can be taken as a definition of smoothness, but it is just equivalent to the standard notion of a smooth map with values in a topological

[1] See Problem 10.8 if you want a proof not using the Fredholm determinant.

vector space. Namely if X is a manifold then

(10.13)
$$\begin{aligned}&\mathcal{C}^\infty(X;G^{-\infty}) = \\ &\qquad \{a \in \mathcal{C}^\infty(X \times \overline{\mathbb{R}^{2n}}); a \equiv 0 \text{ at } X \times \mathbb{S}^{2n-1},\ a(x) \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)\ \forall\ x \in X\},\\ &\mathcal{C}^\infty_c(X;G^{-\infty}) = \{a \in \mathcal{C}^\infty(X \times \overline{\mathbb{R}^{2n}}); a \equiv 0 \text{ at } X \times \mathbb{S}^{2n-1},\\ &\qquad a(x) \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)\ \forall\ x \in X, \exists\ K \Subset X \text{ s.t. } a(x) = 0\ \forall\ x \in X \setminus K\}.\end{aligned}$$

Notice that 'compact supports' here means that the actual operator we have in mind, which is $\mathrm{Id} + a$, reduces to the identity outside a compact set.

The two spaces in (10.13) (they are the same if X is compact) are groups. They are in fact examples of gauge groups (with an infinite-dimensional target group), where the composite of a and b is the map $a(x)b(x)$ given by composition in $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Two elements $a_0,\ a_1 \in \mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}})$ are said to be homotopic (in fact smoothly homotopic, but that is all we will use) if there exists $a \in \mathcal{C}^\infty_c(X \times [0,1]_t;G^{-\infty}_{\text{iso}})$ such that $a_0 = a\big|_{t=0}$ and $a_1 = a\big|_{t=1}$. Clearly if b_0 and b_1 are also homotopic in this sense then a_0b_0 is homotopic to a_1b_1, with the homotopy just being the product of homotopies. This gives the group property in the following definition:-

Definition 10.1. *For any manifold*

$$\mathrm{K}^1_c(X) = \mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}}(\mathbb{R}^n))/\sim \tag{10.14}$$

is the group of equivalence classes of elements under homotopy.

Now, we need to check that this is a reasonable definition, and in particular see how is it related to K-theory in the usual sense. To misquote Atiyah, K-theory is the topology of linear algebra. So, the basic idea is that $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is just a version of $\mathrm{GL}(N,\mathbb{C})$ where $N = \infty$ (but smoother than the usual topological versions). To make this concrete, recall that finite rank elements are actually dense in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Using the discussion of the harmonic oscillator in Chapter 4 we can make this even more concrete. Let $\pi_{(N)}$ be the projection onto the span of the first N eigenvalues of the harmonic oscillator (so if $n > 1$ it is projecting onto space of dimension a good deal larger than N, but no matter). Thus $\pi_{(N)} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is an operator of finite rank, exactly the sum of the dimensions of these eigenspaces. Then, from the discussion in Chapter 4

$$\begin{gathered} f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow \pi_{(N)}f \to f \text{ in } \mathcal{S}(\mathbb{R}^n) \text{ as } N \to \infty,\\ A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \Longrightarrow \pi_{(N)}A\pi_{(N)} \to A \text{ in } \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \text{ as } N \to \infty.\end{gathered} \tag{10.15}$$

The range of $\pi_{(N)}$ is just a finite dimensional vector space, so isomorphic to $\mathbb{C}^M$ (where M depends on N and n, the simplest case is $n = 1$ since then $M = N$). We will choose a fixed linear isomorphism to $\mathbb{C}^M$ by choosing a particular basis of eigenfunctions of the harmonic oscillator. If $a \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then $\pi_{(N)}a\pi_{(N)}$ becomes a linear operator on $\mathbb{C}^M$, so an element of the matrix algebra.

Proposition 10.1. *The 'finite rank elements' in $\mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$, those for which $\pi_{(N)}A = A\pi_{(N)} = A$ for some N, are dense in $\mathcal{C}^\infty_c(X;G^{-\infty}(\mathbb{R}^n))$.*

These elements are really to be thought of as finite rank perturbations of the identity.

Proof. This just requires a uniform version of the argument above, which in fact follows from the pointwise version, to show that

$$A \in \mathcal{C}_c^\infty(X; \Psi_{\text{iso}}^{-\infty}) \Longrightarrow \pi_{(N)} A \pi_{(N)} \to A \text{ in } \mathcal{C}_c^\infty(X; \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)). \tag{10.16}$$

From this it follows that if $A \in \mathcal{C}_c^\infty(X; G_{\text{iso}}^{-\infty}(\mathbb{R}^n))$ (meaning if you look back, that $\operatorname{Id} + A$ is invertible) then $\operatorname{Id} + \pi_{(N)} A$ is invertible for N large enough (since it vanishes outside a compact set). □

Corollary 10.1. *The groups $\mathrm{K}_c^1(X)$ are independent of n, the dimension of the space on which the group acts (as is already indicated by the notation).*

In fact this shows that $\pi_{(N)} a \pi_{(N)}$ and a are homotopic in $\mathcal{C}_c^\infty(X; G_{\text{iso}}^{-\infty}(\mathbb{R}^n))$ provided N is large enough. Thus each element of $\mathrm{K}_c^1(X)$ is represented by a finite rank family in this sense (where the order N may depend on the element and the identity needs to be added). Any two elements can then be represented by finite approximations for the same N. Thus there is a natural isomophism between the groups corresponding to different n's by finite order approximation. In fact this approximation argument has another very important consequence.

Proposition 10.2. *For any manifold $\mathrm{K}_c^1(X)$ is an Abelian group, i.e. the group product is commutative.*

Proof. In view of the preceeding result it suffices to take $n = 1$ so N and the rank of $\pi_{(N)}$ are the same. As shown above, given two elements $[a]$, $[b] \in \mathrm{K}_c^1(X)$ we can choose representatives a, $b \in \mathcal{C}_c^\infty(X; G_{\text{iso}}^{-\infty}(\mathbb{R}^n))$ such that $\pi_{(N)} a = a \pi_{(N)} = a$ and $\pi_{(N)} b = b \pi_{(N)} = b$. Thus they are represented by elements of $\mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ for some large N (so the actual element is $\operatorname{Id}_{(N)} + \pi_{(N)} a \pi_{(N)}$). Now, the range of $\pi_{(2N)}$ contains two N dimensional spaces, the ranges of $\pi_{(N)}$ and $\pi_{(2N)} - \pi_{(N)}$. Since we are picking bases in each, we can identify these two N dimensional spaces and then represent an element of the $2N$-dimensional space as a 2-vector of N-vectors. This decomposes $2N \times 2N$ matrices as 2×2 matrices with $N \times N$ matrix elements. In fact this tensor product of the 2×2 and $N \times N$ matrix algebras gives the same product as $2N \times 2N$ matrices (as follows easily from the definitions). Now, consider a rotation in 2 dimensions, represented by the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{10.17}$$

This rotates the standard basis e_1, e_2 to e_2, $-e_1$ as θ varies from 0 to $\pi/2$. If we interpret it as having entries which are multiples of the identity as an $N \times N$ matrix, and then conjugate by it, we get a curve

$$\begin{aligned} a(x,\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \operatorname{Id}_N \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} a\cos^2\theta + \sin^2\theta & (\operatorname{Id} - a)\sin\theta\cos\theta \\ (\operatorname{Id} - a)\sin\theta\cos\theta & \cos^2\theta + a\sin^2\theta \end{pmatrix}. \end{aligned} \tag{10.18}$$

This is therefore an homotopy between a represented as an $N \times N$ matrix and the same element acting on the *second* N dimensional subspace, i.e. it becomes

$$\begin{pmatrix} \operatorname{Id}_N & 0 \\ 0 & a \end{pmatrix}. \tag{10.19}$$

This commutes with the second element which acts only in the first N dimensional space, so, because of homotopy equivalence, the product in $\mathrm{K}^1_{\mathrm{c}}(X)$ is commutative. □

So now we see that $\mathrm{K}^1_{\mathrm{c}}(X)$ is an Abelian group associated quite naturally to the space X. I should say that the notation is not quite standard. Namely the standard notation would be $\mathrm{K}^1(X)$, without any indication of the 'compact supports' that are involved in the definition. I prefer to put this in explicitly. Of course if X is compact it is not necessary.

Lemma 10.1. *Any proper smooth map $f : X \longrightarrow Y$ induces a homomorphism $f^* : \mathrm{K}^1_c(Y) \longrightarrow \mathrm{K}^1_c(X)$ by composition; the map f^* only depends on the homotopy class of f in proper smooth maps.*

This makes $\mathrm{K}^1_{\mathrm{c}}$ into a contraviant functor on the category of manifolds and proper maps to the category of abelian groups, if you like to think in those terms.

Proof. If $a \in \mathcal{C}^\infty_c(Y; G^{-\infty}_{\mathrm{iso}})$ then $f^*a = a \circ f \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}})$ where the compactness of the support is a consequence of the assumed properness of the map – that $f^{-1}(K) \Subset X$ for any $K \Subset Y$. Homotopies lift to homotopies, so it is straightforward to check that this is a homomorphism at the level of $\mathrm{K}^1_{\mathrm{c}}$ and that it only depends on the homotopy class of f. □

Thus, since it is contravariant, 'pull-back' is the natural operation on K-theory. The index theory that we discuss in Chapter 12 is concerned with the 'wrong-way' map, i.e. push-forward, for K-theory.

Lemma 10.2 (Excision). *The inclusion of any open set $i : U \longrightarrow X$ induces a natural map*

$$i_! : \mathrm{K}^1_c(U) \longrightarrow \mathrm{K}^1_c(X). \tag{10.20}$$

Proof. Any smooth map with compact support $a \in \mathcal{C}^\infty_c(U; G^{-\infty})$ can be extended as the identity to give a smooth map $\tilde{a} \in \mathcal{C}^\infty_c(X; G^{-\infty})$ so fixed by the properties $\tilde{a} = a$ on U, $\tilde{A} = 0$ on $X \setminus U$. Homotopies also extend in this way so this induces the natural map (10.20). □

A fundamental role is played below by the following result computing the odd K-theory of the product $\mathbb{S} \times X$ of a general manifold and a circle.

Proposition 10.3. *For any manifold the natural projection, $\pi : X \times \mathbb{S} \longrightarrow X$, the inclusion $\iota : X \times \mathbb{R} \longrightarrow X \times \mathbb{S}$ given by the 1-point compactification of $\mathbb{R}$ and the inclusion $p_1 : X \ni x \longmapsto (x, 1) \in X \times \mathbb{S}$, lead to a split short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}^1_c(X \times \mathbb{R}) \arrow[r, "\iota_!"] & \mathrm{K}^1_c(X \times \mathbb{S}) \arrow[r, "p_1^*"] & \mathrm{K}^1_c(X) \arrow[r] \arrow[l, bend left, "\pi^*"] & 0 \end{tikzcd} \tag{10.21}$$

and hence an isomorphism

$$\mathrm{K}^1_c(X \times \mathbb{S}) = \mathrm{K}^1_c(X \times \mathbb{R}) \oplus \mathrm{K}^1_c(X) \tag{10.22}$$

Proof. Certainly $\pi \circ p_1 = \mathrm{Id}_X$ so $p_1^* \circ \pi^* = \mathrm{Id}$ shows that p_1^* must be surjective and π^* injective. Since $1 \in \mathbb{S}$ is not in the image of ι, every class in the image of $\iota_!$ has a representative which is equal to the identity on the image of p_1, so pulls back to zero in $\mathrm{K}^1_{\mathrm{c}}(X)$, so $p_1^* \circ \iota_! = 0$.

Since an element in $\mathcal{C}_c^\infty(X\times\mathbb{S};G^{-\infty})$ which vanishes at $X\times\{1\}$ is homotopic through such elements to one which vanishes near $X\times\{1\}$ (and with supports uniformly compact) this sequence corresponds to the short exact sequence of groups

$$\begin{aligned}(10.23)\quad & \big\{a\in\mathcal{C}_c^\infty(X\times\mathbb{S};G_{\mathrm{iso}}^{-\infty});a(x,1)=0\ \forall\ x\in X\big\}\longrightarrow\\ &\qquad\qquad \mathcal{C}_c^\infty(X\times\mathbb{S};G_{\mathrm{iso}}^{-\infty})\longrightarrow\mathcal{C}_c^\infty(X;G_{\mathrm{iso}}^{-\infty}).\end{aligned}$$

Under homotopy this becomes the direct sum decomposition (10.22). □

Thus there are two Abelian groups $\mathrm{K}_\mathrm{c}^1(X)$ and $\mathrm{K}_\mathrm{c}^1(X\times\mathbb{R})$ associated to the manifold X with direct sum naturally $\mathrm{K}_\mathrm{c}^1(\mathbb{S}\times X)$. As we shall see below it is perfectly natural to *define* the even K-theory of X to be $\mathrm{K}_\mathrm{c}^0(X)=\mathrm{K}_\mathrm{c}^1(X\times\mathbb{R})$ (although the notation ${\mathrm{K}_\mathrm{c}}^{-2}(X)$ would be even better) and to denote the sum of the two terms as

$$\mathrm{K}_\mathrm{c}^*(X)=\mathrm{K}_\mathrm{c}^1(\mathbb{S}\times X). \tag{10.24}$$

We will not do this now, only because it is potentially confusing and instead will give a more standard definition of $\mathrm{K}_\mathrm{c}^0(X)$ and then define a natural index map (the isotropic index)

$$\mathrm{Ind}_{\mathrm{iso}}:\mathrm{K}_\mathrm{c}^1(X\times\mathbb{R})\xrightarrow{\simeq}\mathrm{K}_\mathrm{c}^0(X). \tag{10.25}$$

If you know a little algebraic topology, you will see that the discussion above starts from the premise that $G_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ is a *classifying space* for odd K-theory. So this is true by fiat. The corresponding classifying space for even K-theory is then the pointed loop group, the set of maps

$$G_{\mathrm{iso,sus}}^{-\infty}(\mathbb{R}^n)=\{a\in\mathcal{C}^\infty(\mathbb{S};G_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n);a(1)=\mathrm{Id}\}. \tag{10.26}$$

10.3. Computations

Let us pause for a moment to compute some simple cases. Namely

LEMMA 10.3.

$$\mathrm{K}^1(\{pt\})=\{0\},\ \mathrm{K}_c^1(\mathbb{R})=\mathbb{Z},\ \mathrm{K}^1(\mathbb{S})=\mathbb{Z}. \tag{10.27}$$

PROOF. The first two of these statements follow directly from the next two results. The third is a direct consequence of (10.22) and the first two. □

LEMMA 10.4. *The group $G_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ is connected.*

PROOF. If $a\in G_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$, the curve $[0,1]\ni t\longmapsto(1-t)a+t\pi_{(N)}a\pi_{(N)}$ lies in $G_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ for N sufficiently large. Thus it suffices to show that $\mathrm{GL}(n,\mathbb{C})$ is connected for large N; of course[2]

$$\mathrm{GL}(N,\mathbb{C})\text{ is connected for all }N\geq 1. \tag{10.28}$$

□

PROPOSITION 10.4. *A closed loop in $\gamma:S\longrightarrow G_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ is contractible (homotopic through loops to a constant loop) if and only if the composite map*

$$\tilde{\gamma}=\det\circ\gamma:\mathbb{S}\longrightarrow\mathbb{C}^* \tag{10.29}$$

[2]See Problem 10.12

is contractible, so

$$\pi_1(G^{-\infty}_{\text{iso}}(\mathbb{R}^n)) = \mathbb{Z} \tag{10.30}$$

with the identification given by the winding number of the Fredholm determinant.

Proof. Again, as in the previous proof but now a loop can be deformed into $\mathrm{GL}(N,\mathbb{C})$ so it is certainly enough to observe that[3]

$$\pi_1\left(\mathrm{GL}(N,\mathbb{C})\right) = \mathbb{Z} \text{ for all } N \geq 1. \tag{10.31}$$

□

An explicit generator of $\pi_1(G^{-\infty}_{\text{iso}})$ is given by the stabilization of the loop into $\mathrm{GL}(N,\mathbb{C}) = \mathbb{C} \setminus \{0\}$ which is the identity map on the circle embedded in $\mathbb{C}$:

$$\gamma(\theta) = e^{2\pi i\theta}. \tag{10.32}$$

10.4. Vector bundles

The notion of a complex vector bundle was briefly discussed earlier in Section 6.2. Recall from there the notion of a bundle isomorphism and that a bundle is said to be trivial, over some set K, if there is a bundle isomorphism from its restriction to K to $K \times \mathbb{C}^k$. The direct sum of vector bundles and the tensor product are also briefly discussed there.

To see that there is some relationship between K-theory as discussed above and vector bundles consider $\mathrm{K}^1(X)$ for a compact manifold, X. First note that if V is a complex vector bundle over X and $e : V \longrightarrow V$ is a bundle isomorphism, then e defines an element of $\mathrm{K}^1(X)$. To see this we first observe we can always find a complement to V.

Proposition 10.5. *Any vector bundle V which is trivial outside a compact subset of X can be complemented to a trivial bundle, i.e. there exists a vector bundle E, also trivial outside a compact set, and a bundle isomorphism*

$$V \oplus E \longrightarrow X \times \mathbb{C}^N. \tag{10.33}$$

Proof. This follows from the local triviality of V. Choose a finite open cover U_i of X with M elements in which one set is $U_0 = X \setminus K$ for K compact and such that V is trivial over each U_i. Then choose a partition of unity subordinate to U_i – so only the $\phi_0 \in \mathcal{C}^\infty(X)$ with support in U_0 does not have compact support. If $f_i : V\big|_{U_i} \longrightarrow \mathbb{C}^k \times U_i$ is a trivialization over U_i (so the one over U_0 is given by the assumed triviality outside a compact set) consider

$$F : V \longrightarrow X \times \mathbb{C}^{kM},\ u(x) \longmapsto \bigoplus_{i=1}^{M} f_i(\phi_i(u(x))). \tag{10.34}$$

This embeds V as a subbundle of a trivial bundle of dimension $N = kM$ since the map F is smooth, linear on the fibres and injective. Then we can take E to be the orthocomplement of the range of F which is identified with V. □

Thus, a bundle isomorphism e of V can be extended to a bundle isomorphism $e \oplus \mathrm{Id}_E$ of the trivial bundle. This amounts to a map $X \longrightarrow \mathrm{GL}(MN,\mathbb{C})$ which can then be extended to an element of $\mathcal{C}^\infty(X; G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ and hence gives an element of $\mathrm{K}^1_{\mathrm{c}}(X)$ as anticipated. It is straightforward to check that the element defined in

[3] Proof in Problem 10.13

$\mathrm{K}^1(X)$ does not depend on choices made in its construction, only on e (and through it of course on V).

This is one connection between bundles and $\mathrm{K}^1_{\mathrm{c}}$. There is another, similar, connection which is more important. Namely from a class in $\mathrm{K}^1_{\mathrm{c}}(X)$ we can construct a bundle over $\mathbb{S} \times X$. One way to do this is to observe that Proposition 10.5 associates to a bundle V a smooth family of projections $\pi_V \in \mathcal{C}^\infty_c(X; M(N, \mathbb{C}))$ which is trivial outside a compact set, in the sense that it reduces to a fixed projection there. Namely, π_V is just (orthogonal) projection onto the range of V. We will need to think about equivalence relations later, but certainly such a projection defines a bundle as well, namely its range.

For the following construction choose a smooth function $\Theta : \mathbb{R} \longrightarrow [0, \pi]$ which is non-decreasing, constant with the value 0 on some $(-\infty, -T]$, constant with value $\pi/2$ on $[-T/2, T/2]$ and constant with the value π on $[T, \infty)$, for some $T > 0$, and strictly increasing otherwise. It may also be convenient to assume that Θ is 'odd' in the sense that

$$\Theta(-t) = \pi - \Theta(t). \tag{10.35}$$

This is just a function which we can used to progressively 'rotate' through angle π but staying constant initially, near the middle and near the end. If $a \in \mathrm{GL}(N, \mathbb{C})$, consider the rotation matrix

$$S(\theta, a) = \begin{pmatrix} \cos(\theta)\,\mathrm{Id}_N & -\sin(\theta)a^{-1} \\ \sin(\theta)a & \cos(\theta)\,\mathrm{Id}_N \end{pmatrix} \in \mathrm{GL}(2N, \mathbb{C}). \tag{10.36}$$

This is invertible, in fact

$$\begin{gathered} S(\theta, a)S(\theta', a) = S(\theta + \theta', a),\ S(0, a) = \mathrm{Id}, \\ \frac{d}{d\theta}S(\theta, a) = S(\theta + \frac{\pi}{2}, a) = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix} S(\theta, a), \end{gathered} \tag{10.37}$$

Now, for a compact manifold X, consider $a \in \mathcal{C}^\infty(X; \mathbb{C}^N)$ which is everywhere invertible then

$$\mathbb{R} \times X \ni (t, x) \longmapsto R_a(t, x) = S(\Theta(t), a(x)) \tag{10.38}$$

has inverse $R_a(-t, x)$ and is equal to the identity in $|t| > T$. We will use this to construct a bundle on $\mathbb{R} \times X$ which is trivial for $t > 0$. The idea is that $R_a(t, x)$ 'rotates by $\pi/2$' as t runs over $(-\infty, 0)$. Set

$$\Pi_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \Pi'_a(t, x) = \begin{cases} R_a(t, x)\Pi_\infty R_a(-t, x) & t \le 0 \\ R_{\mathrm{Id}}(t, x)\Pi_\infty R_{\mathrm{Id}}(-t, x) & t \ge 0. \end{cases} \tag{10.39}$$

Clearly, $\Pi'_a(t, x)$ is smooth in $t \le 0$, and in $t \ge 0$, and is constant outside a compact set. In fact Π'_a is globally smooth, since near $t = 0$, $\Theta(t) = \pi/2$, by construction, so

$$\Pi'_a(0, x) = \begin{pmatrix} 0 & a^{-1} \\ -a & 0 \end{pmatrix} \Pi_\infty \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \tag{10.40}$$

is independent of a and hence smooth. Thus in fact $\Pi'_a(t, x)$ is constant near $t = \pm\infty$ where it takes the value Π_∞, which is projection onto the first $\mathbb{C}^N$.

Note, for later reference that

$$\Pi'_a(t,x) = \begin{pmatrix} \cos^2(\Theta(t))\,\mathrm{Id}_N & \cos(\Theta(t))\sin(\Theta(t))a^{-1}(x) \\ \sin(\Theta(t))\cos(\Theta(t))a(x) & \sin^2(\Theta(t)) \end{pmatrix},\ t \le 0 \tag{10.41}$$

has entries linear in a and a^{-1}.

Notice that if the conjugating matrix in (10.40) did not jump as it does at $t = 0$, but for instance we continued conjugating by $R_a(t,x)$ in $t \ge 0$ instead of switching to $a = \mathrm{Id}$, then the bundle which is the range of the family of projections would be globally isomorphic to the range of Π_∞, with $R_a(t,x)$ being the global isomorphism. In particular if $a = \mathrm{Id}$ this is indeed the case, so that at least $a = \mathrm{Id}$ corresponds to a trivial bundle.

This was all under the assumption that X is compact and the construction will not quite work if it is not, since then since then Π'_a outside a compact set, even when $a = \mathrm{Id}$. To cover the non-compact case we need to 'undo' the twisting at infinity in X which we do with a global isomorphism (not constant at infinity!) and consider instead

$$\Pi_a(t,x) = R_{\mathrm{Id}}(-t,x)\Pi'_a(t,x)R_{\mathrm{Id}}(t,x). \tag{10.42}$$

In case X is compact this is a global isomorphism, constant outside a compact set, and so gives the same bundle up to isomorphism. In the form (10.42) the projection is actually constant in $t \ge 0$.

Lemma 10.5. *An element $a \in \mathcal{C}^\infty(X; \mathrm{GL}(N,\mathbb{C}))$ equal to the identity outside a compact set defines, through* (10.42)*, a smooth family of matrices with values in the projections, $\Pi_a \in \mathcal{C}^\infty(\mathbb{R}\times X; M(2n,\mathbb{C}))$, which is constant outside a compact subset and so defines a vector bundle over $\mathbb{R}\times X$ which is trivial outside a compact set.*

We will see below that this is one of the important identification maps for K-theory that we need to understand, in fact it leads to (10.1).

So, by now it should not be so surprising that the K-groups introduced above are closely related to the 'Grothendieck group' constructed from vector bundles. The main issue is the equivalence relation.

Definition 10.2. *For a manifold X, $\mathrm{K}_c(X)$ is defined as the set of equivalence classes of pairs of complex vector bundles (V, W), both trivial outside a compact set and with given trivializations a, b there, under the relation $(V_1, W_1; a_1, b_1) \sim (V_2, W_2; a_2, b_2)$ if and only if there is a bundle S and a bundle isomorphism*

$$T : V_1 \oplus W_2 \oplus S \longrightarrow V_2 \oplus W_1 \oplus S \tag{10.43}$$

which is equal to $(a_2 \oplus b_2)^{-1}(a_1 \oplus b_2) \oplus \mathrm{Id}_S$ outside some compact set.

Note that if X is compact then the part about the trivializations is completely void, then we just have pairs of vector bundles (V, W) and the equivalence relation is the existence of a stabilizing bundle S and a bundle isomorphism (10.43).

This is again an Abelian group with the group structure given at the level of pairs of bundles (V_i, W_i), $i = 1, 2$ by[4]

$$[(V_1, W_1)] + [V_2, W_2)] = [(V_1 \oplus V_2, W_1 \oplus W_2)] \tag{10.44}$$

[4] See Problem 10.11 for the details.

with the trivializations $(a_1 \oplus a_2), (b_1 \oplus b_2)$. In particular $[(V, V)]$ is the zero element for any bundle V (trivial outside a compact set).

The equivalence relation being (stable) bundle isomorphism rather than some sort of homotopy may seen strange, but it is actually more general.

LEMMA 10.6. *If V is a vector bundle over $[0,1]_t \times X$ which is trivial outside a compact set then $V_0 = V\big|_{t=0}$ and $V_1 = V\big|_{t=1}$ are bundle isomorphic over X with an isomorphism which is trivial outside a compact set.*

PROOF. The proof is 'use a connection and integrate'. We can do this explicitly as follows. First we can complement V to a trivial bundle so that it is identified with a constant projection outside a compact set, using Proposition 10.5. Let the family of projections be $\pi_V(t, x)$ in $M \times M$ matrices. We want to differentiate sections of the bundle with respect to t. Since they are M-vectors we can do this, but we may well not get sections this way. However defining the (partial) connection by

$$\nabla_t v(t) = v'(t) - \pi'_V v(t) \Longrightarrow (\mathrm{Id} - \pi_V)\nabla_t v(t) = ((\mathrm{Id} - \pi_V)v(t))' = 0 \tag{10.45}$$

if $\pi_V v = v$, i.e. if v is a section. Now, by standard results on the existence, uniquenss and smoothness of solutions to differential equations, the condition $\nabla_t v(t) = 0$ fixes a unique section with $v(0) = v_0 \in V_0$ fixed. Then define $F : V_0 \longrightarrow V_1$ by $Fv_0 = v(1)$. This is a bundle isomorphism. $\square$

PROPOSITION 10.6. *For any manifold X the construction in Lemma 10.5 gives the 'clutching' isomorphism*

$$\mathrm{clu} : \mathrm{K}^1_c(X) \ni [a] \longrightarrow [(\Pi_a, \Pi_a^\infty)] = \mathrm{K}_c(\mathbb{R} \times X) \tag{10.46}$$

where Π_a^∞ is the constant projection to which Π_a restricts outside a compact set.

PROOF. The range of the projection Π_a in Lemma 10.5 fixes an element of $\mathrm{K}_\mathrm{c}(\mathbb{R} \times X)$ but we need to see that it is independent of the choice of a representing $[a] \in \mathrm{K}^1_\mathrm{c}(X)$. A homotopy of a gives a bundle over $[0,1] \times X$ and then Lemma 10.6 shows that the resulting bundles are isomorphic. Stabilizing a, i.e. enlarging it by an identity matrix adds a constant projection to Π_a and the same projection projection to Π_a^∞. Thus the map in (10.46) is well defined. So we need to show that it is an isomorphism. First we should show that it is additive. Recall that the addition in $\mathrm{K}^1_\mathrm{c}(X)$ is defined *either* by composition in $G^{-\infty}_{\mathrm{iso}}$ *or* by taking the direct sum. The direct sum of two bundle isomorphisms valued in $\mathrm{GL}(N, \mathbb{C})$ is then a bundle isomorphism in $\mathrm{GL}(2N, \mathbb{C})$ and the construction above leads to the corresponding direct sum of the two projections valued in $2N \times 2N$ matrices, giving a $4N \times 4N$ projection and this is the addition in K^0_c so clu is a homomorphism.

If V is a bundle over $\mathbb{R} \times X$ which is trivial outside a compact set, we can embed it as in Proposition 10.5 so it is given by a family of projections π_V (this of course involves a bundle isomorphism). Now, using the connection as in (10.45) we can define an isomorphism of the trivial bundle π_V^∞. Namely, integrating from $t = -T$ to $t = T$ defines an isomorphism a. The claim is that $(\Pi_a, \Pi_a^\infty) = (V, V^\infty)$. I leave the details to you, there is some help in Problem 10.10. Conversely, this construction recovers a from Π_a so shows that (10.46) is injective and surjective. $\square$

10.5. Isotropic index map

Now, (10.46) is part of Bott periodicity. The remaining part is that, for any manifold X there is a natural isomorphism

$$\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)\longrightarrow \mathrm{K}_{\mathrm{c}}(X). \tag{10.47}$$

If we regard this as an identification (and one has to be careful about orientations here) then it means that we have identified

$$\mathrm{K}^0_{\mathrm{c}}(X)=\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)=\mathrm{K}_{\mathrm{c}}(X)=\mathrm{K}_{\mathrm{c}}(\mathbb{R}^2\times X) \tag{10.48}$$

as is discussed more below. For the moment what we will work on is the definition of the map in (10.47). This is the 'isotropic' (or 'Toeplitz'[5]) index map.

So, we get to the start of the connection of this stuff with index theory. An element of $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)$ is represented by a map from $\mathbb{R}\times X$ to $\mathrm{GL}(N,\mathbb{C})$, for some N, and with triviality outside a compact set. In particular this map reduces to the identity near $\pm\infty$ in $\mathbb{R}$ so we can join the ends using the radial compactification of $\mathbb{R}\longmapsto\mathbb{S}$ and get a map

$$\tilde{a}\in\mathcal{C}^\infty(\mathbb{S}\times X;\mathrm{GL}(N,\mathbb{C})),\ \tilde{a}=\mathrm{Id}\ \text{ near }\{1\}\times X\text{ and outside a compact set.}$$

This indeed is essentially the implied definition of $\mathrm{K}^0_{\mathrm{c}}(X)$ before (10.25). Now, we can interpret $\tilde{a}$ as the principal symbol of an elliptic family in $\Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ depending smoothly on $x\in X$ (and reducing to the identity outside a compact set). Let's start with the case $X=\{\mathrm{pt}\}$ so there are no parameters.

Proposition 10.7. *If $A\in\Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ is elliptic with principal symbol $a=\sigma_0(A)\in\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$ then the index of A is given by the winding number of the determinant of the symbol*

$$\mathrm{Ind}_{\mathrm{iso}}(A)=-\operatorname{wn}(\det(a))=-\frac{1}{2\pi i}\int_{\mathbb{S}}\operatorname{tr}(a^{-1}\frac{da}{d\theta})d\theta \tag{10.49}$$

and if $a=\mathrm{Id}$ near $\{1\}\in\mathbb{S}$ then $\mathrm{Ind}_{\mathrm{iso}}(A)=0$ if and only if $[a]=0\in\mathrm{K}^1_{\mathrm{c}}(\mathbb{S})$.

Proof. ****Expand This follows from Proposition 10.4. First, recall what the winding number is. Then check that it defines the identification (10.30). Observe that the index is stable under homotopy and stabilization and that the index of a product is the sum of the indices. Then check one example with index 1, namely for the annihilation operator will suffices. For general A with winding number m, compose with m factors of the creation operator – the adjoint of the annihilation operator. This gives an operator with symbol for which the winding number is trivial. By Proposition 10.4 it can be deformed to the identity after stabilization, so its index vanishes and (10.49) follows. □

Now for the analytic step that allows us to define the full (isotropic) index map.

Proposition 10.8. *If $a\in\mathcal{C}^\infty_c(\mathbb{R}\times X;\mathrm{GL}(N,\mathbb{C}))$ (so it reduces to the identity outside a compact set) then there exists $A\in\mathcal{C}^\infty(X;\Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N))$ with $\sigma_0(A)=a$, A constant in $X\setminus K$ for some compact K and such that* $\operatorname{null}(A)$ *is a (constant) vector bundle over X.*

[5]See Problem **??** for this alternative approach.

Proof. We can choose a $B \in \mathcal{C}^\infty(X; \Psi^0_{\text{iso}}(\mathbb{R}; \mathbb{C}^N))$ with $\sigma(B) = a$ by the surjectivity of the symbol map. Moreover, taking a function $\psi \in \mathcal{C}^\infty(X)$ which is equal to 1 outside a compact set in X but which vanishes where $a \neq \mathrm{Id}$, $(1-\psi)B + \psi\,\mathrm{Id}$ has the same principal symbol and reduces to Id outside a compact set.

The problem with this initial choice is that the dimension of the null space may change from point to point. However, we certainly have a parametrix $G_B \in \mathcal{C}^\infty(X; \Psi^0_{\text{iso}}(\mathbb{R}; \mathbb{C}^N))$ which we can take to be equal to the identity outside a compact set, by the same method, and which then satisfies

$$G_B B = \mathrm{Id} + R_1,\ BG_B = \mathrm{Id} + R_2,\ R_i \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\text{iso}}(\mathbb{R}; \mathbb{C}^N)). \tag{10.50}$$

So, recall the finite rank projection $\pi_{(N)}$ onto the span of the first N eigenspaces. We know that $R_1\pi_{(N)} \to R_1$ in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}; \mathbb{C}^N)$ and this is true uniformly on X since the support in X is compact. So, if N is large enough $\sup_{x\in X} \|R_1(x)(\mathrm{Id} - \pi_{(N)})\| < \frac{1}{2}$. Composing the first equation in (10.50) on the right with $\mathrm{Id} - \pi_{(N)}$ we find that

$$G_B B(\mathrm{Id} - \pi_{(N)}) = (\mathrm{Id} + R_1(\mathrm{Id} - \pi_{(N)}))(\mathrm{Id} - \pi_{(N)}) \tag{10.51}$$

where the fact that $\mathrm{Id} - \pi_{(N)}$ is a projection is also used. Now

$$(\mathrm{Id} + R_1(\mathrm{Id} - \pi_{(N)}))^{-1} = \mathrm{Id} + S_1$$

where $S_1 \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\text{iso}}(\mathbb{R}))$ by the openness of $G^{-\infty}_{\text{iso}}(\mathbb{R})$. So if we set $A = B(\mathrm{Id} - \pi_{(N)})$ and $G = (\mathrm{Id} + S_1)G_B$ we see that

$$GA = \mathrm{Id} - \pi_{(N)}. \tag{10.52}$$

In particular the null space of $A(x)$ for each x is exactly the span of $\pi_{(N)}$ – it certainly annihilates this set but can annihilate no more in view of (10.52). Moreover A has the same principal symbol as B and is constant outside a compact set in X. □

Now, once we have chosen A as in Proposition 10.8 it follows from the constancy of the index that family $A(x)^*$ also has null spaces of constant finite dimension, and indeed these define a smooth bundle over X which, if X is not compact, reduces to $\pi_{(N)}$ near infinity – since $A = \mathrm{Id} - \pi_{(N)}$ there. Thus we arrive at the isotropic index map.

Proposition 10.9. *If A is as in Proposition 10.8 the the null spaces of $A^*(x)$ form a smooth vector bundle $\mathrm{Nul}(A^*)$ over X defining a class $[(\pi_{(N)}, \mathrm{Nul}(A^*))] \in \mathrm{K}_c(X)$ which depends only on $[a] \in \mathrm{K}^1_c(\mathbb{R} \times X)$ and so defines an additive map*

$$\mathrm{Ind}_{\text{iso}} : \mathrm{K}^1_c(\mathbb{R} \times X) \longrightarrow \mathrm{K}_c(X). \tag{10.53}$$

Proof. In the earlier discussion of isotropic operators it was shown that an elliptic operator has a generalized inverse. So near any particular point $\bar{x} \in X$ we can add an element $E(\bar{x}) \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}; \mathbb{C}^N)$ to $G(\bar{x})$ so that $H(\bar{x}) = G(\bar{x}) + E(\bar{x})$ is a generalized inverse, $H(\bar{x})A(\bar{x}) = \mathrm{Id} - \pi_{(N)}$, $A(\bar{x}_H(\bar{x}) = \mathrm{Id} - \pi'(\bar{x})$ where $\pi'(\bar{x})$ is a finite rank projection onto a subspace of $\mathcal{S}(\mathbb{R})$. Then $H(x) = G(x) + E(\bar{x})$ is still a parametrix nearby and

$$H(x)A(x) = \mathrm{Id} - \pi_{(N)},\ A(x)H(x) = \mathrm{Id} - p(x) \text{ near } \bar{x} \tag{10.54}$$

where $p(x)$ must have constant rank. Indeed, it follows that $p(x)\pi'(\bar{x})$ is a smooth bundle isomorphism, near $\bar{x}$, from the range of $\pi'(\bar{x})$ to the null space of A^*. This shows that the null spaces of the $A^*(x)$ form a bundle, which certainly reduces to $\pi_{(N)}$ outside a compact set. Thus

$$[(\pi_{(N)}, \mathrm{null}(A^*))] \in \mathrm{K_c}(X). \tag{10.55}$$

Next note the independence of this element of the choice of N. It suffices to show that increasing N does not change the class. In fact increasing N to $N+1$ replaces A by $A(\mathrm{Id} - \pi_{(N+1)})$ which has null bundle increased by the trivial line bundle $(\mathrm{Id}_{(N+1)} - \pi_{(N)})$. The range of A then decreases by the removal of the trivial bundle $A(x)(\mathrm{Id}_{(N+1)} - \pi_{(N)})$ and $\mathrm{null}(A^*)$ increases correspondingly. So the class in (10.55) does not change.

To see that the class is independent of the choice of A, for fixed a, consider two such choices. Initially the choice was of an operator with a as principal symbol, two choices are smoothly homotopic, since $tA + (1-t)A'$ is a smooth family with constant symbol. The same construction as above now gives a pair of bundles over $[0,1] \times X$, trivialized outside a compact set, and it follows from Lemma 10.6 that the class is constant. A similar discussion shows that homotopy of a is just a family over $[0,1] \times X$ so the discussion above applies to it and shows that the bundles can be chosen smoothly, again from Lemma 10.6 the class is constant. $\square$

It is important to understand what the index tell us.

PROPOSITION 10.10. *If $a \in \mathcal{C}_c^\infty(\mathbb{R} \times X; \mathrm{GL}(N,\mathbb{C}))$ then $\mathrm{Ind_{iso}}(a) = 0$ if and only if there is a family $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N))$ with $\sigma_0(A) = a$ which is constant outside a compact set in X and everywhere invertible.*

PROOF. The definition of the index class above shows that a may be quantized to an operator with smooth null bundle and range bundle such with $\mathrm{Ind_{iso}}(a)$ represented by $(\pi_{(N)}, p')$ where p' is the null bundle of the adjoint. If A can be chosen invertible this class is certainly zero. Conversely, if the class vanishes then after stabilizing with a trivial bundle $\pi_{(N)}$ and p' become bundle isomorphic. This just means that they are isomorphic for sufficiently large N with the isomorphism being the trivial one near infinity. However this isomorphism is itself an element of $\mathcal{C}^\infty(X; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N))$ which is trivial near infinity. Adding it to A gives an invertible realization of the symbol, proving the Proposition. $\square$

10.6. Bott periodicity

Now to the proof of Bott periodicity. Choose a 'Bott' element, which in this case is a smooth function

$$\beta(t) = e^{i\Theta(t)} \Longrightarrow \begin{cases} \beta : \mathbb{R} \longrightarrow \mathbb{C}^*,\ \beta(t) = 1 \text{ for } |t| > T, \\ \arg\beta(t) \text{ increasing over } (0, 2\pi) \text{ for } t \in (-T, T) \end{cases} \tag{10.56}$$

where Θ satisfies (10.35) and the preceeding conditions. Thus β has winding number one but is constant near infinity.

We first show

PROPOSITION 10.11. *The map* (10.53) *is surjective with explicit left inverse generated by mapping a smooth projection (constant near infinity) to*

$$(\pi_V, \pi_V^\infty) \longmapsto \beta(t)^{-1}\pi_V + (\mathrm{Id} - \pi_V) \in \mathcal{C}_c^\infty(\mathbb{R} \times X; \mathrm{GL}(N,\mathbb{C})). \tag{10.57}$$

Proof. The surjectivity follows from the existence of a left inverse, so we need to investigate (10.57). Observe that $\beta(t)^{-1}$, when moved to the circle, is a symbol with winding number 1. By Proposition 10.7 we may choose an elliptic operator $b \in \Psi^0_{\mathrm{iso}}(\mathbb{R})$ which has a one-dimensional null space and has symbol in the same class in $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R})$ as β^{-1}. In fact we could take the annihilation operator, normalized to have order 0. Then we construct an elliptic family $B_V \in \Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ by setting

$$B_V = \pi_V(x)b + (\mathrm{Id} - \pi_V(x)),\ x \in X. \tag{10.58}$$

The null space of this family is clearly $\pi_V \otimes N$, where N is the fixed one-dimensional vector space null(b). Thus indeed

$$\mathrm{Ind}_{\mathrm{iso}}(B_V) = [(\pi_V, \pi_V^\infty)] \in \mathrm{K}_{\mathrm{c}}(X). \tag{10.59}$$

This proves the surjectivity of $\mathrm{Ind}_{\mathrm{iso}}$, the index map in this isotropic setting. $\square$

With some danger of repeating myself, if X is compact the 'normalizing term' at infinity π_V^∞ is dropped. You will now see why we have been dragging this non-compact case along, it is rather handy even if interest is in the compact case.

This following proof that $\mathrm{Ind}_{\mathrm{iso}}$ is injective is a variant of the 'clever' argument of Atiyah (maybe it is *very* clever – look at the original proof by Bott or the much more computational, but actually rather enlightening, argument in [**1**]).

Proposition 10.12. *For any manifold X, the isotropic index map in* (10.47), (10.53) *is an isomorphism*

$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}^1_c(\mathbb{R} \times X) \xrightarrow{\simeq} \mathrm{K}_c(X). \tag{10.60}$$

Proof. Following Proposition 10.11 only the injectivity of the map remains to be shown. Rather than try to do this directly we use another carefully chosen homotopy.

So, we need to show that if $a \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \mathrm{GL}(N,\mathbb{C}))$ has $\mathrm{Ind}_{\mathrm{iso}}(a) = 0$ then $0 = [a] \in \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}_s \times X)$. As a first step we use the construction of Proposition 10.6 and Lemma 10.5 to construct the image of $[a]$ in $\mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X)$. It is represented by the projection-valued matrix

$$\Pi_a(t,s,x) \in \mathcal{C}^\infty_c(\mathbb{R}^2 \times X; M(2N,\mathbb{C})) \tag{10.61}$$

which is constant near infinity. Then we use the surjectivity of the index map in the case

$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}_{\mathrm{c}}(\mathbb{R} \times (\mathbb{R}^2 \times X)) \longrightarrow \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X) \tag{10.62}$$

and the explicit lift (10.58) to construct

$$\begin{gathered} e \in \mathcal{C}^\infty_c(\mathbb{R}^2 \times X; \mathrm{GL}(2N,\mathbb{C})),\ e(r,t,s,x) = \beta(r)\Pi_a(t,s,x) + (\mathrm{Id} - \Pi_a(t,s,x)), \\ \mathrm{Ind}_{\mathrm{iso}}(e) = [\Pi_a, \Pi_a^\infty] \in \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X). \end{gathered} \tag{10.63}$$

Here the 'r' variable is the one which is interpreted as the variable in the circle at infinity on $\mathbb{R}^2$ to turn e into a symbol and hence a family of elliptic operators with the given index. However we can rotate between the variables r and s, which is an homotopy replacing $e(r,t,s,x)$ by $e(-s,t,r,x)$. Since the index map is homotopy invariant, this symbol must give the same index class. Now, the third variable here is the argument of a, the original symbol. So the quantization map just turns a and a^{-1} which appears in the formula for Π_a – see (10.41) – into any operator with these symbols. By Proposition 10.10 a (maybe after a little homotopy) is the symbol of an invertible family. Inserting this in place of a and its inverse for a^{-1} gives an invertible family of operators with symbol $e(-s,t,r,x)$[6]. Thus $\mathrm{Ind}_{\mathrm{iso}}(e)=0$, but this means that

$$0=[(\Pi_a,\Pi_a^\infty)]\in\mathrm{K_c}(\mathbb{R}^2\times X)\Longrightarrow 0=[a]\in\mathrm{K_c^1}(\mathbb{R}\times X). \tag{10.64}$$

This shows the injectivity of the isotropic index map and so that (10.60) is an isomorphism. □

What does this tell us? Well, as it turns out, lots of things! For one thing the normalization conditions extend to all Euclidean space:-

$$\mathrm{K_c^1}(\mathbb{R}^k)=\begin{cases}\{0\} & k\text{ even}\\ \mathbb{Z} & k\text{ odd,}\end{cases}\quad \mathrm{K_c^0}(\mathbb{R}^k)=\begin{cases}\mathbb{Z} & k\text{ even}\\ \{0\} & k\text{ odd.}\end{cases} \tag{10.65}$$

This in turn means that we understand a good deal more about $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

THEOREM 10.1 (Bott periodicity). *The homotopy groups $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ are*

$$\pi_j(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n))=\begin{cases}\{0\} & k\ \textit{even}\\ \mathbb{Z} & k\ \textit{odd.}\end{cases} \tag{10.66}$$

Indeed Bott proved this rather directly using Morse theory.

10.7. Toeplitz index map

Although the map from $\mathrm{K_c^1}(\mathbb{R}\times X)$ to $\mathrm{K_c}(X)$ has been discussed here in terms of the quantization of symbols to isotropic pseudodifferential operators it could equally, and more conventionally, be done by quantization to 'Toeplitz operators'. The advantage of the isotropic quantization is that it extends directly to higher dimensions. The Toeplitz algebra is the 'compression' of the pseudodifferential algebra on the circle to the positive Fourier components, some form of the Hardy space. This is discussed in § 6.9. In the Toeplitz context, $\pi_{(N)}$ is projection onto the span of the first N exponentials $\exp(il\theta)$, $1\le l\le N$.

PROPOSITION 10.13. *If $A\in\mathcal{C}^\infty(X;\Psi^0_{\mathrm{To}}(\mathbb{S};\mathbb{C}^k))$ is an elliptic family of Toeplitz operators, which is constant outside a compact subset of X and has $\sigma(A)(1,x)\equiv\mathrm{Id}_{k\times k}$ then for N sufficiently large $A(\mathrm{Id}-\pi_{(N)})$ and its adjoint have null spaces forming a smooth vector bundle over X, the class $[(\mathrm{Nul}(A),\mathrm{Nul}(A^*))]\in\mathrm{K}^0_c(X)$ depends only on the class in of the symbol in $\mathrm{K}^1_c(\mathbb{R}\times\mathbb{S})$ and the map so defined*

$$\mathrm{Ind_{To}}:\mathrm{K}^1_c(\mathbb{R}\times X)\longrightarrow\mathrm{K}^0_c(X) \tag{10.67}$$

is equal to the isotropic index map discussed above.

Notice that the assumption that the symbol of A is equal to the identity at $\theta=1$ on the circe, for all $x\in X$, means that it can be interpreted (after a little deformation) as defining an element in the compactly supported K-group on the left in (10.67), where $\mathbb{R}$ is identified with $\mathbb{S}\setminus\{1\}$ by the map

$$\mathbb{R}\ni t\longmapsto\exp(i\Theta(t)),\ \Theta\in\mathcal{C}^\infty(\mathbb{R}),\ \Theta'(t)\ge0,\ \Theta(t)=0,\ t<<0,\ \Theta(t)=2\pi,\ t>>0. \tag{10.68}$$

where the orientation is important.

[6]See Problem 10.1 for some more details

Problem 10.1. Go through the argument for the stability of the null bundle and the independence of choices, it is essentially the same as for the isotropic case but using the properties of the Toeplitz algebra, and smoothing operators on the circle instead.

Proof. The proof of the stability of the index etc, leading to the map (10.67) is essentially the same as in the isotropic case so is omitted. It remains to show that quantization by Toeplitz operators gives the same index map as quantization by istropic operators.

The shift operator, which is multiplication by $e^{-i\theta}$ followed by projection back onto the Hardy projection, is elliptic and has index 1 as a Toeplitz operator. Its symbol is homotopic, after the identification (10.68), with the symbol of the annihilation operator in the isotropic algebra (after change of order using the square root of the harmonic oscillator), which also has index 1 and is the Bott element. Thus the two indexes agree on this element, with X a point. The argument of sujectivity for the isotropic index above, which involves twisting the annihilation operator with a bundle on X applies equally well in the Toeplitz setting. Thus both maps are surjective and the injectivity of the isotropic index shows that these element span $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X)$, so the two maps give the same isomorphism. □

10.8. The isotropic-semiclassical index (or quantization) maps

Especially since the geometric version of the odd index plays a considerable role in the proof of the index theorem of Atiyah and Singer below, we next discuss the 'odd' version of the isotropic index theorem which arises from the semiclassical limit for isotropic operators. This is used in the next section to obtain the Thom isomorphism in K-theory.

We shall show that for any even dimensional Euclidean space the symbol map for isotropic smoothing operators leads to the isomorphism in (10.5):

$$\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso}} : \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X) \tag{10.69}$$

for any manifold X. This is consistent with the other Bott periodicity constructions, as is shown below.

Proposition 10.14. *If $a \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times X; \mathrm{GL}(N,\mathbb{C}))$ has compact support, in the sense that $a = \mathrm{Id}$ outside a compact set, the there exists $A' \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\mathrm{sl\,iso}}(\mathbb{R}^n); \mathbb{C}^N)$ such that $\sigma_{\mathrm{sl}}(A') = a - \mathrm{Id}_N$ and then for small $\epsilon > 0$ $[\mathrm{Id}_N + A_\epsilon] \in \mathrm{K}^1_c(X)$ depends only on $[a] \in \mathrm{K}^1_c(\mathbb{R}^{2n} \times X)$ and gives the isomorphism* (10.69).

Proof. The main step is the existence of the semiclassical family, reducing to the identity outside a compact set, but this is shown in Chapter 4.

The fact that $[A_\epsilon]$, for $\epsilon > 0$ so small that A_δ is invertible for all $0 < \delta < \epsilon$, only depends on a follows from the homotopy equivalence of all possible semiclassical quantizations. The independence of choices follows from similar arguments to those above – homotopies induce homotopies and stability leads to stability. □

The even isotropic-semiclassical index map is defined in a similar way using the quantization of projections.

Proposition 10.15. *If $p \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times X; M(N,\mathbb{C}))$ is a family of projections which takes a fixed value p_∞ outside some compact set then there exists $Q \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\mathrm{sl\,iso}}(\mathbb{R}^n); \mathbb{C}^N)$ such that $p_\infty + Q$ is a family of projections with*

$p_\infty + \sigma_{\mathrm{sl}}(Q) = p$ and for $\epsilon > 0$ sufficiently small $[p_\infty + Q_\epsilon] \in \mathrm{K}^0_c(X)$ only depends on $[p] \in \mathrm{K}^0_c(\mathbb{R}^{2n} \times X)$ and this leads to the isomorphism (10.6).

PROOF. □

It is important that these two maps are consistent with each other, under iteration and with Bott periodicity, as discussed in the preceeding section.

LEMMA 10.7. *The isotropic-semiclassical quantization maps and the clutching map* (10.46) *lead to the commutative diagrammes of isomorphisms*

$$\begin{tikzcd} \mathrm{K}^1_c(\mathbb{R}^{2n} \times X) \arrow[r, "\mathrm{clu}"] \arrow[d, "\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso}}"'] & \mathrm{K}^0_c(\mathbb{R} \times \mathbb{R}^{2n} \times X) \arrow[d, "\mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso}}"] \\ \mathrm{K}^1_c(X) \arrow[r, "\mathrm{clu}"'] & \mathrm{K}^0_c(\mathbb{R} \times X). \end{tikzcd} \tag{10.70}$$

and

$$\begin{tikzcd} \mathrm{K}^0_c(\mathbb{R}^2 \times \mathbb{R}^{2n} \times X) \arrow[r, "\mathrm{clu}^{-1}"] \arrow[d, "\mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso}}"'] & \mathrm{K}^1_c(\mathbb{R} \times \mathbb{R}^{2n} \times X) \arrow[r, "\mathrm{Ind}_{\mathrm{iso}}"] \arrow[d, "\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso}}"'] & \mathrm{K}^0_c(\mathbb{R}^{2n} \times X) \arrow[d, "\mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso}}"] \\ \mathrm{K}^0_c(\mathbb{R}^2 \times X) \arrow[r, "\mathrm{clu}^{-1}"'] & \mathrm{K}^1_c(\mathbb{R} \times X) \arrow[r, "\mathrm{Ind}_{\mathrm{iso}}"] & \mathrm{K}^0_c(X). \end{tikzcd} \tag{10.71}$$

PROOF. This commutativity of (10.70) is immediate from the explicit formula for the clutching construction in (10.42) and (10.41). Namely, if A is an isotropic-semiclassical quantization of a then A^{-1} quantizes a^{-1} and inserting A and A^{-1} into (10.41) gives an isotropic-semiclassical quantization of Π_a to a family of projections defining Π_A.

The left square in (10.71) is (10.70) for X replaced by $\mathbb{R} \times X$ and written backwards. The commutativity of the right square follows from the formula for the inverse of the isotropic index. Again this is given by an explicit formula, lift a projection to an invertible family, as in (10.57). Thus the commutativity of the square with the horizontal maps inverted follows and since these are isomorphisms (10.71) is also commutative. □

LEMMA 10.8. *For any manifold X*

$$\begin{aligned} \mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso}} &= (\mathrm{Ind}_{\mathrm{iso}} \circ \mathrm{clu}^{-1})^n : \mathrm{K}^0_c(\mathbb{R}^{2n} \times X) \longrightarrow \mathrm{K}^0_c(X), \\ \mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso}} &= (\mathrm{clu}^{-1} \circ \mathrm{Ind}_{\mathrm{iso}})^n : \mathrm{K}^1_c(\mathbb{R}^{2n} \times X) \longrightarrow \mathrm{K}^1_c(X) \end{aligned} \tag{10.72}$$

are the Bott periodicity maps and for any N and M and either parity p, the diagramme

$$\begin{tikzcd} & \mathrm{K}^{\mathrm{p}}_c(X^{2M} \times X) \arrow[dr, "\mathrm{Ind}^p_{\mathrm{iso}}"] & \\ \mathrm{K}^{\mathrm{p}}_c(X^{2N+2M} \times X) \arrow[ur, "\mathrm{Ind}^p_{\mathrm{iso}}"] \arrow[rr, "\mathrm{Ind}^p_{\mathrm{iso}}"] & & \mathrm{K}^{\mathrm{p}}_c(X) \end{tikzcd} \tag{10.73}$$

commutes.

PROOF. To prove (10.73) □

10.9. Complex and symplectic bundles

In Chapter 4 the algebra of istropic pseudodifferential on a symplectic vector space F is discussed. For example the algebra of operators of order 0 is just a non-commutative product on the space of smooth functions on the radial compactification of F, $\mathcal{C}^\infty(\overline{F})$. This product varies smoothly with the symplectic form used to define it. Now suppose that $E \longrightarrow X$ is a real vector bundle over a manifold X which has a symplectic structure, that is a section

$$\begin{gathered}\omega \in \mathcal{C}^\infty(X; \Lambda^2 F'),\\ v \in F_x,\ \omega_x(v,w) = 0\ \forall\ w \in F_x \Longrightarrow v = 0\end{gathered} \tag{10.74}$$

is given. Then the isotropic algebras on the fibre combine to a smooth bundle of algebras. It is this bundle of algebras which we will use to discuss the Thom isomorphism. Since the Thom isomorphism in K-theory is usually thought of in terms of complex bundles, not realy symplectic bundles, we recall the relationship between them here.

Recall that any complex vector space F has an underlying real vector space, $F_{\mathbb{R}}$, which is the same set with only real multiplication allowed. Then multiplication by i on F becomes a real isomorphism $J : F_{\mathbb{R}} \longrightarrow \mathbb{R}_{\mathbb{R}}$ with the property that $J^2 = -\operatorname{Id}$. Conversely, on a real vector space with such an isomorphism, complex multiplication, defined with multiplication by $z = \alpha + i\beta$ being $\alpha + \beta J$, is a complex vector space with the original real vector space underlying it.

Lemma 10.9. *A real vector bundle of even rank admits a complex structure if and only if it admits a symplectic structure and the homotopy classes of these structures are in 1-1 correspondence.*

If X is not compact, this correspondence of complex or symplectic structures extends to those which are trivialized outside a compact set.

Proof. This is really just the corresponding construction in linear algebra. Any complex vector space F admits an Hermitian structure, a sequilinear positive definite form:

$$\begin{gathered}h : F \times F \longrightarrow \mathbb{C},\\ h(z_1v_1 + z_2v_2, w) = z_1h(v_1,w) + z_2h(v_2,w),\ h(v,w) = \overline{h(w,v)},\ h(v,v) \geq 0,\ h(v,v) = 0 \Longrightarrow v = 0.\end{gathered} \tag{10.75}$$

To see this, just take the Euclidean inner product with respect to a basis. The imaginary part of h,

$$\omega_h(v,w) = \Im h(v,w) \tag{10.76}$$

is a symplectic form on $F_{\mathbb{R}}$. Moreover $h(v,w) = \omega_h(v, Jw) + i\omega(v,w)$ so the Hermitian structure can be recovered from the symplectic strucure and J. Conversely, if V is a real vector space with symplectic form ω_V then choosing a real Euclidean structure g on V defines a linear map $J' : V \longrightarrow V$ by

$$\begin{gathered}\omega_V(v, J'w) = g(v,w) \Longrightarrow\\ \omega_V(v, J'w) = g(v,w) = g(w,v) = \omega_V(w, J'v) = -\omega_V(Jv, w).\end{gathered} \tag{10.77}$$

Thus $g(J'v,w) = \omega_V(J'v, J'w) = -\omega_V(J'w, J'v) = -g(J'w, v) = -g(v, J'w)$ shows that J' is skew-adjoint with respect to g and $g((J')^2v, w) = -g(J'v, J'w)$ shows that its square is negative definite and self-adoint. Thus $-(J')^2 = A^2$ where A is a positive definite real self-adjoint matrix, with respect to g, which commutes with J' (since iJ' is self-adjoint and its eigenvectors are eigenvectors of A^2 and hence A. Thus $J = A^{-1}J'$ is a complex structure, $J^2 = -\operatorname{Id}$.

For a symplectic vector bundle, this construction can be carried out smoothly, simply by choosing a smooth family of real metrics on the fibres. The construction of J from J' is determined and hence is easily seen to yield a smooth homorphism J of the real bundle, and hence a smooth complex structure. Moreover both the construction of a complex structure from the symplectic and of the symplectic structure from the complex can lift to homotopies, since they can be carried out smoothly in parameters. □

10.10. Thom isomorphism

The even semiclassical isotropic index map is shown above to generate an isomorphism

$$\operatorname{Ind}_{\mathrm{iso,sl}} : \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X) \tag{10.78}$$

for any manifold X. Here the product can be interpreted as a trivial even-rank bundle over X. The Thom isomorphism extends this to the bundles discussed in the previous section.

Proposition 10.16. *If $F \longrightarrow X$ is an even-rank real vector bundle over X, trivial outside a compact set and with a symplectic structure constant outside a compact set then semiclassical isotropic quantization gives an isomorphism*

$$\begin{gathered}\mathrm{Thom} : \mathrm{K}^0_c(F) \longrightarrow \mathrm{K}^0_c(X) \text{ with inverse} \\ \mathrm{K}^0_c(X) \ni [\mathbb{V}] \longmapsto [\mathbb{V} \otimes b_E] \in \mathrm{K}^0_c(E)\end{gathered} \tag{10.79}$$

where $b_E \in \mathrm{K}^0_{\mathrm{c}}(E)$ is the Bott element corresponding to the harmonic oscillator.

Proof. We first show that isotropic quantization of projections on the fibres descends to an index map in the bundle case (10.79) in the bundle case. Certainly an element of $\mathrm{K}^0_{\mathrm{c}}(F)$ is represented by a smooth map $F \longrightarrow M(N, \mathbb{C})$ for some N with values in the projections and constant outside a compact subset of F (which of course projects to a compact subset of X). Mainly we just need to show that the previous discussion extends smoothly to this case and also that there is a smooth 'Bott element' $\beta_F \in \mathrm{K}^0_{\mathrm{c}}(F)$, so represented by a family of projections, such that $\operatorname{Ind}(F) = [\mathbb{C}]$ is a trivial one-dimensional bundle. Then the second line in (10.79) gives a left inverse of the index,

$$\operatorname{Ind}(\pi^*[\mathbb{V}] \otimes \beta_E) = [\mathbb{V}] \in \mathrm{K}^0_{\mathrm{c}}(X). \tag{10.80}$$

As for the original isotropic index, this proves that the index map is surjective for any symplectic bundle F as in the statement above. So only the injectivity remains to be shown.

If F is a real vector bundle with symplectic structure then it is shown above that it can be realized as the underlying real vector bundle for a complex vector bundle with the symplectic structure being the imaginary part of an Hermitian structure on E. If F is trivial, with constant symplectic structure outside a compact set, then E can be taken to be trivial with complex Hermitian structure outside a compact set. Then E can be embedded as a subbundle of a trivial complex bundle $\mathbb{C}^N$ with constant inclusion outside a compact set. Extending the Hermitian structure to the whole bundle, as a direct sum, and constant outside a compact set, shows that

F can be complemented to a trivial bundle with direct symplectic forms constant outside a compactum. Let the complementary bundle be G so $F \oplus G = \mathbb{R}^{2k}$ for some k. Now we have maps

$$
\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2k} \times X) \arrow[rr, "\mathrm{Ind}"] \arrow[dr, "\mathrm{Ind}_G"'] & & \mathrm{K}^0_{\mathrm{c}}(X). \\
& \mathrm{K}^0_{\mathrm{c}}(F) \arrow[ur, "\mathrm{Ind}_F"'] &
\end{tikzcd}
\tag{10.81}
$$

We claim that this diagramme commutes. *** This is supposed to be done back in the isotropic chapter, namely that when quantizing a projection on the product of two vector spaces one can first quantize in one subspace and then the other. For the moment the more complicated case of the adiabatic limit has already been done so this should be clear enough.

From the commutativity of (10.81) it follows that Ind_F is an isomorphism. Indeed, the bottom two are injective and top is known to be an isomorphism from the preceeding discussion. Thus Ind_F must also be surjective and hence is an isomorphism and Ind_G is Ind_F for a different bundle and base. □

10.11. Chern forms

I would not take this section seriously yet, I am going to change it.

Let's just think about the finite-dimensional groups $\mathrm{GL}(N, \mathbb{C})$ for a little while. Really these can be replaced by $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, as I will do below, but it may be a strain to do differential analysis and differential topology on such an infinite dimensional manifold, so I will hold off for a while.

Recall that for a Lie group G the tangent space at the identity (thought of as given by an equivalence to second order on curves through Id), $\mathfrak{g}$, has the structure of a Lie algebra. In the case of most interest here, $\mathrm{GL}(n, \mathbb{C}) \subset M(N, \mathbb{C})$ is an open subset of the algebra of $N \times N$ matrices, namely the complement of the hypersurface where $\det = 0$. Thus the tangent space at Id is just $M(N, \mathbb{C})$ and the Lie algebra structure is given by the commutator

$$[a, b] = ab - ba,\ a, b \in \mathfrak{gl}(N, \mathbb{C}) = M(N, \mathbb{C}). \tag{10.82}$$

At any other point, g, of the group the tangent space may be naturally identified with $\mathfrak{g}$ by observing that if $c(t)$ is a curve through g then $g^{-1}c(t)$ is a curve through Id with the equivalence relation carrying over. This linear map from T_gG to $\mathfrak{g}$ is herlpfully denoted

$$g^{-1}dg : T_gG \longrightarrow \mathfrak{g}. \tag{10.83}$$

In this notation 'dg' is the differential of the identity map of G at g. This 'Maurier-Cartan' form as a well-defined 1-form on G with values in $\}$ – which is a fixed vector space.

The fundamental property of this form is that

$$d(g^{-1}dg) = -\frac{1}{2}[g^{-1}dg, g^{-1}dg]. \tag{10.84}$$

In the case of $\mathrm{GL}(N, \mathbb{C})$ this can be checked directly, and written slightly differently. Namely in this case as a 'function' 'g' is the identity on G but thought of as the canonical embedding $\mathrm{GL}(N,\mathbb{C}) \subset M(N,\mathbb{C})$. Thus it takes values in $M(N,\mathbb{C})$, a vector space, and we may differentiate directly to find that

$$d(g^{-1}dg) = -dgg^{-1}dg \wedge dg \tag{10.85}$$

where the product is that in the matrix algebra. Here we are just using the fact that $dg^{-1} = -g^{-1}dgg^{-1}$ which comes from differentiating the defining identity $g^{-1}g = \mathrm{Id}$. Of course the right side of (10.85) is antisymmetric as a function on the tangent space $T_gG \times T_gG$ and so does reduce to (10.84) when the product is repalced by the Lie product, i.e. the commutator.

Since we are dealing with matrix, or infinite matrix, groups throughout, I will use the 'non-intrinsic' form (10.85) in which the product is the matrix product, rather than the truly intrinsic (and general) form (10.84).

Proposition 10.17 (Chern forms). *If* tr *is the trace functional on* $N \times N$ *matrices then on* $\mathrm{GL}(N,\mathbb{C})$,

$$\begin{gathered} \mathrm{tr}((g^{-1}dg)^{2k}) = 0 \ \forall\ k \in \mathbb{N}, \\ \beta_{2k-1} = \mathrm{tr}((g^{-1}dg)^{2k-1}) \text{ is closed } \forall\ k \in \mathbb{N}. \end{gathered} \tag{10.86}$$

Proof. This is the effect of the antisymmetry. The trace idenitity, $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ means precisely that tr vanishes on commutators. In the case of an even number of factors, for clarity evaluation on $2k$ copies of $T_g\,\mathrm{GL}(N,\mathbb{C})$, given for $a_i \in M(N,\mathbb{C})$, $i = 1,\dots,2k$, by the sum over

$$\begin{gathered} \mathrm{tr}((g^{-1}dg)^{2k})(a_1,a_2,\dots,a_{2k}) = \sum_e \mathrm{sgn}(e)\,\mathrm{tr}(g^{-1}a_{e(1)}g^{-1}a_{e(2)}\dots g^{-1}a_{e(2k)}) = \\ -\sum_e \mathrm{sgn}(e)\,\mathrm{tr}(g^{-1}a_{e(2k)}g^{-1}a_{e(1)}\dots g^{-1}a_{e(2k-1)}) = -\mathrm{tr}((g^{-1}dg)^{2k})(a_1,a_2,\dots,a_{2k}). \end{gathered} \tag{10.87}$$

In the case of an odd number of factors the same manipulation products a trivial identity. However, notice that

$$g^{-1}dgg^{-1} = -d(g^{-1}) \tag{10.88}$$

is closed, as is dg. So in differentiating the odd number of wedge products each pair $g^{-1}dgg^{-1}dg$ is closed, so (tr being a fixed functional)

$$d\beta_{2k-1} = \mathrm{tr}(dg^{-1})(g^{-1}dgg^{-1}dg)^{2k-2}) = -\mathrm{tr}((g^{-1}dg)^{2k}) = 0 \tag{10.89}$$

by the previous discussion. □

Now, time to do this in the infinite dimensional case. First we have to make sure we know that we are talking about.

Definition 10.3 (Fréchet differentiability). *A function on an open set of a Fréchet space,* $O \subset F$, $f : O \longrightarrow V$, *where* V *is a locally convex topological space (here it will also be Fréchet, and might be Banach) differentiable at a point* $u \in O$ *if there exists a continuous linear map* $D : F \longrightarrow V$ *such that for each continuous seminorm* $\|\cdot\|_\alpha$ *on* V *there is a continuous norm* $\|\cdot\|_i$ *on* F *such that for each* $\epsilon > 0$ *there exists* $\delta > 0$ *for which*

$$\|v\|_i < \delta \Longrightarrow \|f(u+v) - f(u) - Tv\|_\alpha \le \epsilon\|v\|_i. \tag{10.90}$$

This is a rather strong definition of differentiability, stronger than the Gâteaux definition – which would actually be enough for most of what we want, but why not use the stronger condition when it holds?

Proposition 10.18. *The composition of smoothing operators defines a bilinear smooth map*

$$\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \times \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n), \|ab\|_k \le C_k\|a\|_{k+N}\|b\|_{k+N} \tag{10.91}$$

(where the kth norm on u is for instance the $\mathcal{C}^k$ norm on $\langle z\rangle^k u$ and inversion is a smooth map

$$G^{-\infty}_{\text{iso}}(\mathbb{R}^n) \longrightarrow G^{-\infty}_{\text{iso}}(\mathbb{R}^n). \tag{10.92}$$

Proof. I did not define smoothness above, but it is iterated differentiability, as usual. In fact linear maps are always differentiable, as follows immediately from the definition. The same is true of jointly continuous bilinear maps, so the norm estimates in (10.91) actually prove the regularity statement. The point is that the derivative of a bilinear map P at $(\bar a, \bar b)$ is the linear map

$$Q_{\bar a,\bar b}(a,b) = P(a,\bar b)+P(\bar a,b), P(\bar a+a,\bar b+b)-P(\bar a,\bar b)-Q_{\bar a,\bar b}(a,b) = P(a,b). \tag{10.93}$$

The bilinear estimates themselves follow directly by differentiating and estimating the integral composition formula

$$a\circ b(z,z') = \int a(z,z'')b(z'',z')dz''. \tag{10.94}$$

The shift in norm on the right compared to the left is to get a negative factor of $\langle z''\rangle$ to ensure integrability.

Smoothness of the inverse map is a little more delicate. Of course we do know what the derivative at the point g, evaluated on the tangent vector a is, namely $g^{-1}ag^{-1}$. So to get differentiability we need to estimate

$$(g+a)^{-1} - g^{-1} + g^{-1}ag^{-1} = g^{-1}a\left(\sum_{k\ge 0}(-1)^{k+1}g^{-1}(ag^{-1})^k\right)ag^{-1}. \tag{10.95}$$

This is the Neumann series for the inverse. If a is close to 0 in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then we know that $\|a\|_{L^2}$ is small, i.e. it is bounded by some norm on $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus the series on the right converges in bounded operators on $L^2(\mathbb{R}^n)$. However the smoothing terms on both sides render the whole of the right side smoothing and with all norms small in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ when a is small.

This proves differentiability, but in fact infinite differentiability follows, since the differentiability of g^{-1} and the smoothness of composition, discussed above, shows that $g^{-1}ag^{-1}$ is differentiable, and allows one to proceed on inductively. □

We also know that the trace functional extends to $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ as a trace functional, i.e. vanishing on commutators. This means that the construction above of Chern classes on $\mathrm{GL}(N,\mathbb{C})$ extends to $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

Proposition 10.19. *(Universal Chern forms) The statements* (10.86) *extend to the infinite-dimensional group $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ to define deRham classes $[\beta_{2k-1}]$ in each odd dimension.*

In fact these classes generate (not span, you need to take cup products as well) the cohomology, over $\mathbb{R}$, of $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

Proof. We have now done enough to justify the earlier computations in this setting. □

PROPOSITION 10.20. *If X is a manifold and $a \in \mathcal{C}_c^\infty(X; G_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ then the forms $a^*\beta_{2k-1}$ define deRham classes in $\mathrm{H}_c^{2k+1}(X;\mathbb{R})$ which are independent of the homotopy class and so are determined by $[a] \in \mathrm{K}_c^1(X)$. Combining them gives the (odd) Chern character*

$$\mathrm{Ch}_o([a]) = \sum_k c_{2k-1} a^*\beta_{2k-1}. \tag{10.96}$$

the particular constants chosen in (10.96) corresponding to multiplicativity under tensor products, which will be discussed below.

PROOF. The independence of the (smooth) homotopy class follows from the computation above. Namely if $a_t \in \mathcal{C}_c^\infty(X\times[0,1]; G_{\text{iso}}^{-\infty}(\mathbb{R}^b n)$ then $B_{2k-1} = a_t^*\beta_{2k-1}$ is a closed $(2k-1)$-form on $X \times [0,1]$. If we split it into the two terms

$$B_{2k-1} = b_{2k-1}(t) + \gamma_{2k-1}(t) \wedge dt \tag{10.97}$$

where $b_{2k-1}(t)$ and $\gamma_{2k-1}(t)$ are respectively a t-dependent $2k-1$ and $2k-2$ form, then

$$\begin{gathered} dB_{2k-1} = 0 \Longleftrightarrow \frac{\partial}{\partial t} b_{2k-1}(t) = d_X\gamma_{2k-2}(t) \text{ and hence} \\ b(1)_{2k-1} - b(0)_{2k-1} = d\mu_{2k-2},\ \mu_{2k-2} = \int_0^1 dt\gamma_{2k-2}(t) \end{gathered} \tag{10.98}$$

shows that $b(1)_{2k-1}$ and $b(0)_{2k-1}$, the Chern forms of a_1 and a_0 are cohomologous. □

The even case is very similar. Note above that we have defined even K-classes on X as equivalence classes under homotopy of elements $a \in \mathcal{C}_c^\infty(X; G_{iso,\text{sus}}^{-\infty}(\mathbb{R}^n)$. The latter group consists of smooth loops in $G_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ starting and ending at Id. This means there is a natural (smooth) map

$$T : G_{\text{iso,sus}}^{-\infty}(\mathbb{R}^n) \times \mathbb{S} \longrightarrow G_{\text{iso}}^{-\infty}(\mathbb{R}^n),\ (a,\theta) \longmapsto a(\theta). \tag{10.99}$$

This map may be used to pull back the Chern forms discussed above to the product and integrate over $\mathbb{S}$ to get forms in even dimensions:-

$$\beta_{2k} = \int_0^{2\pi} \operatorname{tr}(g^{-1}dg)^{2k+1},\ k = 0, 1, \dots. \tag{10.100}$$

PROPOSITION 10.21. *The group $G_{\text{iso,sus}}^{-\infty}(\mathbb{R}^n)$ has an infinite number of components, labelled by the 'index' β_0 in (10.100), the other Chern forms define cohomology classes such that for any map*

$$\mathrm{Ch}([a]) = \sum_{k=0}^\infty c_{2k} a^*\beta_{2k} \tag{10.101}$$

defines a map $\mathrm{K}_c^0(X) \longrightarrow \mathrm{H}^{\text{even}}(X)$.

The range of this map spans the even cohomology, this is a form of a theorem of Atiyah-Hurzebruch.

If $f : X \longrightarrow Y$ is a smooth map then it induces a pull-back operation on vector bundles (see Problem 10.2) and this in turn induces an operation

PROBLEM 10.2.

$$f^* : \mathrm{K}(Y) \longrightarrow \mathrm{K}(X). \tag{10.102}$$

Now we can interpret Proposition **??** in a more K-theoretic form.

10.12. Chern character

We have seen above that the 'unnormalized' Chern forms $\mathrm{Tr}((a^{-1}da)^{2k+1})$ are well-defined closed forms on the group $G^{-\infty}$ and allow manipulation in the uisual way. In particular they each pull back to give cohomology classes associated to a given odd K-class on a manifold. It is important for us to understand the behaviour of these forms under the basic maps in K-theory that we have defined above. The most important is the isotropic/Toeplitz index map (10.47). For the moment, we will take X to be compact even though this is not necessary.

The inverse of (10.47) we know explicitly, that is we know how to represent a bundle as the index bundle of a family of isotropic (or Toeplitz) operators. Namely if E is a vector bundle over X then it can be embedded as a subbundle of a trivial bundle so there is a smooth family of projections $\Pi \in \mathcal{C}^\infty(X; M(N, \mathbb{C}))$ such that we may identify $E = \mathrm{Ran}(\Pi)$. Then E (as an element of $K(X)$) is the index bundle for any isotropic family with symbol

$$a(x, \theta) = e^{-i\theta}\Pi(x) + (\mathrm{Id}_N - \Pi(x)),\ \Pi(x)^2 = \Pi(x). \tag{10.103}$$

So, it is naturally of interest to compute the (odd) Chern forms of a on $\mathbb{S} \times X$.

Computing away,

$$\begin{aligned} a^{-1}da &= \left(e^{i\theta}\Pi + \mathrm{Id} - \Pi\right)\left(-ie^{-i\theta}d\theta\Pi + (e^{-i\theta} - 1)d_X\Pi\right) \\ &= e^{i\theta}(e^{-i\theta} - 1)\Pi d_X\Pi + (e^{-i\theta} - 1)(\mathrm{Id} - \Pi)d_X\Pi - id\theta\Pi. \end{aligned} \tag{10.104}$$

As a form on a product manifold we may decompose

$$\mathrm{Tr}((a^{-1}da)^{2k+1}) = d\theta \wedge \alpha + \beta \tag{10.105}$$

where α and β are forms on X depending smoothly on θ. Since we know it is closed it follows that

$$d\theta(\partial_\theta\beta - d_X\alpha) + d_X\beta = 0 \Longrightarrow d_X\beta = 0,\ d_X\alpha = \partial_\theta\beta. \tag{10.106}$$

Expanding α and β in Fourier series

$$\alpha = \sum_{k\in\mathbb{Z}} e^{ik\theta}\alpha_k,\ \beta = \sum_{k\in\mathbb{Z}} e^{ik\theta}\beta_k \tag{10.107}$$

it follows from (10.106) that all the β_k with $k \neq 0$ are exact. In fact all the terms in (10.105) corresponding to $k \neq 0$ are exact since

$$d\theta \wedge e^{ik\theta}\alpha_k + e^{ik\theta}\beta_k = d(\frac{1}{ik}e^{ik\theta}\alpha_k). \tag{10.108}$$

So the only cohomology which can arise comes from the terms α_0 and β_0 since separately $d\alpha_0 = 0$ and $d\beta_0 = 0$.

Problem 10.3. Show that β_0 arising from the Chern form in (10.105) is cohomologous to a constant (i.e. is exact except in form degree 0. What is the constant?

So, we most want to compute α_0. By definition α is the contraction of $\mathrm{Tr}((a^{-1}da)^{2k+1})$ with ∂_θ. Watching out for normalizations it follows from antisymmetry and (10.104) that

$$\alpha = -i\,\mathrm{Tr}\left(\Pi(e^{i\theta}(e^{-i\theta}-1)\Pi(d_X\Pi) + (e^{-i\theta}-1)(\mathrm{Id}-\Pi)(d_X\Pi))^{2k}\right) \tag{10.109}$$

where you should note that for a projection $\Pi(d_X\Pi) = \Pi(d_X\Pi)(\mathrm{Id}-\Pi)$ (meaning that the differential is completely off-diagonal with respect to the projection at that point). So in fact

$$\alpha = -i(e^{ik\theta}(e^{-i\theta}-1)^{2k}\,\mathrm{Tr}\left(\Pi d_X\Pi(\mathrm{Id}-\Pi)d_X\Pi)^k\right). \tag{10.110}$$

Thus

$$\alpha_0 = -\frac{i}{2\pi}\int_0^{2\pi} e^{ik\theta}(e^{-i\theta}-1)^{2k}d\theta\,\mathrm{Tr}(\omega^k),\ \omega = \Pi(d_X\Pi)(\mathrm{Id}-\Pi)(d_X\Pi). \tag{10.111}$$

The constant here can be readily evaluated and is (perhaps)

$$-\frac{i}{2\pi}\int_0^{2\pi} e^{ik\theta}(e^{-i\theta}-1)^{2k}d\theta = -i\frac{(2k)!}{k!}. \tag{10.112}$$

Now, ω is in fact the curvature of a connection on the bundle $E = \mathrm{Ran}(\Pi)$. Namely, d can be applied to sections of $\mathrm{Ran}(\Pi)$ but will not give a new section of the bundle (with values in 1-forms as well), however $\nabla^\Pi s = \Pi ds = \Pi d\Pi s$ is a connection since it distributes over functions

$$\nabla^\Pi(fs) = dfs + f\nabla^\Pi s.$$

The curvature of this connection is easily computed, especially if one uses extension of the distribution law to all forms

$$\nabla(s\alpha) = d\alpha s + (-1)^k\alpha\nabla s,\ \alpha \in \mathcal{C}^\infty(X;\Lambda^k),$$

since then

$$(\nabla^\Pi)^2 s = \Pi d(\Pi ds) = \Pi(d\Pi)(\mathrm{Id}-\Pi)(d\Pi) = \omega. \tag{10.113}$$

Thus for this one connection we see that α_0 is a multiple of $\mathrm{Tr}(\omega^k)$. The basic observation of Chern-Weil theory is

Lemma 10.10. *For any connection ∇ on a complex vector bundle E the forms*

$$\mathrm{tr}(\omega^k) \in \mathcal{C}^\infty(X;\Lambda^{2k}),\ \omega = \nabla^2,$$

are closed and represent a fixed deRham cohomology class.

Proof. The crucial point is that (10.10) is always a closed form. The connection ∇ acts on sections of E but also defines a connection on the bundle $\hom(E)$ of homomorphisms. Namely if $b \in \mathcal{C}^\infty(X;\hom(E))$ then

$$(\nabla b)s = \nabla(bs) - b\nabla s = [\nabla, b]s$$

is a connection. As before it extends to homomorphisms with values in forms and in this sense Bianchi's identity holds

$$\nabla\omega = 0 \Longrightarrow \nabla\omega^k = 0. \tag{10.114}$$

Indeed, (10.114) just comes from the associativity of operators, that $\nabla(\nabla)^2 = (\nabla)^2\nabla$.

Locally on a coordinate patch in X over which the bundle E is trivial, i.e. can be identified with $\mathbb{C}^N$, any connection takes the form $d+\gamma$ where γ is a homomorphism of $\mathbb{C}^N$ with values in 1-forms on X in the open set. Then the connection acting on homomorphisms becomes $\nabla b = db + [\gamma, b]$ and so

$$d\,\mathrm{tr}(\omega^k) = \mathrm{tr}(d\omega^k) = \mathrm{tr}(d\omega^k + [\gamma, \omega^k]) = \mathrm{tr}(\nabla\omega^k) = 0 \tag{10.115}$$

using the trace identity.

Thus, $\mathrm{tr}(\omega^k)$ is a closed form for the curvature of any connection on E. To see that its cohomology class does not depend on which connection is used, observe that any two connections ∇_i $i = 0, 1$ are connected by a smooth path of connections, $\nabla_t = (1-t)\nabla_0 + t\nabla_1$, $t \in [0, 1]$. This 1-parameter family of connections is also a connection on E pulled back from X to $X \times [0, 1]$ in the sense that it defines

$$\nabla s(t, x) = \nabla_t s(t, x) + dt\partial_t s(t, x). \tag{10.116}$$

The Chern form $\mathrm{tr}(\nabla^2)$ is therefore closed as a form on $X \times [0, 1]$ from which it follows that $\mathrm{tr}(\nabla_0^2)$ and $\mathrm{tr}(\nabla_1^2)$, which are its pull-backs to $t = 0$ and $t = 1$, are cohomologous by the analogue of (10.106). □

This means that the cohomology classes

$$\mathrm{Ch}(E, \nabla) = \mathrm{tr}(\omega^k),\ \mathrm{Ch}(E) = [\mathrm{Ch}(E, \nabla)] \in H^{2k}(X) \tag{10.117}$$

are well-defined.

Lemma 10.11. *The Chern forms in* (10.117) *define maps*

$$K(X) \longrightarrow H^{2k}(X),\ k \in \mathbb{N}_0. \tag{10.118}$$

Proof. For the formal difference (E_+, E_-) of two bundles the Chern classes are just the differences. To see that this gives a well-defined map (10.118) we need to check that it respect equivalence classes. Invariance under bundle isomorphisms is obvious enough ****. To see invariance under stability, that $(E_+ \oplus F, E_- \oplus F)$ defines the same class as (E_+, E_-) it suffices to consider the Chern classes of sums of bundles. In fact the Chern classes are additive, since we can always take as connection on a sum the direct sum of connections on the summands. Then the curvature is the direct sum of the curvatures and it follows that

$$\mathrm{Ch}(E \oplus F, \nabla^E \oplus \nabla^F) = \mathrm{Ch}(E, \nabla^E) + \mathrm{Ch}(F, \nabla^F) \tag{10.119}$$

at the level of forms, and hence certainly at the level of cohomolgy. □

It is also straightforward to see what happens to these Chern forms for the tensor product of two bundles. Again on $E \otimes F$ on can take as connection the tensor product of connections on the bundles. Then

$$(\nabla^E \otimes \nabla^F)^2 = (\nabla^E)^2 \otimes \mathrm{Id}_E + \mathrm{Id}_E \otimes (\nabla^F)^2 \tag{10.120}$$

and it follows that the Chern forms decompose (for this connection)

$$\mathrm{tr}_{E\otimes F}(\omega_{E\otimes F})^k = \sum_{j=0}^{k} \binom{k}{j} \mathrm{tr}_E((\omega_E)^j) \wedge \mathrm{tr}((\omega_F)^{k-j}). \tag{10.121}$$

From the properties of the exponential and binomial coefficients it follows that the Chern character, formally a sum of all the Chern forms,

$$\mathrm{Ch}(E) = \mathrm{tr}(\exp(\omega)) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{tr}(\omega^k) \in \mathcal{C}^\infty(X; \Lambda^*) \tag{10.122}$$

defines a map which is both additive and multiplicative

(10.123)
$$\mathrm{Ch} : \mathrm{K}^0(X) \longrightarrow H^{\mathrm{even}}(X),\ \mathrm{Ch}(e+f) = \mathrm{Ch}(e) + \mathrm{Ch}(f),\ \mathrm{Ch}(ef) = \mathrm{Ch}(e) \wedge \mathrm{Ch}(f)$$

in cohomolgy (where you might prefer to think of wedge as the cup product). The basic normalization ensures that the constant terms is the (effective) rank of the bundle. A second normalization is possible, multiplying the curvature by a constant. This is frequently chosen so that the term of degree 2 is integral, i.e. is in the image of the integral cohomology.

Now, having normalized the even Chern character, consider the second map involved in Bott periodicity. Namely the injection

$$\mathrm{K}^1(X) \longrightarrow \mathrm{K}^0(\mathbb{S} \times X). \tag{10.124}$$

Here we use an element $a \in \mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ to define a vector bundle over $\mathbb{S} \times X$ by 'clutching'. The bundle can be defined in terms of its global section, so set, for $\epsilon > 0$ small,

(10.125)
$$\mathcal{C}^\infty(\mathbb{S} \times X; E_a) = \left\{ s \in \mathcal{C}^\infty([0, 2\pi + \epsilon); \mathbb{C}^N); s(t + 2\pi, x) = a(x)s(t, x),\ t \in [0, \epsilon) \right\}.$$

Problem 10.4. Go through the proof that there is a smooth vector bundle over $\mathbb{S} \times X$ such that $\mathcal{C}^\infty(\mathbb{S} \times X; E_a)$, as defined in (10.125), is the space of global sections. Hint:- Define the fibre as a quotient of the putative space of sections.

We wish to consider the Chern character of the bundle E_a and related it to a sum of forms on X. To do so we need to choose a connection on E_a; this can be thought of as a differential operator on sections. Namely if $\rho \in \mathcal{C}^\infty(\mathbb{R})$ has $\rho(t) = 1$ in $t < 1$ and $\rho(t) = 0$ in $t > \pi$ then

$$\nabla s(t) = d_X s + dt \partial_t s + \rho(t) a^{-1} das \tag{10.126}$$

is a well-defined operator

$$\nabla : \mathcal{C}^\infty(\mathbb{S} \times X; E_a) \longrightarrow \mathcal{C}^\infty(\mathbb{S} \times X; E_a \otimes \Lambda^1). \tag{10.127}$$

Indeed, if $\epsilon > 0$ is small enough, $\rho(t) = 1$ in $t < \epsilon$ and

$$\nabla s(2\pi + t, x) = ds(2\pi + t, x) = das(t, x) = a(\nabla s(t, x)). \tag{10.128}$$

It is convenient to choose ρ so that $\rho' \leq 0$.

Lemma 10.12. *The bundle E_a is isomorphic to the range of Π_a in Lemma 10.5.*

Proof. We proceed to show that E_a can be embedded as a subbundle of $\mathbb{C}^{2N}$ as a bundle over $\mathbb{S} \times X$. Consider $E_a \oplus E_{a^{-1}}$. This is defined by the same construction as E_a with a replaced by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{10.129}$$

acting on $\mathbb{C}^{2N}$. It was shown above that this matrix is trivial as an odd K-class, i.e. can be connected to the identity. This can be done explicitly, for instance the family B(r)=

$$\begin{pmatrix} a\cos(\theta(r)) & \mathrm{Id}_N \sin(\theta(r)) \\ -\mathrm{Id}_N \sin(\theta(r)) & a^{-1}\cos(\theta(r)) \end{pmatrix} \tag{10.130}$$

connects it to

$$\begin{pmatrix} 0 & \mathrm{Id}_N \\ -\mathrm{Id}_N & 0 \end{pmatrix} \tag{10.131}$$

if $\theta : [0,1] \longrightarrow [0,2\pi]$ is weakly increasing and constant at 0 and 2π near the end points. Reversing the curve with a replaced by the identity connects (10.129) to the identity.

Now, to embed E_a as a subbundle of $\mathbb{C}^{2N}$ it suffices to consider the bundle $E_{B(t)}$ over $\mathbb{S} \times X \times I$ where $B(r)$, for $r \in I$, is the curve connecting (10.129) to the identity. Thus $E_{B(r)}$ is $E_a \oplus E_{a^{-1}}$ at one end of the interval and $\mathbb{C}^{2N}$ at the other. Choosing a connection on $E_{B(r)}$ and integrating from E_a and integrating from one side to the other embeds E_a as a subbundle of $\mathbb{C}^{2N}$.

It remains to show that this subbundle is isomorphic to the range if Π_a as defined in before Lemma 10.5; to do so consider in more detail the connection on $E_{B(r)}$. From (10.126) the ∂_r component of the connection is

(10.132)

$$\nabla_{\partial_r} s = \partial_r s + \rho(t) B(r)^{-1}(\partial_r B(r)) s = \begin{cases} \partial_r s + \rho(t)\Theta'(r) A_1(x) s & r \in [0, \frac{\pi}{2}] \\ \partial_r s + \rho(t)\Theta'(-r) A_2 s & r \in [\frac{\pi}{2}, \pi] \end{cases}$$

$$A_1(x) = \begin{pmatrix} 0 & a^{-1}(x) \\ -a(x) & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & \mathrm{Id}(x) \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

The induced connection on homomorphisms acts by conjugation, so the projection in $\mathbb{C}^{2N}$ which gives the embedding is the solution of

$$\partial_r \Pi(r) + \rho(t)\Theta'(r)[A(x), \Pi(r)] = 0,\ \Pi(0) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}. \tag{10.133}$$

We will do this in two stages, corresponding to the two subintervals for $B(r)$. It is natural to look for $\Pi(r) = Q(r)\Pi(0)Q(r)^{-1}$ with Q invertible. Then the differential condition (10.133) can be replaced by

(10.134)

$$\partial_r Q(r) + \rho(t)\Theta'(r) A_1(x) Q(r) = 0 \Longrightarrow \partial_r (Q(r)^{-1}) + \rho(t)\Theta'(r) Q(r)^{-1} A_1(x) = 0,$$

where $Q(0) = \mathrm{Id}$. This is satisfied by

$$Q(r) = S(\rho(t)\Theta(r), -a^{-1}) \tag{10.135}$$

where $S(\theta, a)$ is defined in (10.38).

Thus after the first interval of integration the projection is

$$S(2\pi\rho(t), -a^{-1})\Pi(0)S(-2\pi\rho(t), -a^{-1}). \tag{10.136}$$

In the second interval of the homotopy, a is replaced by the identity so E_a is embedded in $\mathbb{C}^{2N}$ through the projection

$$S(-2\pi\rho(t), -\mathrm{Id})S(2\pi\rho(t), -a^{-1})\Pi(0)S(-2\pi\rho(t), -a^{-1})S(2\pi\rho(t), -\mathrm{Id}). \tag{10.137}$$

This is the same as $\Pi_a(t, x)$ in (10.39) except that all the signs are wrong at once!**** Better try to get the orientations right! $\square$

Now the curvature of this connection over $\mathbb{S} \times X$ is

$$\nabla^2 = (d + \rho(t) a^{-1} da)^2 = \rho'(t) dt a^{-1} da + (\rho^2(t) - \rho(t))(a^{-1} da)^2. \tag{10.138}$$

The Chern character form of E_a with respect to this connection is

$$\operatorname{tr}\sum_k \frac{1}{k!}(\rho'(t)dta^{-1}da + (\rho^2(t)-\rho(t))(a^{-1}da)^2)^k. \tag{10.139}$$

From this even-degree sum of closed forms on $\mathbb{S}\times X$ we can extract an odd-degree sum of forms on X by integration over $\mathbb{S}$. Changing variable from t to $\rho(t)$ gives

$$\begin{aligned}\operatorname{Ch}^{\mathrm{odd}}(a) &= \sum_k \frac{1}{(k-1)!}\int_0^1 a^{-1}da((\rho^2(t)-\rho(t))(a^{-1}da)^2))^{k-1}\\ &= \int_0^1 \operatorname{tr}(a^{-1}da\exp(w(s)))ds,\ w(s)=s(s-1)(a^{-1}da)^2.\end{aligned} \tag{10.140}$$

Proposition 10.22. *The odd Chern character, defined by* (10.140)*, gives an additive map*

$$\mathrm{K}^1(X) \longrightarrow \mathrm{H}^{\mathrm{odd}}(X) \tag{10.141}$$

which has the multiplicative property

$$\operatorname{Ch}^{\mathrm{odd}}(a\otimes \mathrm{Id}_E) = \operatorname{Ch}^{\mathrm{odd}}(a)\wedge \operatorname{Ch}(E) \tag{10.142}$$

for any vector bundle E over X.

Proof. This follows directly from the discussion above. The multiplicativity in (10.142) is a consequence of the fact that if $a\otimes \mathrm{Id}_E$ is used to define a bundle over $\mathbb{S}\times X$ following the clutching construction above then the resulting bundle is $E_a\otimes E$. Then (10.142) is a consequence of the multiplicativity of the even Chern character under tensor products. □

In fact it is rather useful to generalize the formula in (10.140) by allowing a to be an isomorphism of a general bundle F over X, rather than a trivial bundle. Then a defines a class by stabilization, meaning that if F is complemented to a trivial bundle then a is extended by the identity on the complement. Proceeding directly the space of global sections of the new bundle over $\mathbb{S}\times X$ is defined by the obvious replacement of (10.125):

$$\mathcal{C}^\infty(\mathbb{S}\times X;E_a) = \{s\in\mathcal{C}^\infty([0,2\pi+\epsilon);F);s(t+2\pi,x)=a(x)s(t,x),\ t\in[0,\epsilon)\}. \tag{10.143}$$

The trivial connection d in (10.126) can then be replaced by a connection ∇^F on F and used in the same way to define a connection

$$\nabla s = (\nabla^F + \rho(t)a^{-1}\nabla a)s. \tag{10.144}$$

The formula for the odd Chern character in this more general setting is due (I believe) to Fedosov (beware of possible sign errors below, to say the least)

$$\begin{aligned}\operatorname{Ch}^{\mathrm{odd}}(a) &= \int_0^1 \operatorname{tr}(a^{-1}\nabla a\exp(w(s)))ds,\\ &\qquad w(s) = (1-s)\omega_F + sa^{-1}\omega_F a + s(s-1)(a^{-1}da)^2.\end{aligned} \tag{10.145}$$

Problem 10.5. Go through the derivation of (10.145) and correct it as necessary!

Problem 10.6. Formula (10.140) normalizes the constants in (10.96); what are they?

Going back to the discussion at the beginning of this section we can now deduce the 'Toeplitz index in cohomology'.

PROPOSITION 10.23. *Under the isotropic/Toeplitz index map* (10.47)*,*

$$\mathrm{Ch}(\mathrm{Ind}(a)) = -\frac{1}{2\pi i}\int_{\mathbb{S}} \mathrm{Ch}^{\mathrm{odd}}(a). \tag{10.146}$$

Of course this is consistent with (10.142) since we know that if E is a bundle over X then $\mathrm{Ind}(a\otimes \mathrm{Id}_E) = \mathrm{Ind}(a)\otimes E$, where this should really be thought of as products in K-theory.

PROOF. Check the constants, I haven't. *** □

I also should discuss here the extension to non-compact manifolds. This is quite straightforward.

10.13. Todd class

Now, we need to go on and see the effect on the Chern character, i.e. in cohomology, of the Thom isomorphism; whoops it isn't there yet ***. Thus, if E is a complex (or symplectic) vector bundle over X then there is an isomorphism

$$\mathrm{Thom} : \mathrm{K}^0_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^0(X) \tag{10.147}$$

which is given by the isotropic index map.

PROPOSITION 10.24. *If E is a complex vector bundle over X then there is a cohomology class* $\mathrm{Td}(E)\in \mathrm{H}^{\mathrm{even}}(E)$ *such that under the Thom isomorphism*

$$\mathrm{Ch}(\mathrm{Thom}(f)) = \int_{E/X} \mathrm{Ch}(f)\wedge \mathrm{Td}(E). \tag{10.148}$$

Note that this Todd class $\mathrm{Td}(E)$ represents a 'twisting' in the behaviour of K-theory as opposed to cohomology under push-forward.

PROOF. We are supposed to know by now that the inverse of (10.147) is given by 'twisting with the Bott element'. That is, we know there is an element $\beta_E\in \mathrm{K}^0_{\mathrm{c}}(E)$, the Bott element, represented by a family of harmonic oscillators, which has index class, $\mathrm{Thom}(\beta_E)$, a trivial 1-dimensional line bundle.

Consider first the case that E is a trivial line bundle, hence a trivial bundle with fibre $\mathbb{R}^2$ as a real space. Thus we know about Bott periodicity and in fact we get a commutative diagramme

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{R}^2) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Ch}"] & \mathrm{K}^1_{\mathrm{c}}(X\times\mathbb{R}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Ch}^{\mathrm{odd}}"] & \mathrm{K}^0_{\mathrm{c}}(X) \arrow[d, "\mathrm{Ch}"] \\ \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(X\times\mathbb{R}^2) \arrow[r, "\simeq"] & \mathrm{H}^{\mathrm{odd}}_{\mathrm{c}}(X\times\mathbb{R}) \arrow[r, "\simeq"] & \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(X). \end{tikzcd} \tag{10.149}$$

The top row we know to be isomorphisms and the two bottom maps are also isomorphisms, given by integration. We have defined the odd Chern character so that the left square commutes. We also know that the Bott element, the symbol $e^{-i\theta}$ on the circle, induces an element of $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)$ which is mapped to the trivial line by the index map, the second map on the top, and has Chern character equal to 1. The commutativity of the right square then follows from the multiplicativity of the Chern character in (10.142). This proves (10.148) in the case that E is a trivial line bundle.

Since we have not assumed that X is compact here the case of a general rank n trivial complex or rank $2n$ real bundle follows by iteration of (10.149); again the Todd class is 1.

Now, as with the Thom isomorphism for K-theory, we pass to the general case by complementing a complex bundle E to a trivial bundle $E \subset \mathbb{C}^N$ with complementary bundle F. Then we know we have isomorphisms in K-theory and cohomology leading to a commutative diagramme

$$
\begin{tikzcd}
\mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(E) \arrow[dddrrr, bend left, "\int_{E/X}"] & & & \\
& \mathrm{K}^0_{\mathrm{c}}(E) \arrow[ul, dashed, "Ch'"'] \arrow[dr, "\mathrm{Ind}"] \arrow[dd, "\otimes\beta_F"] & & \\
& & \mathrm{K}^0_{\mathrm{c}}(X) \arrow[r, "\mathrm{Ch}"] & \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(X) \\
& \mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{C}^N) \arrow[ur, "\mathrm{Ind}"] \arrow[dl, "\mathrm{Ch}"'] & & \\
\mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(X\times\mathbb{C}^N) \arrow[uuuu, "\int_F"] \arrow[uurrr, bend right, "\int"'] & & &
\end{tikzcd}
\tag{10.150}
$$

Here all three inner maps and all three outer maps are isomorphisms. The inner triangle commutes and the outer triangle also commutes, being fibre integration. The quadrangle towards the lower right commutes, this being the case of a trivial bundle just discussed. Thus the diagramme without the dotted arrow is commutative. Moreover there is only one way to get the left quadrangle to commute, namely by defining

$$
\mathrm{Ch}'(e) = \int_F \mathrm{Ch}(e \otimes \beta_F) \tag{10.151}
$$

where the integral is over the fibres of F. Then the whole diagramme commutes and gives us the formula in cohomology that we want. On the other hand, $\mathrm{Ch}(e\otimes\beta_F) = \pi^*\,\mathrm{Ch}(e) \otimes \mathrm{Ch}(\beta_F)$ where π is the projection from $\mathbb{C}^N \times X$ to E along the fibres of F. Since

$$
\int_f \pi^* a \wedge b = a \wedge \int_F b \tag{10.152}
$$

for any form b on $\mathbb{C}^N \times X$ with compact support relative to the fibres of F, the integral being fibre integration, we conclude that

$$
\mathrm{Ch}'(e) = \mathrm{Ch}(e) \wedge \mathrm{Td}(E),\ \ \mathrm{Td}(E) = \int_F \mathrm{Ch}(\beta_F) \tag{10.153}
$$

with the Todd class being, by definition, a form on the total space of E, but not with compact support. $\square$

10.14. Stabilization

In which operators with values in $\Psi^{-\infty}_{\mathrm{iso}}$ are discussed.

10.15. Delooping sequence

The standard connection between even and odd classifying groups.

10.16. Looping sequence

The quantized connection between classifying groups.

10.17. $\mathcal{C}^*$ algebras

10.18. K-theory of an algeba

10.19. The norm closure of $\Psi^0(X)$

10.20. Problems

Problem 10.7. There is a natural adjoint map on $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ so we could also look at the unitary subgroup

$$\mathrm{U}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \left\{A \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n); (\mathrm{Id}+A)^{-1} = \mathrm{Id}+A^*\right\}. \tag{10.154}$$

Show that the natural inclusion induces an homotopy equivalence, so there is a natural identification

$$\mathrm{K}^1_{\mathrm{c}}(X) \simeq \mathcal{C}^\infty_c(X; U^{-\infty}_{\mathrm{iso}})/\sim \tag{10.155}$$

where the equivalence relation is again homotopy.

Problem 10.8. Remind yourself of the proof that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is open. Since $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a group, it suffices to show that a neighbourhood of $0 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a neighbourhood of the identity. Show that the set $\|A\|_{\mathcal{B}(L^2)} < \frac{1}{2}$, given by the operator norm, fixes an open neighbourhood of $0 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ (this is the L^2 continuity estimate). The inverse of $\mathrm{Id}+A$ for A in this set is given by the Neumann series and the identity (which follows from the Neumann series)

$$(\mathrm{Id}+A)^{-1} = \mathrm{Id}+B = \mathrm{Id}-A + A^2 - ABA \tag{10.156}$$

in which *a prioiri* $B \in \mathcal{B}(L^2)$ shows that $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ by the 'corner' property of smoothing operators (meaning $ABA' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ if $A, A' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \mathcal{B}(L^2)$.

Problem 10.9. Additivity of the map (10.46).

Problem 10.10. Details that (10.46) is an isomorphism.

Problem 10.11. Check that (10.44) is well-defined, meaning that if (V_1, W_2) is replaced by an equivalent pair then the result is the same. Similarly check that the operation is commutative and that it make $\mathrm{K}(X)$ into a group.

Problem 10.12. Check that you do know how to prove (10.28). One way is to use induction over N, since it is certainly true for $N = 1$, $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$. Proceeding by induction, note that an element $a \in \mathrm{GL}(N, \mathbb{C})$ is fixed by its effect on the standard basis, e_i. Choose $N-1$ elements ae_j which form a basis together with e_1. The inductive hypothesis allows these elements to be deformed, keeping their e_1 components fixed, to e_k, $k > 1$. Now it is easy to see how to deform the resulting basis back to the standard one.

PROBLEM 10.13. Prove (10.31). Hint:- The result is very standard for $N = 1$. So proceed by induction over N. Given a smooth curve in $\mathrm{GL}(N, \mathbb{C})$, by truncating its Fourier series at high frequencies one gets, by the openness of $\mathrm{GL}(N, \mathbb{C})$, a homotopic curve which is real-analytic, denote it $a(\theta)$. Now there can only be a finite number of points at which $e_1 \cdot a(\theta)e_1 = 0$. Moreover, by deforming into the complex near these points they can be avoided, since the zeros of an analytic function are isolated. Thus after homotopy we can assume that $g(\theta) = e_1 \cdot a(\theta)e)1 \neq 0$. Composing with a loop in which e_1 is roatated in the complex by $1/g(\theta)$, and e_2 in the opposite direction, one reduces to the case that $e_1 \cdot a(\theta)e)1 = 0$ and then easily to the case $a(\theta)e_1 = e_1$, then induction takes over (with the determinant condition still holding). Thus it is enough to do the two-dimensional case, which is pretty easy, namely e_1 rotated in one direction and e_2 by the inverse factor.

CHAPTER 11

Hochschild homology

11.1. Formal Hochschild homology

The Hochschild homology is defined, formally, for any associative algebra. Thus if $\mathcal{A}$ is the algebra then the space of *formal* k-chains, for $k \in \mathbb{N}_0$ is the $(k+1)$-fold tensor product

$$\mathcal{A}^{\otimes(k+1)} = \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}. \tag{11.1}$$

The 'formal' here refers to the fact that for the 'large' topological algebras we shall consider it is wise to replace this tensor product by an appropriate completion, usually the 'projective' tensor product. At the formal level the differential defining the cohomolgy is given in terms of the product, $\star$, by

$$\begin{gathered} b(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = b'(a_0 \otimes a_1 \otimes \cdots \otimes a_k) + (-1)^k (a_0 \star a_k) \otimes a_1 \otimes \cdots \otimes a_{k-1}, \\ b'(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_k. \end{gathered} \tag{11.2}$$

Lemma 11.1. *Both the partial map, b', and the full map, b, are differentials, that is*

$$(b')^2 = 0 \text{ and } b^2 = 0. \tag{11.3}$$

Proof. This is just a direct computation. From (11.2) it follows that

$$\begin{aligned} &(b')^2(a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m) \\ &= \sum_{j=2}^{m-1} \sum_{p=0}^{j-2} (-1)^j (-1)^p (\cdots \otimes a_{p+1} \star a_p \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_m) \\ &\quad - \sum_{j=1}^{m-1} (\cdots \otimes a_{j+1} \star a_j \star a_{j-1} \otimes \cdots) - \sum_{j=0}^{m-2} (\cdots \otimes a_{j+21} \star a_{j+1} \star a_j \star \otimes \cdots) \\ &+ \sum_{j=0}^{m-3} \sum_{p=j+2}^{m-1} (-1)^j (-1)^{p-1} (a_0 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_{p+1} \star a_p \otimes \cdots) = 0. \end{aligned} \tag{11.4}$$

Similarly, direct computation shows that

$$\begin{aligned}(b-b')b'(a_0\otimes\cdots\otimes a_m) &= (-1)^{m-1}(a_1\star a_0\star a_m\otimes\cdots a_{m-1})\\ &+\sum_{i=1}^{m-2}(-1)^{i+m-1}(a_0\star a_m\otimes\cdots\otimes a_{i+1}\star a_i\otimes\cdots)+(a_0\star a_m\star a_{m-1}\otimes\cdots),\\ b'(b-b')(a_0\otimes\cdots\otimes a_m) &= (-1)^m(a_1\star a_0\star a_m\otimes\cdots a_{m-1})\\ &+\sum_{i=1}^{m-2}(-1)^{i+m}(a_0\star a_m\otimes\cdots\otimes a_{i+1}\star a_i\otimes\cdots)\text{ and}\\ (b-b')^2(a_0\otimes\cdots\otimes a_m) &= -(a_0\star a_m\star a_{m-1}\otimes\cdots)\end{aligned}$$

so

$$(b-b')b'+b'(b-b')=-(b-b')^2. \tag{11.5}$$

□

The difference between these two differentials is fundamental, roughly speaking b' is 'trivial'.

Lemma 11.2. *For any algebra with identity the differential b' is acyclic, since it satifies*

$$b's+sb'=\mathrm{Id}\ \textit{where} \tag{11.6}$$

$$s(a_0\otimes\cdots\otimes a_m)=\mathrm{Id}\otimes a_0\otimes\cdots\otimes a_m. \tag{11.7}$$

Proof. This follows from the observation that

$$b'(\mathrm{Id}\otimes a_0\otimes\cdots\otimes a_m)=a_0\otimes\cdots\otimes a_m+\sum_{i=1}^{m}(-1)^i(\mathrm{Id}\otimes\cdots a_i\star a_{i-2}\otimes\cdots). \tag{11.8}$$

□

Definition 11.1. *An associative algebra is said to be* H-unital *if its b' complex is acyclic.*

Thus the preceeding lemma just says that every unital algebra is H-unital.

11.2. Hochschild homology of polynomial algebras

Consider the algebra $\mathbb{C}[x]$ of polynomials in n variables[1], $x\in\mathbb{R}^n$ (or $x\in\mathbb{C}^n$ it makes little difference). This is not a finite dimensional algebra but it is filtered by the finite dimensional subspaces, $P_m[x]$, of polynomials of degree at most m;

$$\mathbb{C}[x]=\bigcup_{m=0}^{\infty}P_m[x],\ P_m[x]\subset P_{m+1}[x].$$

Furthermore, the Hochschild differential does not increase the total degree so it is enough to consider the formal Hochschild homology.

The chain spaces, given by the tensor product, just consist of polynomials in $n(k+1)$ variables

$$(\mathbb{C}[x])^{\hat{\otimes}(k+1)}=\mathbb{C}[x_0,x_1,\ldots,x_k],\ x_j\in\mathbb{R}^n.$$

[1]The method used here to compute the homology of a polynomial algebra is due to Sergiu Moroianu; thanks Sergiu.

Furthermore composition acts on the tensor product by

$$p(x_0)q(x_1) = p \otimes q \longmapsto p(x_0)q(x_0)$$

which is just restriction to $x_0 = x_1$. Thus the Hochschild differential can be written

$$b : \mathbb{C}[x_0, \dots, x_k] \longrightarrow \mathbb{C}[x_0, \dots, x_{k-1}],$$

$$(bq)(x_0, x_1, \dots, x_{k-1}) = \sum_{j=0}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0).$$

One of the fundamental results on Hochschild homology is

THEOREM 11.1. *The Hochschild homology of the polynomail algebra in n variables is*

$$\mathrm{HH}_k(\mathbb{C}[x]) = \mathbb{C}[x] \otimes \Lambda^k(\mathbb{C}^n), \tag{11.9}$$

with the identification given by the map from the chain spaces

$$\mathbb{C}[x_0, \dots, x_k] \ni q \longrightarrow \sum_{1 \le j_i \le n} \frac{\partial}{\partial x_1^{j_1}} \cdots \frac{\partial}{\partial x_k^{j_k}} p\big|_{x=x_0=x_1=\cdots=x_k} dx_1^{j_1} \wedge \cdots \wedge dx_k^{j_k}.$$

Note that the appearance of the original algebra $\mathbb{C}[x]$ on the left in (11.9) is not surprising, since the differential commutes with multilplication by polynomails in the first variable, x_0

$$b(r(x_0)q(x_0, \dots, x_k)) = r(x_0)(bq(x_0, \dots, x_k)).$$

Thus the Hochschild homology is certainly a module over $\mathbb{C}[x]$.

PROOF. Consider first the cases of small k. If $k = 0$ then b is identically 0. If $k = 1$ then again

$$(bq)(x_0) = q(x_0, x_0) - q(x_0, x_0) = 0$$

vanishes identically. Thus the homology in dimension 0 is indeed $\mathbb{C}[x]$.

Suppose that $k > 1$ and consider the subspace of $\mathbb{C}[x_0, x_1, \dots, x_k]$ consisting of the elements which are independent of x_1. Then the first two terms in the definition of b cancel and

$$(bq)(x_0, x_1, \dots, x_{k-1}) = \sum_{j=2}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0),\ \partial_{x_1} q \equiv 0.$$

It follows that bq is also independent of x_1. Thus there is a well-defined subcomplex on polynomails independend of x_1 given by

$$\mathbb{C}[x_0, x_2, \dots, x_k] \ni q \longmapsto (\tilde{b}q)(x_0, x_2, \dots, x_{k-1}) \\ = \sum_{j=2}^{k-1} (-1)^j p(x_0, x_2, x_2, x_3 \dots, x_{k-1}) + \sum_{j=3}^{k-1} (-1)^j \\ p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) + (-1)^k q(x_0, x_2, \dots, x_{k-1}, x_0)$$

The reordering of variables $(x_0, x_2, x_3, \dots, x_k) \longrightarrow (x_2, x_3, \dots, x_k, x_0)$ for each k, transforms $\tilde{b}$ to the reduced Hochschild differential b' acting in k variables. Thus $\tilde{b}$ is acyclic.

Similarly consider the subspace of $\mathbb{C}[x_0, x_1, \dots, x_k]$ consisting of the polynomials which vanish at $x_1 = x_0$. Then the first term in the definition of b vanishes and the action of the differential becomes

$$\begin{aligned}(bq)(x_0, x_1, \dots, x_{k-1}) &= p(x_0, x_1, x_1, x_2, \dots, x_{k-1}) + \\ &\sum_{j=2}^{k-1}(-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ &\quad + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0), \text{ if } b(x_0, x_0, x_2, \dots) \equiv 0.\end{aligned} \tag{11.10}$$

It follows that bq also vanishes at $x_1 = x_0$.

By Taylor's theorem any polynomial can be written uniquely as a sum

$$q(x_0, x_1, x_2, \dots, x_k) = q_1'(x_0, x_1, x_2, \dots, x_k) + q''(x_0, x_2, \dots, x_k)$$

of a polynomial which vanishes at $x_1 = x_0$ and a polynomial which is independent of x_1. From the discussion above, this splits the complex into a sum of two subcomplexes, the second one of which is acyclic. Thus the Hochschild homology is the same as the homology of b, which is then given by (11.10), acting on the spaces

$$\{q \in \mathbb{C}[x_0, x_1, \dots, x_k]; q(x_0, x_1, \dots) = 0\}. \tag{11.11}$$

This argument can be extended iteratively. Thus, if $k > 2$ then b maps the subspace of (11.11) of functions independent of x_2 to functions independent of x_2 and on these subspaces acts as b' in $k-2$ variables; it is therefore acyclic. Similar it acts on the complementary spaces given by the functions which vanish on $x_2 = x_1$. Repeating this argument shows that the Hochschild homology is the same as the homology of b acting on the smaller subspaces

$$\begin{gathered}\{q \in \mathbb{C}[x_0, x_1, \dots, x_k]; q(\dots, x_{j-1}, x_j, \dots) = 0,\ j = 1, \dots, k\}, \\ (bq)(x_0, x_1, \dots, x_{k-1}) = (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0).\end{gathered} \tag{11.12}$$

Note that one cannot proceed further directly, in the sense that one cannot reduce to the subspace of functions vanishing on $x_k = x_0$ as well, since this subspace is not linearly independent of the previous ones[2]

$$x_k - x_0 = \sum_{j=0}^{k-1}(x_{j_1} - x_j).$$

It is precisely this 'non-transversality' of the remaining restriction map in (11.12) which remains to be analysed.

Now, let us we make the following change of variable in each of these reduced chain spaces setting

$$y_0 = x_0,\ y_1 = x_j - x_{j-1}, \text{ for } j = 1, \dots, k.$$

Then the differential can be written in terms of the pull-back operation

$$E_P : \mathbb{R}^{nk} \hookrightarrow \mathbb{R}^{n(k+1)},\ E_P(y_0, y_1, \dots, y_{k-1}) = (y_0, y_1, \dots, y_{k-1}, -\sum_{j=1}^{k-1} y_j),$$

$$bq = (-1)^k E_P^* q,$$

[2]Hence Taylor's theorem cannot be applied.

The variable $x_0 = y_0$ is a pure parameter, so can be dropped from the notation (and restored at the end as the factor $\mathbb{C}[x]$ in (11.9)). Also, as already noted, the degree of a polynomial (in all variables) does not increase under any of these pull-back operations, in fact they all preserve the total degree of homogeneity so it suffices to consider the differential b acting on the spaces of homogeneous polynomials which vanish at the origin in each factor

$$Q^m_k = \{q \in \mathbb{C}^m[y_1, \dots, y_k]; q(sy) = s^m q(y),\ q(y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_k) = 0\}$$
$$b : Q^m_k \longrightarrow Q^m_{k-1},\ bq = (-1)^* E^*_P q.$$

To analyse this non-transversality further, let $J_i \subset \mathbb{C}[y_1, \dots, y_k]$ be the ideal generated by the n monomials y^l_i, $l = 1, \dots, n$. Thus, by Taylor's theorem,

$$J_i = \{q \in \mathbb{C}[y_1, \dots, y_k]; q(y_1, y_2, \dots, y_{j-1}, 0, y_j, y_k) = 0.$$

Similary set

$$J_P = \{q \in \mathbb{C}[y_1, \dots, y_k]; q(y_1, \dots, -\sum_{j=1}^{k-1} y_j) = 0)$$

For any two ideals I and J, let $I \cdot J$ be the span of the products. Thus for these particular ideals an element of the product is a sum of terms each of which has a factor vanishing on the corresponding linear subspace. For each k there are $k+1$ ideals and, by Taylor's theorem, the intersection of any k of them is equal to the span of the product of those k ideals. For the k coordinate ideals this is Taylor's theorem as used in the reduction above. The general case of any k of the ideals can be reduced to this case by linear change of coordinates. The question then, is structure of the intersection of all $k+1$ ideals. The proof of the theorem is therefore completed by the following result. □

LEMMA 11.3. *The intersection $Q^m_k \cap J_P = Q^m \cdot J_P$ for every $m \neq k$ and*

$$Q^k_k \cap J_P = \Lambda^k(\mathbb{C}^n). \tag{11.13}$$

PROOF. When $m < k$ the ideal Q^m_k vanishes, so the result is trivial.

Consider the case in (11.13), when $m = k$. A homogeneous polynomial of degree k in k variables (each in $\mathbb{R}^n$) which vanishes at the origin in each variable is necessarily linear in each variable, i.e. is just a k-multilinear function. Given such a multilinear function $q(y_1, \dots, y_k)$ the condition that $bq = 0$ is just that

$$q(y_1, \dots, y_{k-1}, -y_1 - y_2 - \cdots - y_{k-1}) \equiv 0. \tag{11.14}$$

Using the linearity in the last variable the left side can be expanded as a sum of $k-1$ functions each quadratic in one variables y_j and linear in the rest. Thus the vanishing of the sum implies the vanishing of each, so

$$q(y_1, \dots, y_{k-1}, y_j) \equiv 0\ \forall\ j = 1, \dots, k-1.$$

This is the statement that the multlinear function q is antisymmetric between the jth and kth variables for each $j < k$. Since these exchange maps generate the permutation group, q is necessarily totally antisymmetric. This proves the isomorphism (11.13) since $\Lambda^k(\mathbb{C}^n)$ is the space of complex-valued totally antisymmetric k-linear forms.[3]

Thus it remains to consider the case $m \geq k+1$. Consider a general element $q \in Q^m_k \cap J_P$. To show that it is in $Q^m_k \cdot J_P$ we manipulate it, working modulo $Q^m_k \cap J_P$,

[3]Really on the dual but that does not matter at this stage.

and use induction over k. Decompose q as a sum of terms q_l, each homogeneous in the first variable, y_1, of degree l. Since q vanishes at $y_1 = 0$ the first term is q_1, linear in y_1. The condition $bq = 0$, i.e. $q \in J_P$, is again just (11.14). Expanding in the last variable shows that the only term in bq which is linear in y_1 is

$$q_1(y_1, \dots, y_{k-1}, -y_2 - \dots - y_{k-1}).$$

Thus the coefficient of $y_{1,i}$, the ith component of y_1 in q_1, is an element of Q^{m-1}_{k-1} which is in the ideal $J_P(\mathbb{R}^{k-1})$, i.e. for $k-1$ variables. This ideal is generated by the components of $y_2 + \dots + y_k$. So we can proceed by induction and suppose that the result is true for less than k variables for all degrees of homogeneity. Writing $y_2 + \dots + y_k = (y_1 + y_2 + \dots + y_k) - y_1$ It follows that, modulo $Q^m_k \cdot J_P$, q_1 can be replaced by a term of one higher homogeneity in y_1. Thus we can assume that $q_i = 0$ for $i < 2$. The same argument now applies to q_2; expanded as a polynomial in y_1 the coefficients must be elements of $Q^{m-2}_{k-1} \cap J_P$. Thus, unless $m-2 = k-1$, i.e. $m = k+1$, they are, by the inductive hypothesis, in $Q^{m-2}_{k-1} \cdot J_P(\mathbb{R}^{k-1})$ and hence, modulo $Q^m_k \cdot J_P$, q_2 can be absorbed in q_3. This argument can be continued to arrange that $q_i \equiv 0$ for $i < m-k+1$. In fact $q_i \equiv 0$ for $i > m-k+1$ by the assumption that $q \in Q^m_k$.

Thus we are reduced to the assumption that $q = q_{m-k+1} \in Q^m_k \cap J_P$ is homogeneous of degree $m-k+1$ in the first variable. It follows that it is multilinear in the last $k-1$ variables. The vanishing of bq shows that it is indeed totally antisymmetric in these last $k-1$ variables. Now for each non-zero monomial consider the map $J : \{1, 2, \dots, n\} \longrightarrow \mathbb{N}_0$ such that $J(i)$ is the number of times a variable $y_{l,i}$ occurs for some $1 \le l \le k$. The decomposition into the sum of terms for each fixed J is preserved by b. It follows that we can assume that q has only terms corresponding to a fixed map J. If $J(i) > 1$ for any i then a factor $y_{1,i}$ must be present in q, since it is antisymmetric in the other $k-1$ variables. In this case it can be written $y_{1,i}q'$ where $bq' = 0$. Since q' is necessarily in the product of the indeals $J_2 \cdot \dots J_k \cdot J_P$ it follows that $q' \in Q^m \cdot J_P$. Thus we may assume that $J(i) = 0$ or 1 for all i. Since the extra variables now play no rôle we may assume that $n = m$ is the degree of homogeneity and each index i occurs exactly once.

For convenience let us rotate the last $k-1$ variables so the last is moved to the first position. Polarizing q in the first variable, it can be represented uniquely as an n-multilinear function on $\mathbb{R}^n$ which is symmetric in the first $n-k+1$ variables, totally antisymmetric in the last $k-1$ and has no monomial with repeated index. Let $M_{k-1}(n)$ be the set of such multilinear funtions. The vanishing of bq now corresponds to the vanishing of the symmetrization of q in the first $n-k+2$ variables. By the antisymmetry in the second group of variables this gives a complex

$$M_n(n) \xrightarrow{b_n} M_{n-1}(n) \xrightarrow{b_{n-1}} \dots \qquad \xrightarrow{b_2} M_1(n) \xrightarrow{b_1} M_0 \xrightarrow{b_0} 0.$$

The remaining step is to show that this is exact.

Observe that $\dim(M_k(n)) = \binom{n}{k}$ since there is a basis of $M_k(n)$ with elements labelled by the subsets $I \subset \{1, \dots, n\}$ with k elements. Indeed let ω be a non-trivial k-multilinear function of k variables and let ω_I be this function on $\mathbb{R}^k \subset \mathbb{R}^n$ identified as the set of variables indexed by I. Then if $a \in M_0(n-k)$ is a basis of this 1-dimensional space and a_I is this function on the complementary $\mathbb{R}^{n-k}$ the tensor products $a_I\omega_I$ give a basis. Thus there is an isomorphism

$$M_k \ni q = \sum_{I\subset\{1,\dots,n\},|I|=k} c_I a_I \otimes \omega_I \longmapsto \sum_{I\subset\{1,\dots,n\},|I|=k} c_I \otimes \omega_I \in \Lambda^k(\mathbb{R}^n).$$

Transfered to the exterior algebra by this isomorphism the differential b is just contraction with the vector $e_1 + e_2 + \cdots + e_n$ (in the first slot). A linear transformation reducing this vector to e_1 shows immediately that this (Koszul) complex is exact, with the null space of b_k on $\Lambda^k(\mathbb{R}^n)$ being spanned by those ω_I with $1 \in I$ and the range of b_{k+1} spanned by those with $1 \notin I$. The exactness of this complex completes the proof of the lemma. □

11.3. Hochschild homology of $\mathcal{C}^\infty(X)$

The first example of Hochschild homology that we shall examine is for the commutative algebra $\mathcal{C}^\infty(X)$ where X is any $\mathcal{C}^\infty$ manifold (compact or not). As noted above we need to replace the tensor product by some completion. In the present case observe that for any two manifolds X and Y

$$\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(Y) \subset \mathcal{C}^\infty(X \times Y) \tag{11.15}$$

is dense in the $\mathcal{C}^\infty$ topology. Thus we simply declare the space of k-chains for Hochschild homology to be $\mathcal{C}^\infty(X^{k+1})$, which can be viewed as a natural completion[4] of $\mathcal{C}^\infty(X)^{\otimes(k+1)}$. Notice that the product of two functions can be written in terms of the tensor product as

$$a \cdot b = D^*(a \otimes b),\ a, b \in \mathcal{C}^\infty(X),\ D : X \ni z \longmapsto (z, z) \in X^2. \tag{11.16}$$

The variables in X^{k+1} will generally be denoted $z_0,\ z_1, \dots, z_k$. Consider the 'diagonal' submanifolds

$$D_{i,j} = \{(z_0, z_1, \dots, z_k); z_i = z_j\},\ i, j = 0, \dots, m,\ i \neq j. \tag{11.17}$$

We shall use the same notation for the natural embedding of X^k as each of these submanifolds, at least for $j = i + 1$ and $i = 0$, $j = m$,

$$D_{i,i+1}(x_0, \dots, z_{m-1}) = (z_0, \dots, z_i, z_i, z_{i+1}, \dots, z_{m-1}) \in D_{i,i+1},\ i = 0, \dots, m-1$$
$$D_{m,0}(z_0, \dots, z_{m-1}) = (z_0, \dots, z_{m-1}, z_0).$$

Then the action of b' and b on the tensor products, and hence on all chains, can be written

$$b'\alpha = \sum_{i=0}^{m-1} (-1)^i D^*_{i,i+1}\alpha,\ b\alpha = b'\alpha + (-1)^m D^*_{m,0}\alpha. \tag{11.18}$$

---

[4]One way to justify this is to use results on smoothing operators. For finite dimensional linear spaces V and W the tensor product can be realized as

$$V \otimes W = \hom(W', V)$$

the space of linear maps from the dual of W to V. Identifying the topological dual of $\mathcal{C}^\infty(X)$ with $\mathcal{C}_c^{-\infty}(X; \Omega)$, the space of distributions of compact support, with the weak topology, we can identify the *projective* tensor product $\mathcal{C}^\infty(X)\hat{\otimes}\mathcal{C}^\infty(X)$ as the space of continuous linear maps from $\mathcal{C}_c^{-\infty}(X; \Omega)$ to $\mathcal{C}^\infty(X)$. These are precisely the smoothing operators, corresponding to kernels in $\mathcal{C}^\infty(X \times X)$.

THEOREM 11.2. *The differential b' is acyclic and the homology*[5] *of the complex*

$$\dots \xrightarrow{b} \mathcal{C}^\infty(X^{k+1}) \xrightarrow{b} \mathcal{C}^\infty(X^k) \xrightarrow{b} \dots \tag{11.19}$$

is naturally isomorphic to $\mathcal{C}^\infty(X;\Lambda^*)$.

Before proceeding to the proof proper we note two simple lemmas.

LEMMA 11.4. [6]*For any $j = 0, \dots, m-1$, each function $\alpha \in \mathcal{C}^\infty(X^{k+1})$ which vanishes on $D_{i,i+1}$ for each $i \le j$ can be written uniquely in the form*

$$\alpha = \alpha' + \alpha'',\ \alpha',\ \alpha'' \in \mathcal{C}^\infty(X^{k+1})$$

where α'' vanishes on $D_{i,i+1}$ for all $i \le j+1$ and α' is independent of z_{j+1}.

PROOF. Set $\alpha' = \pi_{j+1}^*(D_{j,j+1}^*\alpha)$ where $\pi_j : X^{k+1} \longrightarrow X^k$ is projection off the jth factor. Thus, essentially by definition, α' is independent of z_{j+1}. Moreover, $\pi_{j+1}D_{j,j+1} = \mathrm{Id}$ so $D_{j,j+1}^*\alpha' = D_{j,j+1}^*\alpha$ and hence $D_{j,j+1}^*\alpha'' = 0$. The decomposition is clearly unique, and for $i < j$,

$$D_{j,j+1} \circ \pi_{j+1} \circ D_{i,i+1} = D_{i,i+1} \circ F_{i,j} \tag{11.20}$$

for a smooth map $F_{i,j}$, so α' vanishes on $D_{i,i+1}$ if α vanishes there. □

LEMMA 11.5. *For any finite dimensional vector space, V, the k-fold exterior power of the dual, $\Lambda^k V^*$, can be naturally identified with the space of functions*

$$\begin{split}\big\{u \in \mathcal{C}^\infty(V^k); u(sv) = s^k v,\ s \ge 0,\ u \restriction (V^i \times \{0\} \times V^{k-i-1}) = 0 \textit{ for } i = 0, \dots, k-1 \\ \textit{and } u \restriction G = 0,\ G = \{(v_1, \dots, v_k) \in V^k; v_1 + \dots + v_k = 0\}\big\}.\end{split} \tag{11.21}$$

PROOF. The homogeneity of the smooth function, u, on V^k implies that it is a homogeneous polynomial of degree k. The fact that it vanishes at 0 in each variable then implies that it is multlinear, i.e. is linear in each variable. The vanishing on G implies that for any j and any $v_i \in V$, $i \ne j$,

$$\sum_{i \ne j} u(v_1, \dots, v_{j-1}, v_i, v_{j+1}, \dots, v_k) = 0. \tag{11.22}$$

Since each of these terms is quadratic (and homogeneous) in the corresponding variable v_i, they must each vanish identically. Thus, u vanishes on $v_i = v_j$ for each $i \ne j$; it is therefore totally antisymmetric as a multlinear form, i.e. is an element of $\Lambda^k V^*$. The converse is immediate, so the lemma is proved. □

PROOF OF THEOREM 11.2. The H-unitality[7] of $\mathcal{C}^\infty(X)$ follows from the proof of Lemma 11.61 which carries over *verbatim* to the larger chain spaces.

By definition the Hochschild homology in degree k is the quotient

$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = \big\{u \in \mathcal{C}^\infty(X^{k+1}); bu = 0\big\}/b\mathcal{C}^\infty(X^{k+2}). \tag{11.23}$$

The first stage in identifying this quotient is to apply Lemma 11.4 repeatedly. Let us carry through the first step separately, and then do the general case.

[5]This homology is properly referred to as the *continuous* Hochschild homology of the topological algebra $\mathcal{C}^\infty(X)$.

[6]As pointed out to me by Maciej Zworski, this is a form of Hadamard's lemma.

[7]Meaning here the *continuous* H-unitality, that is the acyclicity of b' on the chain spaces $\mathcal{C}^\infty(X^{k+1})$.

For $j = 0$, consider the decomposition of $u \in \mathcal{C}^\infty(X^{k+1})$ given by Lemma 11.4, thus

(11.24)
$$u = u_0 + u_{(1)},\ u_0 \in \pi_1^*\mathcal{C}^\infty(X^k),\ u_{(1)} \in J_1^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); u \restriction D_{0,1} = 0\big\}.$$

Now each of these subspaces of $\mathcal{C}^\infty(X^{k+1})$ is mapped into the corresponding subspace of $\mathcal{C}^\infty(X^k)$ by b; i.e. they define subcomplexes. Indeed,

$$u \in \pi_1^*\mathcal{C}^\infty(X^k) \Longrightarrow D_{0,1}^*u = D_{1,2}^*u \text{ so}$$
$$u = \pi_1^*v \Longrightarrow bu = \pi_1^*Bv,\ B^*v = -\sum_{i=1}^{k-1}(-1)^iD_{i,i+1}^*u + (-1)^kD_{k-1,0}^*v.$$

For the other term

$$bu_{(1)} = \sum_{i=1}^{k-1}(-1)^iD_{i,i+1}^*u_{(1)} + (-1)^kD_{k,0}^*u_{(1)} \Longrightarrow bu_{(1)} \in J_1^{(k-1)}. \tag{11.25}$$

Thus, $bu = 0$ is equivalent to $bu_0 = 0$ and $bu_{(1)} = 0$. From (11.3), defining an isomorphism by

$$E_{(k-1)} : \mathcal{C}^\infty(X^k) \longrightarrow \mathcal{C}^\infty(X^k),\ E_{(k-1)}v(z_1, \dots, z_k) = v(z_2, \dots, z_k, z_1), \tag{11.26}$$

it follows that

$$B = -E_{(k-1)}^{-1}b'E_{(k-1)} \tag{11.27}$$

is conjugate to b'. Thus B is acyclic so in terms of (11.24)

$$bu = 0 \Longrightarrow u - u_{(1)} = bw,\ w = \pi_1^*v'. \tag{11.28}$$

As already noted this is the first step in an inductive procedure, the induction being over $1 \le j \le k$ in Lemma 11.4. Thus we show inductively that

$$\begin{aligned} &bu = 0 \Longrightarrow u - u_{(j)} = bw,\\ &\qquad u_{(j)} \in J_j^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); u \restriction D_{i,i+1} = 0,\ 0 \le i \le j-1\big\}. \end{aligned} \tag{11.29}$$

For $j = 1$ this is (11.28). Proceeding inductively we may suppose that $u = u_{(j)}$ and take the decomposition of Lemma 11.4, so

$$u_{(j)} = u' + u_{(j+1)},\ u_{(j+1)} \in J_{j+1}^{(k)},\ u' = \pi_{j+1}^*v \in J_j^{(k)}. \tag{11.30}$$

Then, as before, $bu_{(j)} = 0$ implies that $bu' = 0$. Furthermore, acting on the space $\pi_{j+1}^*\mathcal{C}^\infty(X^k) \cap J_{(j)}^k$, b is conjugate to b' acting in $k + 1 - j$ variables. Thus, it is again acyclic, so $u_{(j)}$ and $u_{(j+1)}$ are homologous as Hochschild k-cycles.

The end point of this inductive procedure is that each k-cycle is homologous to an element of

$$J^{(k)} = J_k^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); D_{i,i+1}^*u = 0,\ i \le i \le k-1\big\}. \tag{11.31}$$

Acting on this space $bu = (-1)^kD_{k,0}^*u$, so we have shown that

(11.32)
$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = M^{(k)}/\big(M^{(k)} \cap b\mathcal{C}^\infty(X^{k+1})\big),\ M^{(k)} = \big\{u \in J^{(k)}; D_{k,o}^*u = 0\big\}.$$

Now consider the subspace

$$\tilde{M}^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); \\ u = \sum_{\text{finite},\ 0\le j\le k-1} (f(z_j) - f(z_{j+1}))\, u_{f,j},\ u_{f,j} \in M^{(k)},\ f \in \mathcal{C}^\infty(X).\big\}. \tag{11.33}$$

If $u = (f(z_j) - f(z_{j+1}))v$, with $v \in M^{(k)}$ set

$$w(z_0, z_1, \dots, z_j, z_{j+1}, z_{j+2}, \dots, z_{k+1}) \\ = (-1)^j (f(z_j) - f(z_{j+1}))v(z_0, \dots, z_j, z_{j+2}, z_{j+3}, \dots, z_k). \tag{11.34}$$

Then, using the assumed vanishing of v, $bw = u$.[8] Thus all the elements of $\tilde{M}^{(k)}$ are exact.

Let us next compute the quotient $M^{(k)}/\tilde{M}^{(k)}$. Linearizing in each factor of X around the submanifold $z_0 = z_1 = \dots = z_k$ in V^k defines a map

$$\mu : M^{(k)} \ni u \longrightarrow u' \in \mathcal{C}^\infty(X; TX \otimes \cdots \otimes T^*X). \tag{11.35}$$

The map is defined by taking the term of homogeneity k in a normal expansion around the submanifold. The range space is therefore precisely the space of sections of the k-fold tensor product bundle which vanish on the subbundle defined in each fibre by $v_1 + \cdots + v_k = 0$. Thus, by Lemma 11.5, μ actually defines a sequence

$$0 \longrightarrow \tilde{M}^{(k)} \hookrightarrow M^{(k)} \xrightarrow{\mu} \mathcal{C}^\infty(X; \Lambda^k X) \longrightarrow 0. \tag{11.36}$$

Lemma 11.6. *For any k,* (11.36) *is a short exact sequence.*

Proof. So far I have a rather nasty proof by induction of this result, there should be a reasonably elementary argument. Any offers? □

From this the desired identification, induced by μ,

$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = \mathcal{C}^\infty(X; \Lambda^k X) \tag{11.37}$$

follows, once is is shown that no element $u \in M^{(k)}$ with $\mu(u) \neq 0$ can be exact. This follows by a similar argument. Namely if $u \in M^{(k)}$ is exact then write $u = bv$, $v \in \mathcal{C}^\infty(X^k)$ and apply the decomposition of Lemma 11.4 to get $v = v_0 + v_{(1)}$. Since $u = 0$ on $D_{1,0}$ it follows that $bv_0 = 0$ and hence $u = bv_{1)}$. Proceeding inductively we conculde that $u = bv$ with $v \in M^{(k+1)}$. Now, $\mu(bv) = 0$ by inspection. □

11.4. Commutative formal symbol algebra

As a first step towards the computation of the Hochschild homology of the algebra $\mathcal{A} = \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X)$ we consider the formal algebra of symbols with commutative product. Thus,

$$\mathcal{A} = \big\{(a_j)_{j=-\infty}^{\infty}; a_j \in \mathcal{C}^\infty(S^*X; P^{(j)}),\ a_j = 0 \text{ for } j >> 0\big\}. \tag{11.38}$$

Here $P^{(k)}$ is the line bundle over S^*X with sections consisting of the homogeneous functions of degree k on $T^*X \setminus 0$. The multiplication is as functions on $T^*X \setminus 0$, so

$$(a_j) \cdot (b_j) = (c_j),\ c_j = \sum_{k=-\infty}^{\infty} a_{j-k} b_k$$

[8] Notice that $v(z_0, \dots, z_j, z_{j+2}, \dots, z_{k+1})$ vanishes on $z_{i+1} = z_i$ for $i < j$ and $i > j+1$ and also on $z_0 + z_1 + \cdots + z_{k+1} = 0$ (since it is independent of z_{j+1} and $bv = 0$.

using the fact that $P^{(l)} \otimes P^{(k)} \equiv P^{(l+k)}$. We take the completion of the tensor product to be

$$\begin{aligned}(11.39)\quad \mathcal{B}^{(k)} = \big\{u \in \mathcal{C}^\infty((T^*X\setminus 0)^{k+1}); u = \sum_{\text{finite}} u_I,\\ u_I \in \mathcal{C}^\infty(S^*X; P^{(I_0)}\otimes P^{(I_1)}\otimes\cdots\otimes P^{(I_k)}),\ |I| = k\big\}.\end{aligned}$$

That is, an element of $\mathcal{B}^{(k)}$ is a finite sum of functions on the $(k+1)$-fold product of $T^*X\setminus 0$ which are homogeneous of degree I_j on the jth factor, with the sum of the homogeneities being k. Then the Hochschild homology is the cohomology of the subcomplex of the complex for $\mathcal{C}^\infty(T^*X)$

$$(11.40)\qquad \cdots \xrightarrow{b} \mathcal{B}^{(k)} \xrightarrow{b} \mathcal{B}^{(k-1)} \xrightarrow{b} \cdots$$

Theorem 11.3. *The cohomology of the complex* (11.40) *for the commutative product on* $\mathcal{A}$ *is*

$$(11.41)\quad \mathrm{HH}_k(\mathcal{A}) \equiv \big\{\alpha \in \mathcal{C}^\infty(T^*X\setminus 0; \Lambda^k(T^*X); \alpha \textit{ is homogeneous of degree } k\big\}.$$

11.5. Hochschild chains

The completion of the tensor product that we take to define the Hochschild homology of the 'full symbol algebra' is the same space as in (11.39) but with the non-commutative product derived from the quantization map for some Riemann metric on X. Since the product is given as a formal sum of bilinear differential operators it can be take to act on an pair of factors.

$$(11.42)\qquad \dots \xrightarrow{b^{(\star)}} \mathcal{B}^{(k)} \xrightarrow{b^{(\star)}} \mathcal{B}^{(k-1)} \xrightarrow{b^{(\star)}} \dots$$

The next, and major, task of this chapter is to describe the cohomology of this complex.

Theorem 11.4. *The Hochschild homolgy of the algebra,* $\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)$, *of formal symbols of pseudodifferential operators of integral order, identified as the cohomology of the complex* (11.42), *is naturally identified with two copies of the cohomology of* S^*X[9]

$$(11.43)\qquad \mathrm{HH}_k(\mathcal{A};\circ) \equiv H^{2n-k}(S^*X) \oplus H^{2n-1-k}(S^*X).$$

11.6. Semi-classical limit and spectral sequence

The 'classical limit' in physics, especially quantuum mechanics, is the limit in which physical variables become commutative, i.e. the non-commutative coupling between position and momentum variables vanishes in the limit. Formally this typically involves the replacement of Planck's constant by a parameter $h \to 0$. A phenomenon is 'semi-classical' if it can be understood at least in Taylor series in this parameter. In this sense the Hochschild homology of the full symbol algebra is semi-classical and (following [**4**]) this is how we shall compute it.

The parameter h is introduced directly as an isomorphism of the space $\mathcal{A}$

$$L_h : \mathcal{A} \longrightarrow \mathcal{A},\ L_h(a_j)^*_{j=-\infty} = (h^j a_j)^*_{j=-\infty},\ h > 0.$$

[9]In particular the Hochschild homology vanishes for $k > 2\dim X$. For a precise form of the identification in (11.43) see (**??**).

Clearly $L_h \circ L_{h'} = L_{hh'}$. For $h \neq 1$, L_h is *not* an algebra morphism, so induces a 1-parameter family of products

$$\alpha \star_h \beta = (L_h^{-1})(L_h\alpha \star L_h\beta). \tag{11.44}$$

In terms of the differential operators, associated to quantization by a particular choice of Riemann metric on X this product can be written

$$\alpha \star_h \beta = (c_j)_{j=-\infty}^{*},\ c_j = \sum_{k=0}^{*}\sum_{l=-*}^{*} h^k P_k(a_{j-l-k}, b_l). \tag{11.45}$$

It is important to note here that the P_k, as differential operators on functions on T^*X, do only depend on k, which is the difference of homogeneity between the product $a_{j-l+k}b_l$, which has degree $j+k$ and c_j, which has degree j.

Since $\mathcal{A}$ with product $\star_h$ is a 1-parameter family of algebras, i.e. a deformation of the algebra $\mathcal{A}$ with product $\star = \star_1$, the Hochschild homology is 'constant' in h. More precisely the map L_h induces a canonical isomorphism

$$L_h^* : \mathrm{HH}_k(\mathcal{A};\star_h) \equiv \mathrm{HH}_k(\mathcal{A};\star).$$

The dependence of the product on h is smooth, so it is reasonable to expect the cycles to have smooth representatives as $h \to 0$. To investigate the consider Taylor series in h and define

$$\begin{aligned} F_{p,k} = \big\{\alpha \in \mathcal{B}^{(k)};\ \exists\ \alpha(h) \in \mathcal{C}^\infty([0,1)_h;\mathcal{B}^{(k)}) \text{ with } \alpha(0) = \alpha \text{ and}\\ b_h\alpha \in h^p\mathcal{C}^\infty([0,1)_h;\mathcal{B}^{k-1})\big\}, \end{aligned} \tag{11.46}$$

$$\begin{aligned} G_{p,k} = \big\{\alpha \in \mathcal{B}^{(k)};\ \exists\ \beta(h) \in \mathcal{C}^\infty([0,1)_h;\mathcal{B}^{(k+1)}) \text{ with}\\ b_h\beta(h) \in h^{p-1}\mathcal{C}^\infty([0,1)_h;\mathcal{B}^{(k)} \text{ and } (t^{-p+1}b_h\beta)(0) = \alpha\big\}. \end{aligned} \tag{11.47}$$

Here b_h is the differential defined by the product $\star_h$.

Notice that the $F_{p,k}$ decrease with increasing p, since the condition becomes stronger, while $G_{p,k}$ increases with p, the condition becoming weaker.[10] We define the 'spectral sequence' corresponding to this filtration by

$$E_{p,k} = F_{p,k}/G_{p,k}.$$

These can also be defined successively, in the sense that if

$$\begin{gathered} F'_{p,k} = \big\{u \in E_{p-1,k}; u = [u'],\ u' \in F_{p,k}\big\}\\ G'_{p,k} = \big\{e \in E_{p-1,k}; u = [u'],]u' \in G_{p,k}\big\}\\ \text{then } E_{p,k} \equiv F'_{p,k}/G'_{p,k}. \end{gathered}$$

The basic idea[11] of a spectral sequence is that each $E_p = \bigoplus_k E_{p,k}$, has defined on it a differential such that the next spaces, forming E_{p+1}, are the cohomology space for the complex. This is easily seen from the definitions of $F_{p,k}$ as follows. If $\alpha \in F_{p,k}$ let $\beta(t)$ be a 1-parameter family of chains as in the defintion. Then consider

$$\gamma(t^{-p}b_h\beta)(0) \in \mathcal{B}^{(k-1)}. \tag{11.48}$$

[10] If $\alpha \in G_{p,k}$ and $\beta(h)$ is the 1-parameter family of chains whose existence is required for the definition then $\beta'(h) = h\beta(h)$ satisfies the same condition with p increased to show that $\alpha \in G_{p+1,k}$.

[11] Of Leray I suppose, but I am not really sure.

This depends on the choice of β, but only up to a term in $G_{p,k-1}$. Indeed, let $\beta'(t)$ is another choice of extension of α satisfying the condition that $b_h\beta' \in h^p\mathcal{C}^\infty([0,1);\mathcal{B}^{(k-1)}$ and let γ' be defined by (11.48) with β replaced by β'. Then $\delta(t) = t^{-1}(\beta(h) - \beta'(h))$ satisfies the requirements in the definition of $G_{p,k-1}$, i.e. the difference $\gamma' - \gamma \in G_{p,k-1}$. Similarly, if $\alpha \in G_{p,k}$ then $\gamma \in G_{p,k}$.[12] The map so defined is a differential

$$b_{(p)} : E_{p,k} \longrightarrow E_{p,k-1},\ b^2_{(p)} = 0.$$

This follows from the fact that if $\mu = b_{(p)}\alpha$ then, by definition, $\mu = (t^{-p}b_h\beta)(0)$, where $\alpha = \beta(0)$. Taking $\lambda(t) = t^{-p}b_h\beta(t)$ as the extension of μ it follows that $b_h\lambda = 0$, so $b_{(p)}\mu = 0$.

Now, it follows directly from the definition that $F_{0,k} = E_{0,k} = \mathcal{B}^{(k)}$ since $G_{0,k} = \{0\}$. Furthermore, the differential $b_{(0)}$ induced on E_0 is just the Hochschild differential for the limiting product, $\star_0$, which is the commutative product on the algebra. Thus, Theorem 11.3 just states that

$$E_{1,k} = \bigoplus_{k=-\infty}^{*} \big\{u \in \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k); u \text{ is homogeneous of degree } k\big\}.$$

To complete the proof of Theorem 11.4 it therefore suffices to show that

$$E_{2,k} \equiv H^{2n-k}(S^*X) \oplus H^{2n-1-k}(S^*X), \tag{11.49}$$

$$E_{p,k} = E_{2,k},\ \forall\ p \geq 2, \text{ and} \tag{11.50}$$

$$\mathrm{HH}_k(\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)) = \lim_{p\to\infty} E_{p,k}. \tag{11.51}$$

The second and third of these results are usually described, respectively, as the 'degeneration' of the spectral sequence (in this case at the 'E_2 term') and the 'convergence' of the spectral sequence to the desired cohomology space.

11.7. The E_2 term

As already noted, the $E_{1,k}$ term in the spectral sequence consists of the formal sums of k-forms, on $T^*X \setminus 0$, which are homogeneous under the $\mathbb{R}^+$ action. The E_2 term is the cohomology of the complex formed by these spaces with the differential $b_{(1)}$, which we proceed to compute. For simplicity of notation, consider the formal tensor prodoct rather than its completion. As already noted, for any $\alpha \in \mathcal{B}^{(k)}$ the function $b_h\alpha$ is smooth in h and from the definition of b,

$$\begin{aligned}\frac{d}{dh}b_h\alpha(0) = \sum_{i=0}^{k-1}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes P_1(a_{i+1}, a_i) \otimes a_{i+2} \otimes \cdots \otimes a_k \\ + (-1)^k P_1(a_0, a_k) \otimes a_1 \otimes \cdots \otimes a_{k-1},\ \alpha = a_0 \otimes \cdots \otimes a_k.\end{aligned} \tag{11.52}$$

The general case is only more difficult to write, not different.[13] This certainly determines $b_1\alpha$ if α is a superposition of such terms with $b_0\alpha = 0$. Although (11.52) is explicit, it is not given directly in terms of the representation of α, assumed to satisfy $b_0\alpha = 0$ as a form on $T^*X \setminus 0$.

[12]Indeed, α is then the value at $h = 0$ of $\beta(t) = t^{-p+1}b_h\phi(t)$ which is by hypothesis smooth; clearly $b_h\beta \equiv 0$.

[13]If you feel it necessary to do so, resort to an argument by continuity towards the end of this discussion.

To get such an explicit formula we shall use the symplectic analogue of the Hodge isomorphism. Recall that in any local coordinates on X, x_i, $i = 1, \dots, n$, induce local coordinates x_i, ξ_i in the part of T^*X lying above the coordinate patch. In these canonical coordinates the symplectic form (which determines the Poisson bracket) is given by

$$\omega = \sum_{k=1}^{n} d\xi_k \wedge dx_k. \tag{11.53}$$

This 2-form is non-degenerate, i.e. the n-fold wedge product $\omega^n \neq 0$. In fact this volume form fixes an orientation on T^*X. The symplectic form can be viewed as a non-degenerate antisymmetric bilinear form on $T_q(T^*X)$ at each point $q \in T^*X$, and hence by duality as a bilineear form on $T_q^*(T^*X)$. We denote this form in the same way as the Poisson bracket, since with the convention

$$\{a, b\}(q) = \{da, db\}_q$$

they are indeed the same. As a non-degenerate bilinear form on T^*Y, $Y = T^*X$ this also induces a bilinear form on the tensor algebra, by setting

$$\{e_1 \otimes \cdots \otimes e_k, f_1 \otimes \cdots \otimes f_k, \} = \prod_j \{e_j, f_j\}.$$

These bilinear forms are all antisymmetric and non-degenerate and restrict to be non-degnerate on the antisymmetric part, $\Lambda^k Y$, of the tensor algebra. Thus each of the form bundles has a bilinear form defined on it, so there is a natural isomorhism

$$W_\omega : \Lambda_q^k Y \longrightarrow \Lambda_q^{2n-k} Y, \ \alpha \wedge W_\omega \beta = \{\alpha, \beta\}\omega^n, \ \alpha, \beta \in \mathcal{C}^\infty(Y, \Lambda^k Y), \tag{11.54}$$

for each k.

Lemma 11.7. *In canonical coordinates, as in* (11.53)*, consider the basis of k-forms given by all increasing subsequences of length k,*

$$I : \{1, 2, \dots, k\} \longrightarrow \{1, 2, \dots, 2n\},$$

and setting

$$\alpha_I = dz_{I(1)} \wedge dz_{I(2)} \wedge \cdots \wedge dz_{I(k)}, \qquad (z_1, z_2, \dots, z_{2n}) = (x_1, \xi_1, x_2, \xi_2, \dots, x_n, \xi_n). \tag{11.55}$$

In terms of this ordering of the coordinates

$$W_\omega(\alpha_I) = (-1)^{N(I)} \alpha_{W(I)} \tag{11.56}$$

where $W(I)$ is obtained from I by considering each pair $(2p-1, 2p)$ for $p = 1, \dots, n$, erasing it if it occurs in the image of I, inserting it into I if neither $2p-1$ nor $2p$ occurs in the range of I and if exactly one of $2p-1$ and $2p$ occurs then leaving it unchanged; $N(I)$ is the number of times $2p$ appears in the range of I without $2p-1$.

Proof. The Poisson bracket pairing gives, on 1-forms,

$$-\{dx_j, d\xi_j\} = 1 = \{d\xi_j, dx_j\}$$

with all other pairings zero. Extending this to k-forms gives

$$\begin{aligned} &\{\alpha_I, \alpha_J\} = 0 \text{ unless } (I(j), J(j)) = (2p-1, 2p) \text{ or } (2p, 2p-1) \ \forall \ j \text{ and} \\ &\{\alpha_I, \alpha_J\} = (-1)^N, \text{ if } (I(j), J(j)) = (2p-1, 2p) \text{ for } N \text{ values of } j \\ &\qquad\qquad \text{and } (I(j), J(j)) = (2p-1, 2p) \text{ for } N - k \text{ values of } j. \end{aligned}$$

From this, and (11.54), (11.56) follows. □

From this proof if also follows that $N(W(I)) = N(I)$, so $W_\omega^2 = \mathrm{Id}$. We shall let

$$\delta_\omega = W_\omega \circ d \circ W_\omega \tag{11.57}$$

denote the differential operator obtained from d by conjugation,

$$\delta_\omega : \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k) \longrightarrow \mathcal{C}^\infty(T^*X \setminus 0, \Lambda^{k-1}).$$

By construction $\delta_\omega^2 = 0$. The exterior algebra of a symplectic manifold with this differential is called the Koszul complex.[14] All the α_I are closed so

$$\begin{aligned} \delta_\omega(a\alpha_I) &= W_\omega\big(\sum_j \frac{\partial a}{\partial z_j} dz_j\big) \wedge (-1)^{N(I)} \alpha_{W(I)} \\ &= \sum_j \frac{\partial a}{\partial z_j} (-1)^{N(I)} W_\omega(dz_j \wedge \alpha_{W(I)}), \end{aligned} \tag{11.58}$$

Observe that[15]

$$\begin{aligned} W_\omega\big(dz_{2p-1} \wedge \alpha_{W(I)}\big) &= \iota_{\partial/\partial z_{2p}} \alpha_I \\ W_\omega\big(dz_{2p} \wedge \alpha_{W(I)}\big) &= \iota_{\partial/\partial z_{2p-1}} \alpha_I, \end{aligned}$$

where, ι_v denotes contraction with the vector field v. We therefore deduce the following formula for the action of the Koszul differential

$$\delta_\omega(a\alpha_I) = \sum_{i=1}^{2n} \big(H_{z_i} a\big) \iota_{\partial/\partial z_i} \alpha_I. \tag{11.59}$$

Lemma 11.8. *With E_1 identified with the formal sums of homogeneous forms on $T^*X \setminus 0$, the induced differential is*

$$b_{(1)} = \frac{1}{i} \delta_\omega. \tag{11.60}$$

Proof. We know that the bilinear differential operator $2iP_1$ is the Poisson bracket of functions on T^*X. Thus (11.52) can be written

$$\begin{aligned} 2ib_1\alpha = \sum_{i=0}^{k-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes \{a_{i+1}, a_i\} \otimes a_{i+2} \otimes \cdots \otimes a_k \\ + (-1)^k \{a_0, a_k\} \otimes a_1 \otimes \cdots \otimes a_{k-1},\ \alpha = a_0 \otimes \cdots \otimes a_k. \end{aligned} \tag{11.61}$$

The form to which this maps under the identification of E_2 is just

$$\begin{aligned} 21b_1\alpha = \sum_{i=0}^{k-1} (-1)^i a_0 \wedge da_{i-1} \wedge \cdots \wedge d\{a_{i+1}, a_i\} \wedge da_{i+2} \wedge a_k \\ + (-1)^k \{a_0, a_k\} \wedge da_1 \wedge \cdots \wedge da_{k-1} \end{aligned} \tag{11.62}$$

---

[14] Up to various sign conventions of course!

[15] Check this case by case, as the range of I meets the pair $\{2p-1, 2p\}$ in $\{2p-1, 2p\}$, $\{2p-1\}$, $\{2p\}$ or $\emptyset$. Both sides of the first equation are zero in the second and fourth case as are both sides of the second equation in the third and fourth cases. In the remaining four individual cases it is a matter of checking signs.

Consider the basis elements α_I for k-forms. These arise as the images of the corresponding functions in local coordinates on X^{k+1}

$$\tilde{\alpha}_I(z_0, z_1, \dots, z_k) = \sum_\sigma (-1)^{\operatorname{sgn}\sigma}(z_{1,\sigma I(1)} - z_{0,\sigma I(1)}) \\ \times z_{2,\sigma I(1)} - z_{1,\sigma I(1)}) \dots (z_{1,\sigma I(m)} - z_{0,\sigma I(m-1)}).$$

Since these functions are defined in local coordinates they are not globally defined on $(T^*X\setminus 0)^{k+1}$. Nevertheless they can be localized away from $z_0 = \cdots = z_m$ and then, with a coefficient $(a_j(z_0))^*_{j=-\infty}$, $a_j \in \mathcal{C}^\infty(T^*X \setminus 0)$ homogeneous of degree j with support in the coordinate patch, unambiguously define elements of E_1 which we can simply denote as $a(z_0)\tilde{\alpha}_I \in E_1$. These elements, superimposed over a coordinate cover, span E_1. Consider $b_{(1)}\tilde{\alpha}$ given by (11.62). In the sum, the terms with P_1 contracting between indices other than $0, 1$ or $m, 0$ must give zero because the Poisson bracket is constant in the 'middle' variable. Futhermore, by the antisymmetry of $\tilde{\alpha}$, the two remaining terms are equal so

$$\begin{aligned} ib_{(1)}\big(a\tilde{\alpha}_I\big) &= \sum_{\sigma\in\mathcal{P}_k} \big(H_{z_{\sigma I(1)}}a\big)(-1)^{\operatorname{sgn}(\sigma)} dz_{\sigma I(2)} \wedge \cdots \wedge dz_{\sigma I(k)} \\ &= \sum_i \big(H_{z_i}a\big)\iota_{\partial/\partial_i}\alpha_I. \end{aligned}$$

Since this is just (11.59) the lemma follows. □

With this lemma we have identified the differential on the E_1 term in the spectral sequence with the exterior differential operator. To complete the identification (11.49) we need to compute the corresponding deRham groups.

Proposition 11.1. *The cohomology of the complex*

$$\dots \xrightarrow{d} \sum_{j=-\infty}^{*} \mathcal{C}^\infty_{hom(j)}(T^*X \setminus 0; \Lambda^k) \xrightarrow{d} \sum_{j=-\infty}^{*} \mathcal{C}^\infty_{hom(j)}(T^*X \setminus 0; \Lambda^{k+1}) \xrightarrow{d} \dots$$

in dimension k is naturally isomorphic to $H^k(S^*X) \oplus H^{k-1}(S^*X)$.

Proof. Choose a metric on X and let $R = |\xi|$ denote the corresponding length function on $T^*X \setminus 0$. Thus, identifying the quotient $S^*X = (T^*X \setminus 0)/\mathbb{R}^+$ with $\{R = 1\}$ gives an isomorphism $T^*X \setminus 0 \equiv S^*X \times (0,\infty)$. Under this map the smooth forms on $T^*X\setminus 0$ which are homogeneous of degree j are identified as sums

$$\begin{aligned} &\mathcal{C}^\infty_{\mathrm{hom}(j)}(T^*X \setminus 0, \Lambda^k) \ni \alpha_j \\ &= R^j\big(\alpha'_j + \alpha''_j \wedge \frac{dR}{R}\big),\ \alpha'_j \in \mathcal{C}^\infty(S^*X; \Lambda^k),\ \alpha''_j \in \mathcal{C}^\infty(S^*X; \Lambda^{k-1}). \end{aligned} \tag{11.63}$$

The action of the exterior derivative is then easily computed

$$\begin{aligned} d\alpha_j = \beta_j,\ \beta_j = R^j\big(\beta'_j + \beta_j -'' \wedge \frac{dR}{R}\big), \\ \beta'_j = d\alpha'_j,\ \beta''_j = d\alpha''_j + j(-1)^{k-1}\alpha'_j. \end{aligned}$$

Thus a k-form $(\alpha_j)^*_{j=-\infty}$ is closed precisely if it satisfies

$$j\alpha'_j = (-1)^k d\alpha''_j,\ d\alpha'_j = 0 \forall\ j. \tag{11.64}$$

It is exact if there exists a $(k-1)$-form $(\gamma_j)^*_{j=-\infty}$ such that

$$\alpha'_j = d\gamma'_j,\ \alpha''_j = d\gamma''_j + j(-1)^k\gamma'_j. \tag{11.65}$$

Since the differential preserves homogeneity it is only necessary to analyze these equations for each integral j. For $j \neq 0$, the second equation in (11.64) follows from the first and (11.65) then holds with $\gamma'_j = \frac{1}{j}(-1)^k\alpha''_j$ and $\gamma''_j = 0$. Thus the cohomology lies only in the subcomplex of homogeneous forms of degree 0. Then (11.64) and (11.65) become

$$d\alpha'_0 = 0,\ d\alpha''_0 = 0 \text{ and } \alpha'_0 = d\gamma'_0,\ \alpha''_0 = d\gamma''_0$$

respectively. This gives exactly the direct sum of $H^k(S^*X)$ and $H^{k-1}(S^*X)$ as the cohomology in degree k. The resulting isomorphism is independent of the choice of the radial function R, since another choice replaces R by Ra, where a is a smooth positive function on S^*X. In the decomposition (11.63), for $j = 0$, α''_0 is unchanged whereas α'_0 is replaced by $\alpha'_0 + \alpha''_0 \wedge d\log a$. Since the extra term is exact whenever α''_0 is closed it has no effect on the identification of the cohomology. □

Combining Proposition 11.1 and Lemma 11.8 completes the proof of (11.49). We make the identification a little more precise by locating the terms in E_2.

PROPOSITION 11.2. *Under the identification of E_1 with the sums of homogeneous forms on $T^*X \setminus 0$, E_2, identified as the cohomology of δ_ω, has a basis of homogeneous forms with the homogeneity degree j and the form degree k confined to*

$$k - j = \dim X,\ -\dim X \leq j \leq \dim X,\ \dim X \geq 2. \tag{11.66}$$

PROOF. Provided $\dim X \geq 2$, the cohomology of S^*X is isomorphic to two copies of the cohomology of X, one in the same degree and one shifted by $\dim X - 1$.[16] The classes in the first copy can be taken to be the lifts of deRham classes from X, while the second is spanned by the wedge of these same classes with the Todd class of S^*X. This latter, $n-1$, class restricts to each fibre to be non-vanishing. Thus in local representations the first forms involve only the base variable and in the second each terms has the maximum number, $n-1$, of fibre forms. The cohomology of the complex in Proposition 11.1 therefore consists of four copies of $H^*(X)$ consisting of these forms and the same forms wedged with dR/R.

With this decomoposition of the cohomology consider the effect on it of the map W_ω. In each case the image forms are again homogeneous. A deRham class on X in degree l therefore has four images in E_2. One is a form of degree $k_1 = 2n - l$ which is homogeneous of degree $j_1 = n - l$. The second is a form of degree $k_2 = 2n - l - 1$ which is homogeneous of degree $j_2 = n - l - 1$. The third image is of form degree $k_3 = n - l + 1$ and homogeneous of degree $j_3 = -l + 1$ and the final image is of form degree $k_4 = n - l$ and is homogeneous of degree $j_4 = -l$. This gives the relations (11.66). □

11.8. Degeneration and convergence

Now that the E_2 term in the spectral sequence has been explicitly computed, consider the induced differential, $b_{(2)}$ on it. Any homogeneous form representing a class in E_2 can be represented by a Hochshild chain α of the same homogeneity. Thus an element of E_2 in degree k corresponds to a function on $\mathcal{C}^\infty((T^*X)\backslash)^{k+1})$ which is separately homogeneous in each variable and of total homogeneity $k - n$. Furthermore it has an extension $\beta(t)$ as a function of the parameter h, of the same

[16] That is, just as though $S^*X = \mathbb{S}^{n-1} \times X$, where $n = \dim X$.

homogeneity, such that $b_t\beta(t) = t^2\gamma(t)$. Then $b_{(2)}\alpha = [\gamma(0)]$, the class of $\gamma(0)$ in E_2. Noting that the differential operator, P_j, which is the jth term in the Taylor series of the product $\star_h$ reduces homogeneity by j and that b_h depends multilinearly on $\star_h$ it follows tha $b_{(r)}$ must decrease homogeneity by r. Thus if the class $[\gamma(0)]$ must vanish in E_2 by (11.66). We have therefore shown that $b_{(2)} \equiv 0$, so $E_3 = E_2$. The same argument applies to the higher differentials, defining the $E_r \equiv E_2$ for $r \geq 2$, proving the 'degeneration' of the spectral sequence, (11.50).

The 'convergence' of the spectral sequence, (11.51), follows from the same analysis of homogneities. Thus, we shall define a map from E_2 to the Hochschild homology and show that it is an isomorphism.

11.9. Explicit cohomology maps

11.10. Hochschild holomology of $\Psi^{-\infty}(X)$

11.11. Hochschild holomology of $\Psi^{\mathbb{Z}}(X)$

11.12. Morita equivalence

CHAPTER 12

The index theorem and formula

Using the earlier results on K-theory and cohomology the families index theorem of Atiyah and Singer is proved using a variant of their 'embedding' proof. The index formula in cohomology (including of course the formula for the numerical index) is then derived from this.

12.1. Outline

The index theorem of Atiyah and Singer is proved here in K-theory, using the results from Chapter 10 and then the cohomological version is derived from this. Here are the main steps carried out below:-

(1) Fibrations of manifolds, $M \longrightarrow B$, are discussed and shown to be embeddable in a trivial fibration following Whitney's embedding theorem.
(2) The 'semiclassical index' is defined using semiclassical smoothing operators, first for odd K-theory and then for even K-theory; it there is an innovation here, this is it. Both exhibit 'excision'.
(3) The odd and even semiclassical index maps are shown to be related by suspension, using a calculus combining semiclassical smoothing operators and standard pseudodifferential operators.
(4) The odd (and hence the even) semiclassical index is shown to be natural for iterated fibrations.
(5) The group of homotopy classes of sections of the bundle $G^{-\infty}(M/B; E)$ is shown to reduce to $\mathrm{K_c}(B)$ using smooth families of projections approximating the identity.
(6) The notion of an elliptic family of pseudodifferential operators on the fibres of a fibration is introduced and the analytic index $\mathrm{Ind_a} : \mathrm{K_c}(T^*(M/B)) \longleftrightarrow \mathrm{K_c}(B)$ is defined.
(7) The analytic and semiclassical index maps are shown to be equal by defining a combined analytic-semiclassical index which extends both.
(8) The topological index map is defined using embeddings and the Thom isomorphism and is shown to be equal to the analytic and semiclassical index maps.

Subsequently the special case of Dirac operators is treated and the formula for the Chern character of the index bundle is deduced.

Maybe other things will go in here, η forms, determinant bundle etc.

12.2. Fibrations

Instead of just considering families of pseudodifferential operators on a manifold but depending smoothly on parameters in some other manifold we allow 'twisting by the diffeomorphism group' and consider the more general setting of a family of pseudodifferential operators on the fibres of a fibration, so the parameters are the variables in the base of the fibration and the operators act on the fibres, which are diffeomorphic to a fixed manifold. This indeed is the setting for the 'families index theorem' of Atiyah and Singer.

So, first we need a preliminary discussion of fibrations. A map between two manifolds

$$\phi : M \longrightarrow B \tag{12.1}$$

is a fibration, with typical fibre a manifold Z, if it is smooth, surjective and has the 'local product' property:-

(12.2)

Each $b \in B$ has an open neighbourhood $U \subset B$

for which there exists a diffeomorphism F_U giving a commutative diagramme

$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\phi"'] & & Z \times U \arrow[dl, "\pi_U"] \\ & U. & \end{tikzcd}$$

Here of course, π_U is projection onto the second factor. In particular this means that each fibre $\phi^{-1}(b) = Z_b$ is diffeomorphic to Z, and in such a way that the diffeomorphism can be chosen locally to be smooth in $b \in B$. However there is no *chosen* diffeomorphism and of course in general the diffeomorphism cannot be chosen globally smoothly in b – other wise the fibration is trivial in the sense that there exists a diffeomorphism giving a commutative diagramme

$$\begin{tikzcd} M \arrow[rr, "F"] \arrow[dr, "\phi"'] & & Z \times B \arrow[dl, "\pi_B"] \\ & B. & \end{tikzcd} \tag{12.3}$$

I use the notation

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd} \tag{12.4}$$

to denote a fibration, the headless arrow meaning that there is no chosen diffeomorphism onto the fibres; often people put an arrow there.

One standard source of fibrations is the implict function theorem.

Proposition 12.1. [1] *If $\phi : M \longrightarrow B$ is a smooth map between connected smooth compact manifolds which is a submersion, i.e. the differential $\phi_* : T_m M \longrightarrow T_{\phi(m)} B$ is surjective for every $m \in M$, then ϕ is a fibration.*

It is easy to see that this implication can fail if M is not compact.

We will discuss operators on the fibres of a fibration below. First however we consider one of the important steps in the proof of the Atiyah-Singer theorem, namely the embedding of a fibration.

---

[1] See Problem 12.1

Proposition 12.2. *Any fibration of compact manifolds can be embedded in a trivial fibration to give a commutative diagramme*

$$
\begin{tikzcd}
M \arrow[rr, "\iota"] \arrow[dr, "\phi"'] & & \mathbb{R}^M \times B \arrow[dl, "\pi_B"] \\
& B. &
\end{tikzcd}
\tag{12.5}
$$

Proof. Following Whitney, simply embed M in $\mathbb{R}^M$ for some M. This is easy to do, much the same way as vector bundle can be complemented to a trivial bundle.[2] Then let ι be the product of this embedding and ϕ, giving a map into $\mathbb{R}^M \times B$. $\square$

Vector bundles give particular examples of fibrations. There are various standard constructions on fibrations, in particular the fibre product.

Lemma 12.1. *If $\phi_i : M_i \longrightarrow B$, $i = 1, 2$ are two fibrations with the same base and typical fibres Z_i, then*

$$M_1 \times_B M_2 = \{(m_1, m_2) \in M_1 \times M_2; \phi_1(m_1) = \phi_2(m_2)\} \subset M_1 \times M_2$$

is an embedded submanifold and the restriction of $\phi_1 \times \phi_2$ to it gives a fibration

$$
\begin{tikzcd}
Z_1 \times Z_2 \arrow[r, dash] & M_1 \times_B M_2 \arrow[d, "\phi_1 \times \phi_2"] \\
& B.
\end{tikzcd}
\tag{12.6}
$$

Proof. Just look at local trivializations. $\square$

It has become standard to denote 'relative objects' for a fibration, meaning objects on the fibres, using the formal notation M/B for the fibres. Thus $T(M/B)$ is the *fiber tangent bundle*. It is a bundle over the total space M with fibre at $m \in M$ the tangent space to the fibre through m, $\phi^{-1}(\phi(m))$, at m. To see that it is a bundle, just look at local trivializations of the fibration. Its dual bundle is $T^*(M/B)$, with fibre at m the cotangent space for the fibre. This will play a significant role in what we do below.

12.3. Smoothing families

Philosophically, it is often a good idea to think of a space like $\mathcal{C}^\infty(M)$, the smooth functions (or more generally sections of some vector bundle) on the total space of a fibration as an infinite-dimensional bundle over the base. The fibre at b is just $\mathcal{C}^\infty(Z_b)$, the smooth functions on the fibre, and a local trivialization of the fibration gives a local trivialization of this bundle. To be consistent with the notation above I suppose this bundle should be denoted $\mathcal{C}^\infty(M/B) = \mathcal{C}^\infty(M)$ (or $\mathcal{C}^\infty_c(M/B) = \mathcal{C}^\infty_c(M)$ if M is not compact but B is) thought of as a bundle over B.

Next let us consider smoothing operators on the fibres of a fibration from this point of view. Recall that the densities on a manifold form a trivial, but not canonically trivial, real line bundle over the manifold. If this bundle is trivialized

[2]See Problem 12.2 for more details.

then the smoothing operators on Z are identified with the smooth functions (their Schwartz kernels) on $Z \times Z$. Really this is more invariantly written

$$\Psi^{-\infty}(Z) = \mathcal{C}^{\infty}(Z \times Z; \pi_R^*\Omega(Z)) \tag{12.7}$$

where $\pi_R^*\Omega(Z)$ is the density bundle over Z, pulled back to the product under the projection onto to the right-hand factor.

LEMMA 12.2. *For a fibration* (12.4) *the densities bundles on the fibres form a trivial bundle, denoted* $\Omega(M/B)$*, over the total space and the bundle of (compactly-supported) smoothing operators on the fibres may be identified as*

$$\Psi_c^{-\infty}(M/B) = \mathcal{C}_c^{\infty}(M \times_B M; \pi_R^*\Omega(M/B)) \tag{12.8}$$

where π_R *is the right projection from the total space of the fibre product to the total space of the fibration.*

PROOF. Perhaps this is more a definition than a Lemma. The fibre density bundle is just the density bundle for $T(M/B)$. It is then easy to see that an element on the right in (12.8) defines a smoothing operator on each fibre of the fibration and these operators vary smoothly when identified in a local trivialization of the fibration. This leads to the notation on the left. □

Again $\Psi_{\mathrm{c}}^{-\infty}(M/B)$ can be thought of as a (big) bundle over B.

So, now to something a little less formal. As noted above, one case of a fibration is a vector bundle. If we consider a symplectic (or complex) we have discussed the Thom isomoprhism in K-theory above. In doing this we have used, rather extensively, the projections $\pi_{(N)}$ onto the first N eigenspaces of the harmonic oscillators. Since the index theorem is an geometric extension, to a general fibration, of the Thom isomorphism, we need some replacement for these 'exhausting projections' in the general case. Unfortunately there is nothing[3] to take the place of the harmonic oscillators on the fibres. Of course there are similar objects, such as the Laplacians for some family of fibre metrics, but the eigenvalues of such operators are not constant. As a result the eigenspaces are not even smooth and there is not simple replacement for $\pi_{(N)}$. But we really want these, so we have to construct them a little more crudely. I will do this using the embedding construction above; this is a similar argument to the core of the proof of the Atiyah-Singer theorem but in a much simpler setting.

First we note an extension result using these same $\pi_{(N)}$'s, or just $\pi_{(1)}$, the projection onto the ground state of the harmonic oscllator.

PROPOSITION 12.3. *Let* W *be a symplectic vector bundle over a compact manifold* Z *then there is a natural embedding as a subalgebra*

$$\Psi^{-\infty}(Z) \hookrightarrow \dot{\Psi}^{-\infty}(\bar{W}; \Lambda^*), \tag{12.9}$$

into the algebra of smoothing operator on the total space of the bundle of radial of the fibres of W *which vanish to infinite order at the boundary, acting on sections of the exterior algebra, in which an operator on* Z *is identified with an operator on the ground state of the bundle of harmonic oscillators.*

[3] As far as I know, please correct me if you know better.

PROOF. The point here is simply that the bundle of ground states of the (bundle of) harmonic oscillators is canonically trivial. Indeed all these functions (and the projections onto them) are positive, so there is a unique choice of unit length basis. A smoothing operator on the manifold is then lifted to the same smoothing operator acting on this line bundle, so as a smoothing operator on the total space it is projection onto this bundle, followed by the action of the smoothing operator. Clearly this forms a subalgebra as claimed, since the Schwartz functions correspond to the functions vanishing at infinity on the radial compactification. $\square$

Now, suppose the total space of W is mapped diffeomorphically to an open subset of a smooth manifold in such a way that $\mathcal{S}(W)$, the space of functions which are Schwartz on the fibres, is identified with the smooth functions with support in the closure of the image set. Then the algebra on the right in (12.9) is identified as the subalgebra of the smoothing operators on this manifold with supports in the closure of the image.

12.4. Semiclassical index maps

As noted above, the index theorem may be thought of as the essential uniqueness of the push-forward map in K-theory. Given a fibration of manifolds as in (12.4) we will first define a 'semiclassical index map'

$$\mathrm{Ind}_{\mathrm{sl}} : \mathrm{K}^1_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(B). \tag{12.10}$$

In fact we will do this separately for odd and even K-theory and then compare the results. First we need to discuss the family of fibrewise semiclassical algebras on the fibres of ϕ.

In accordance with the general notation for fibrations the space of semiclassical families of smoothing operators is denoted $\Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$ where E is a vector bundle over M. Repeating again the general principal, this is the space of sections (defined explicitly below) of an infinite dimensional bundle over B whose fibre above $b \in B$ consists of the (space of families of) semiclassical smoothing operators on $Z_b = \phi^{-1}(b)$. There is of course a lot more notation like this below.

Since we have defined the semiclassical algebra on sections of any bundle over any manifold, $\Psi^{-\infty}_{\mathrm{sl}}(Z_b; E_b)$ is well defined. Thus $A \in \Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$ consists of an element of $\Psi^{-\infty}_{\mathrm{sl}}(Z_b; E_b)$ for each $b \in B$, where we only need to specify the meaning of smoothness in $b \in B$. Locally in B the notion of smoothness if obvious enough, since the bundle is trivialized and the meaning of smooth dependence on parameters, which is in any case straightforward, is explained in §6.10. It is therefore only necessary to check that this notion is invariant under diffeomorphisms of the fibres, depending smoothly on the base. I ask you to do this in problems below.****

The results derived earlier for the semiclassical algebra can now be restated for fibrations. The most significant one is the existence and behaviour of the semiclassical symbol map. Here we recall that the semiclassical symbol is 'not quite' a function on the fibrewise cotangent bundle. It is a (Schwartz) function on the slightly different bundle denoted ${}^{\mathrm{sl}}T^*(M/B)$ which is discussed in Section 6.10. In particular this bundle is bundle-isomorphic to $T^*(M/B)$ but *not* equal, i.e. not canonically isomorphic, to it.

PROPOSITION 12.4. *For any fibration the algebra of uniformly properly supported smoothing operators on the fibres, $\Psi_{\mathrm{sl}}^{-\infty}(M/B;E)$, gives a short exact, multiplicative, sequence*

(12.11)
$$0 \longrightarrow \epsilon\Psi_{\mathrm{sl}}^{-\infty}(M/B;E) \hookrightarrow \Psi_{\mathrm{sl}}^{-\infty}(M/B;E) \xrightarrow{\sigma_{\mathrm{sl}}} \mathcal{S}({}^{sl}T^*(M/B);\hom(E)) \longrightarrow 0.$$

Recall that $\epsilon\Psi_{\mathrm{sl}}^{-\infty}(M/B;E)$ is just this lazy man's notation for sections which are of the form ϵA where A is another semiclassical family.

PROOF. *** Part of this proof will be shifted back to the section on the semiclassical calculus on a single manifold where ${}^{\mathrm{sl}}T^*Z$ has already been used but not defined.

In §3.9 there is a rather pedantic definition of the cotangent bundle of a manifold. Namely the fibre at a point $p \in M$ is defined to be the 'linearization' of the space of functions vanishing at p, that is the quotient

(12.12)
$$T_p^*M = \{f \in \mathcal{C}^\infty(M); f(p) = 0\}/\{\sum_{\text{finite}} f_i g_i; f_i, g_i \in \mathcal{C}^\infty(M),\ f_i(p) = g_i(p) = 0\}.$$

Suppose we take the product of M and an interval $[0,1]$. Then

$$\pi^*T^*_{(p,\epsilon)}M = \frac{\{f \in \mathcal{C}^\infty(M\times[0,1]); \partial_\epsilon f = 0,\ f(p,\epsilon) = 0\}}{\{\sum\limits_{\text{finite}} f_i g_i; f_i, g_i \in \mathcal{C}^\infty(M\times[0,1]),\ \partial_\epsilon f_i = \partial_\epsilon g_i = 0,\ f_i(p,\epsilon) = g_i(p,\epsilon) = 0\}} \tag{12.13}$$

is a rather complicated-looking definition of the pull-back to $M\times[0,1]$ of the cotangent bundle to M, under the projection $\pi : M\times[0,1] \longrightarrow M$ at $(p,0) \in M\times[0,1]$. The latter is just defined to be T_p^*M and the definition (12.13) is obviously isomorphic to T_p^*M since all the functions are independent of ϵ, that is it is simply the same definition as (12.12); i.e. this discussion appears moronic.

Let us just change this slightly by inserting factors of ϵ^{-1}. Namely set

(12.14)
$$\begin{gathered} {}^{\mathrm{sl}}T^*M_p = \{f \in \mathcal{C}^\infty(M\times(0,1]); \epsilon f \in \mathcal{C}^\infty(M\times[0,1]),\ \partial_\epsilon(\epsilon f) = 0,\ f(p,0) = 0\}/\mathcal{E}, \\ \mathcal{E} = \{h \in \mathcal{C}^\infty((0,1)\times M); \epsilon h = \sum_{\text{finite}} f_i g_i \\ \text{for } f_i, g_i \in \mathcal{C}^\infty(M\times[0,1]),\ \partial_\epsilon f_i = \partial_\epsilon g_i = 0,\ f_i(p) = g_i(p) = 0\}. \end{gathered}$$

Of course this second definition just involves inserting a factor of ϵ. So, given that we know what ϵ is,

$${}^{\mathrm{sl}}T^*M_p \xrightarrow{\times\epsilon} T^*M_p. \tag{12.15}$$

On the other hand, suppose that we think of $[0,1]$ as a compact, connected, nonempty, 1-dimensional manifold with boundary. That is, we permit ourselves to make diffeomorphisms in ϵ. The differential condition $\partial_\epsilon f = 0$ is invariant under diffeomorphisms, although ∂_ϵ itself is not. However, ϵ just becomes a defining function for $0 \in [0,1]$, it could as well be 2ϵ or even $\epsilon T(\epsilon)$ with $T > 0$ and smooth. The result of this is that the isomorphism (12.15) is not well-defined. The left side is well-defined for $[0,1]$ as a manifold and it is always isomorphic to T_p^*M, but it is *not canonically isomorphic* to T^*_pM. The result is that ${}^{\mathrm{sl}}T^*M$ is a well-defined vector bundle over M, bundle isomorphic to T^*M but not canonically so.

Now, the claim is that the semiclassical symbol really gives a function on T^*Z, not as one might naively think, on T^*Z – however the error in so thinking will likely never show up! Notice that this is clear from the definition of the semiclassical symbol in local coordinates, i.e. back on $\mathbb{R}^n$. There we took the kernel of the semiclassical family,

$$\epsilon^{-n}B(\epsilon, z, \frac{z-z'}{\epsilon}), \tag{12.16}$$

changed variable to $Z = \frac{z-z'}{\epsilon}$, restricted the result to $\epsilon = 0$ and then took the Fourier transform to get a function $b(z,\zeta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which is Schwartz in the second variables. Under change of variables we showed before that this transforms as a function on $T^*\mathbb{R}^n$, so in the case of manifolds gives a function on T^*M. However, this depends on knowing precisely what ϵ is. If you think of the variable $\epsilon/2$ instead the resulting function will be $b(z,\zeta/2)$. Note that you might expect a change by an overall factor of 2^n but this does not happen because this is absorbed in the measure when we take the Fourier transform. On the other hand the discussion above shows that after the new identification with ${}^{\mathrm{sl}}T^*M$

$$B \in \Psi^{-\infty}_{\mathrm{sl}}(M) \Longrightarrow \sigma_{\mathrm{sl}}(B) \in \mathcal{S}({}^{\mathrm{sl}}T^*M) \text{ is well-defined.} \tag{12.17}$$

The case of semiclassical families acting on a vector bundle on the total space of a fibration just involves the invariance under diffeomorphisms, and the behaviour under multiplcation by smooth functions, of the semiclassical smoothing algebra. That is, the exact sequence (12.11), including its multiplicativity, just comes from the same result on each fibre. □

One direct way to see why the image space in (12.11) is the right one is to define $\sigma_{\mathrm{sl}}(B)$ by 'oscillatory testing'. This is done in Euclidean space in Problem****. Restating this result more invariantly we get

LEMMA 12.3. *If $A \in \Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$, $u \in \mathcal{C}^\infty(M;E)$ and $f \in \mathcal{C}^\infty(M)$ is real-valued then*

$$\lim_{\epsilon\downarrow 0} e^{-if(z)/\epsilon}B(e^{if(z)/\epsilon}u = \sigma_{\mathrm{sl}}(\frac{df}{\epsilon})u \in \mathcal{C}^\infty(M;E) \tag{12.18}$$

with the limit existing in this space.

Notice that we need to interpret $df/\epsilon \in \mathcal{C}^\infty(M; {}^{\mathrm{sl}}T^*M)$ as a section for (12.18) to make sense. Going back to the formal definition (12.14) we can do this by defining its value at $\bar{z} \in M$ to be the class of $(f(z) - f(\bar{z}))/\epsilon$.

Now, having the semiclassical algebra on the fibres at our disposal we can construct the corresponding index.

PROPOSITION 12.5. *If $a \in \mathcal{C}^\infty_c({}^{sl}T^*(M/B); \hom E)$ is such that $\operatorname{Id} + a$ is everywhere invertible and $A \in \Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$ is uniformly properly supported and has $\sigma_{\mathrm{sl}}(A) = a$ then for $\epsilon > 0$ sufficiently small, $\operatorname{Id} + A(\epsilon) \in G^{-\infty}(M/B; E)$ and $[\operatorname{Id} + A] \in \mathrm{K}^1(B)$ depends only on $[\operatorname{Id} + a] \in \mathrm{K}^1_c(T^*(M/B)$ so defining*

$$\mathrm{Ind}_{\mathrm{sl}} : \mathrm{K}^1_c(T^*(M/B)) \longrightarrow \mathrm{K}_c(B). \tag{12.19}$$

PROOF. Note that we are making the effort here not to assume that the fibres are compact – nor does the base need to be compact. The main point is that the quantized family of operators is invertible for small $\epsilon > 0$ with inverse of the same type. Indeed, the discussion in Section **??** shows that there is no problem in constructing a semiclassical family which inverts the quantization to infinite order, so up to an error term which is a standard family of smoothing operators vanishing to infinte order at $\epsilon = 0$. Here all families are uniformly properly supported and so such a perturbation of the identity is invertible with inverse of the same form. Thus it remains only to show that the K-class defined by this invertible section of $G^{-\infty}(M/B; E)$ is independent of choices. Since any two semiclassical quantizations are homotopic for small enough $\epsilon > 0$, independence of choice and homotopy invariance under deformation of a follows from the same construction. Stability is also immediate, so the map(12.19) is well-defined as desired. $\square$

Having defined this 'odd semiclassical index map' we note that there is also an even version, defined using the discussion of projections in Proposition 3.11. Recall that the K-theory with compact supports of a non-compact space X, in this case ${}^{\mathrm{sl}}T^*(M/B)$, is represented by equivalence classes of smooth families of projections $\pi : X \longrightarrow \mathrm{GL}(N, \mathbb{C})$, where $\pi^2 = \pi$ and π is constant outside a compact set. Equivalence of two such projections π_i corresponds to the existence of maps a, $b : X \longrightarrow M(N, \mathbb{C})$ also constant outside a compact set and such that $a\pi_i b = \pi_2$. This just means that $\pi_2 b \pi_2$ is an isomorphism from the range of π_2 to the range of π_1 with inverse $\pi_2 a \pi_1$.

PROPOSITION 12.6. *The semiclassical quantization of projections to projections, in Proposition 3.11, induces a push-forward, or index, map in even K-theory for any fibration with compact fibres*

$$\mathrm{Ind}^0_{\mathrm{sl}} : \mathrm{K}^0_c(T^*(M/B)) \longrightarrow \mathrm{K}^0(B). \tag{12.20}$$

PROOF. Two semiclassical families of projections with the same symbol are homotopic through projections so the map (12.20), in which the index of π is the formal difference of the pair $P \ominus \pi_\infty$ of its quantization and the constant projection 'at infinity' is well-defined up to homotopy. Finite rank approximation shows that it defines an element of the K-theory B and it is straightforward to show independence of choices. $\square$

12.5. Bott periodicity and the semiclassical index

*** Take $E = \mathbb{C}^N$ below, since we know we can do this in constructing the index maps.

In the preceeding section two versions of the index map, as pushforward in K-theory for a fibration, have been defined. Next we show that they are 'equal'.

PROPOSITION 12.7. *For any fibration with compact fibres, the diagramme*

$$\begin{tikzcd} \mathrm{K}^1_c(\mathbb{R} \times T^*(M/B)) \arrow[r, "\mathrm{Ind}^1_{\mathrm{sl}}"] \arrow[d, "\mathrm{Ind}_{\mathrm{iso}}"] & \mathrm{K}^1_c(\mathbb{R} \times B) \arrow[d, "\mathrm{Ind}_{\mathrm{iso}}"] \\ \mathrm{K}^0_c(T^*(M/B)) \arrow[r, "\mathrm{Ind}^0_{\mathrm{sl}}"] & \mathrm{K}^0(B) \end{tikzcd} \tag{12.21}$$

commutes, where the vertical maps are the realizations of Bott periodicity discussed in (10.53).

The top map is, as indicated, the odd semiclassical index for the fibration $M\times\mathbb{R}\longrightarrow B\times\mathbb{R}$ with an extra factor of $\mathbb{R}$. Clearly the relative cotangent bundle for this fibration is $\mathbb{R}\times T^*(M/B)$.

Of course the problem with proving such a result is that the vertical map are defined by isotropic quantization and the horizontal maps by semiclassical quantization. As usual, the approach adopted here is to construct an algebra of operators which includes both quantizations naturally (i.e. the correspond to the symbol maps). In this case this is relatively straightforward because isotropic quantization is itself rather simple. Thus the algebra $\Psi^0_{\text{iso}}(\mathbb{R}^n)$ arises from a non-commutative product on $\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$. Similarly we now the that the algebra of semiclassical operators on the fibres of M can also be identified with a space of smooth functions on a manifold, namely

(12.22)
$$\Psi^{-\infty}_{\text{sl}}(M/B;E)=\epsilon^{-d}\left\{A\in\mathcal{C}^\infty(M^2_{\text{sl}};\operatorname{Hom}(E)\otimes\Omega_R);A\equiv 0\text{ at }\{\epsilon=0\}\setminus\text{ff}\right\}.$$

Here

$$M^2_{\text{sl}}=[[0,1]\times M^2_\phi,\{0\}\times\Delta] \tag{12.23}$$

is obtained by the blow up of the diagonal at $\epsilon=0$ in the fibre product.

What we want is really the completed tensor product of these two algebras. Thus consider

(12.24)
$$\begin{aligned}&\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\\&\qquad=\left\{A\in\epsilon^{-d}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}\times M^2_{\text{sl}};\operatorname{Hom}(E)\otimes\Omega_R);A\equiv 0\text{ at }\{\epsilon=0\}\setminus\text{ff}\right\}.\end{aligned}$$

As spaces of amplitudes (i.e. before quantization) we can define the spaces of other (real or complex) orders by

$$\Psi^{m,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)=\rho^{-m}\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E) \tag{12.25}$$

where $\rho\in\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$ is a boundary defining function, i.e. a non-vanishing real elliptic symobl of order -1 in the isotropic calculus.

Proposition 12.8. *The space in* (12.24) *is an algebra under (the continuous extension of) the isotropic product on* $\mathbb{R}^n$ *for symbols with values in the smoothing operators on the fibres of* $\phi:M\longrightarrow B$. *There are two short exact symbol sequences which are multiplicative*

(12.26)
$$\Psi^{-1,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\hookrightarrow\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\xrightarrow{\sigma_{\text{iso}}}\mathcal{C}^\infty(\mathbb{S}^{n-1};\Psi^{-\infty}_{\text{sl}}(M/B;E)$$
$$\epsilon\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\hookrightarrow\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\xrightarrow{\sigma_{\text{sl}}}\mathcal{S}({}^{sl}T^*(M/B);\Psi^0_{\text{iso}}(\mathbb{R}^n;E))$$

which have a common double symbol map

(12.27)
$$
\begin{tikzcd}
& \mathcal{C}^\infty(\mathbb{S}^{n-1};\Psi^{-\infty}_{\mathrm{sl}}(M/B;E)) \arrow[dr, "\sigma_{\mathrm{sl}}"] & \\
\Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n\times M/B;E) \arrow[ur, "\sigma_{\mathrm{iso}}"] \arrow[dr, "\sigma_{\mathrm{sl}}"'] & & \mathcal{S}(\mathbb{S}^{n-1}\times{}^{sl}T^*(M/B);\hom(E)), \\
& \mathcal{S}({}^{sl}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E)) \arrow[ur, "\sigma_{\mathrm{iso}}"'] &
\end{tikzcd}
$$

and combine to give a joint symbol sequence

(12.28)
$$
\begin{aligned}
\epsilon\Psi^{-1,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n\times M/B;E) &\hookrightarrow \Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n\times M/B;E)\\
&\xrightarrow{\sigma_{\mathrm{iso}}\oplus\sigma_{\mathrm{sl}}} \mathcal{C}^\infty(\mathbb{S}^{n-1};\Psi^{-\infty}_{\mathrm{sl}}(M/B;E))\oplus\mathcal{S}({}^{sl}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E))
\end{aligned}
$$

which is exact in the centre and has range precisely the subspace satisfying the compatibility condition in (12.27)*, that*

(12.29)
$$\sigma_{\mathrm{sl}}\sigma_{\mathrm{iso}}=\sigma_{\mathrm{iso}}\sigma_{\mathrm{sl}}.$$

Proof. I will do this ***. The main point is that these are just smooth functions and we can do the quantizations separately in each of the spaces treating the other variables as parameters and then reverse the discussion – really just as though it is a finiter rather than a completed tensor product. Everything should work out pretty well. □

Proof of Proposition 12.7. Now – and really this is essentially the same argument as recurs below in the proof of multiplicativity – we consider the quantization procedure determined by this algebra. Starting with a 'double symbol'

(12.30)
$$\tilde a\in\mathcal{S}(\mathbb{R}\times{}^{\mathrm{sl}}T^*(M/B)\text{ s.t. }(\mathrm{Id}+\tilde a)^{-1}=\mathrm{Id}+\tilde b,\ \tilde b\in\in\mathcal{S}(\mathbb{R}\times{}^{\mathrm{sl}}T^*(M/B),$$

we take the radial compactification of the line into $\mathbb{S}$ and so realize $\tilde a$ as an element a of the image space in (12.27),

(12.31)
$$a\in\mathcal{S}(\mathbb{S}\times{}^{\mathrm{sl}}T^*(M/B);\hom(E))\text{ s.t. }(\mathrm{Id}+a)^{-1}=\mathrm{Id}+b,\ b\in\mathcal{S}(\mathbb{S}\times{}^{\mathrm{sl}}T^*(M/B);\hom(E))$$

we proceed to 'quantize' a in two ways. First, we can use semiclassical quantization to choose a family

(12.32)
$$\alpha'\in\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}_{\mathrm{sl}}(M/B;E))\text{ s.t. }\sigma_{\mathrm{sl}}(\alpha')=a.$$

Here the circle just consists of parameters which should be added to both the base and the fibre and so do not contribute at all to the fibre quantization. It follows that $\mathrm{Id}+\alpha'$ is invertible for small $\epsilon>0$ and that, by definition of semiclassical quantization,

(12.33)
$$\mathrm{Ind}_{\mathrm{sl}}([\mathrm{Id}+\tilde a])=[\mathrm{Id}+\alpha']\in\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times B).$$

Secondly we can proceed in the opposite way and construct a family

(12.34)
$$\alpha''\in\mathcal{S}({}^{\mathrm{sl}}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E))\text{ s.t. }\sigma_{\mathrm{iso}}(\alpha'')=a.$$

From the discussion of isotropic quantization, the invertibility of $\operatorname{Id} + a$ means that $\operatorname{Id} + \alpha''$ is a Fredholm family. In fact we know that for N sufficiently large,

$$(\operatorname{Id} + \alpha'')(\operatorname{Id} - \pi_{(N)}) \text{ has null space precisley the range of } \pi_{(N)}, \tag{12.35}$$

where $\pi_{(N)}$ is the projection onto the span of the first N eigenspaces of the harmonic oscillator (extended to act on sections of E ****). Then we can choose a generalized inverse $\operatorname{Id} + \beta''$ of $\operatorname{Id} + \alpha''$ with

$$\begin{gathered}\beta'' \in \mathcal{S}({}^{\mathrm{sl}}T^*(M/B); \Psi^0_{\mathrm{iso}}(\mathbb{R}^n; E)) \text{ and}\\ (\operatorname{Id} + \beta'')(\operatorname{Id} - \pi')(\operatorname{Id} + \alpha'')(\operatorname{Id} - \pi_{(N)}) = (\operatorname{Id} - \pi_{(N)}),\\ (\operatorname{Id} + \alpha'')(\operatorname{Id} - \pi_{(N)})(\operatorname{Id} + \beta'')(\operatorname{Id} - \pi') = (\operatorname{Id} - \pi')\\ \pi'\pi' = \pi',\ \pi' - \pi_{(N)} \in \mathcal{S}({}^{\mathrm{sl}}T^*(M/B); \Psi^0_{\mathrm{iso}}(\mathbb{R}^n; E)).\end{gathered} \tag{12.36}$$

Note that I have written things out this way to avoid having to allow the main families to remain trivial at infinity on ${}^{\mathrm{sl}}T^*(M/B)$ – although there has to be non-triviality there in order to get the errors to be represented by projections in this way. Then, from the definition of the Bott periodicity map by isotropic quantization,

$$\operatorname{Ind}_{\mathrm{iso}}([\operatorname{Id} + \tilde{a}]) = [\pi_{(N)} \ominus \pi'] \in \mathrm{K}^0_{\mathrm{c}}({}^{\mathrm{sl}}T^*(M/B)) \tag{12.37}$$

is the left vertical map in (12.21).

So, as it should be, the semiclassical quantization is easier.

Using the properties of the algebra, we can find a common element $A \in \Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n \times M/B; E)$ such that $\sigma_{\mathrm{sl}}(A) = \alpha''$ and $\sigma_{\mathrm{iso}}(A) = \alpha'$. Moreover, $\operatorname{Id} + A$ has a 'two-sided parameterix' $\operatorname{Id} + B$, $B \in \Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n \times M/B; E)$ which can be constructed so that, as elements of the semiclassical-isotropic algebra

$$\begin{gathered}(\operatorname{Id} + B)(\operatorname{Id} + A) = \operatorname{Id} - \pi_1,\ (\operatorname{Id} + A)(\operatorname{Id} + B) = \operatorname{Id} - \pi_2,\\ \pi_i \in \Psi^{-\infty,-\infty}_{\mathrm{iso-sl}}(M/B; E),\ \pi_i^2 = \pi_i,\ i = 1, 2.\end{gathered} \tag{12.38}$$

Now, of necessity

$$\sigma_{\mathrm{sl}}(\pi_1) = \pi_{(N)},\ \sigma_{\mathrm{sl}}(\pi_2) = \pi'. \tag{12.39}$$

Now, we claim that for $\epsilon > 0$ small,

$$\begin{gathered}\operatorname{Ind}(\operatorname{Id} + A) = [\pi_1 \ominus \pi_2] = \operatorname{Ind}^0_{\mathrm{sl}}(\pi_{(N)}, \pi') = \operatorname{Ind}_{\mathrm{sl}} \operatorname{Ind}^0_{\mathrm{iso}}(\operatorname{Id} + \tilde{a}) \text{ and}\\ \operatorname{Ind}(\operatorname{Id} + A) = \operatorname{Ind}_{\mathrm{iso}}(\operatorname{Id} + \alpha') = \operatorname{Ind}_{\mathrm{iso}} \operatorname{Ind}_{\mathrm{sl}}(\operatorname{Id} + \tilde{a}).\end{gathered} \tag{12.40}$$

The first equality on the top line in (12.40) is the essentially definition of the index in the isotropic algebra (here extended a bit because of the values in the smoothing algebra) because of (12.38). The second equality follows from the definition of the even semiclassical index map and (12.39) and the third equality is the combination of this and (12.37). Similarly the second line in (12.40) follows from the choice of A as a semiclassical quantization of α'. □

12.6. Hilbert bundles and projections

** This section, or parts of it, may need to be moved back into the K-theory chapter.

As is well-known, all infinite-dimensional separable Hilbert spaces (here always over $\mathbb{C}$) are isomorphic. Namely, if one takes as model ℓ_2, the space of square-summable complex sequences, then an isomorphism to a separable infinite-dimensional Hilber space, $\mathcal{H}$, corresponds exactly to a choice of complete orthonormal basis in $\mathcal{H}$ and such can be constructed by the application of the Gram-Schmidt orthonormalization procedure to any countable dense subset. The group $\mathrm{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$ is then an infinite-dimensional analogue of $\mathrm{U}(N)$, the group of $N \times N$ unitary matrices. However, in contrast the the finite dimensional case, $\mathrm{U}(\mathcal{H})$ is contractible. In the infinite-dimensional setting it is necessary to specify the topology in which this contractibility is to take place. The 'serious' theorem here is Kuiper's theorem that the unitary group is contractible in the norm topology. We will only use the weaker result,

PROPOSITION 12.9. *For any infinite-dimensional separable Hilbert space,* $\mathrm{U}(\mathcal{H})$ *is contractible in the strong topology, meaning there is a map*

$$\mathrm{U}(\mathcal{H}) \times [0,1] \ni (U,t) \Longrightarrow U_t \in \mathrm{U}(\mathcal{H}) \text{ s.t. } U_t v \longrightarrow v \text{ in } \mathcal{H} \ \forall \ v \in \mathcal{H},\ U \in \mathrm{U}(\mathcal{H}). \tag{12.41}$$

PROOF. One example of an infinite-dimensional separable Hilbert space is $L^2([0,1])$ with an orthonormal basis given by the exponentials of period 1. Thus it suffices to prove strong contractibility for this example. For given $v \in L^2([0,1])$, $U \in \mathrm{U}(L^2([0,1]))$ and $t \in [0,1]$ set

$$v_t(x) = v(tx),\ x \in [0,1] \text{ and } U_t v(x) = \begin{cases} (Uv_t)(x/t) & 0 \le x \le t \\ v(x) & x > t. \end{cases} \tag{12.42}$$

Thus $U_t v(x)$ is given by the identity operator on $[t,1]$ and by a rescaled version of U on $[0,1]$. Clearly U_t is linear and

$$\int_0^1 |U_t v(x)|^2 = \int_0^t |Uv_t(x/t)|^2 dx + \int_t^1 |v(x)|^2 dx = t\|v_t\|^2_{L^2} + \int_t^1 |v(x)|^2 dx = \|v\|^2_{L^2} \tag{12.43}$$

so U_t is unitary. Similarly

$$\|U_t v - v\|_{L^2([0,1])} = \|U_t v - v\|_{L^2([0,t])} \le \|U_t v\|_{L^2([0,t])} + \|u\|_{L^2([0,t])} \le 2\|v\|_{L^2([0,t]} \to 0 \text{ as } t \to 0 \tag{12.44}$$

shows that $U_t \to \mathrm{Id}$ strongly.

So, it suffices to check that the continuity of the map

$$\mathrm{U}(\mathcal{H}) \times \mathcal{H} \times [0,1] \ni (U,v) \longmapsto U_t v \in \mathcal{H} \tag{12.45}$$

with respect to the strong topology on $\mathrm{U}(\mathcal{H})$. □

Note that the contraction constructed in (12.42) is multiplicative, namely

$$(UV)_t = U_t V_t \tag{12.46}$$

as follows directly from the definition. This should be useful somewhere!

The main application we need of this contractibility is the existence of finite rank approximations to the identity for fibre bundles. One way to do this is to note the following general topological result.

PROPOSITION 12.10. *If* $F : \mathcal{M} \longrightarrow B$ *is a topological fibre bundle over a compact manifold and the typical fibre,* $\mathcal{Z}$ *is contractible, then* F *has a right inverse, i.e. the bundle has a section.*

PROOF. Maybe I will put a detailed proof in somewhere. This is pretty easy using a triangulation but it might be better to have a proof using small geodesic balls. $\square$

Thus, for a fibration $M \longrightarrow B$ consider the Hilbert spaces of square-integrable sections of a vector bundle E over M these combine to give a bundle $L^2(M/B; E)$ over B. Each of the fibres is unitarily equivalent to the fixed Hilbert space $L^2(Z; E\big|_Z)$ and we can consider the bundle $\mathcal{P} \longrightarrow B$ with fibre at b

$$\mathcal{P}_b = \{G : L^2(Z_b; E\big|_{Z_b}) \longrightarrow L^2(Z; E\big|_Z) \text{ unitary} \tag{12.47}$$

consisting of all such unitary equivalences – this is a principal bundle for the action of $\mathrm{U}(L^2(Z; E\big|_Z))$ by composition. In fact the bundle is locally trivial for the norm topology but we have only checked the contractibility of the fibre for the strong topology. Applying Proposition 12.10 we conclude that there is a section of $B \longrightarrow \mathcal{P}$ which is strongly continuous – of course if we used Kuiper's theorem we could show that there is a norm-continuous section.

PROPOSITION 12.11. *For any fibration $M \longrightarrow B$, Hermitian vector bundle E over M and choice of smooth positive fibre density, there are sections $e_i \in \mathcal{C}^\infty(M; E)$, $i \in \mathbb{N}$, which form an orthonormal basis in each fibre.*

PROOF. As discussed before the statement of the Proposition, the bundle $\mathcal{P}$ has a strongly continuous section G. Then for any orthonormal basis f_i of $L^2(Z; E\big|_Z)$ the sections $e'_i = G^{-1}f_i \in \mathcal{C}^0(B; L^2(M/B; E))$ are continuous and form an orthonormal basis at each point. Now, we can approximate such continuous-L^2 sections by smooth sections $e''_i \in \mathcal{C}^\infty(M; E)$ as closely as we wish. In particular we can choose these smooth sections so that

$$\sup_{b \in B} \|e'_i - e''_i\| \le 2^{-i-4},\ i \ge 1. \tag{12.48}$$

Now, we claim that these new sections can in turn be modified to a smooth orthonormal basis. Certainly from (12.48), the operator

$$T_b u = \sum_i \langle u, e'_i(b)\rangle e''_i(b) \text{ has } \|T_b - \operatorname{Id}\|_{L^2} \le \left(\sum_i \|e'_i - e''_i\|^2\right)^{\frac{1}{2}} \le 1/4 \tag{12.49}$$

so is invertible. Thus the finite span of the e''_i is certainly dense and they are independent. Gram-Schmidt orthonormalization therefore gives an orthonormal basis all elements of which are smooth. $\square$

The main use we put such a smooth orthonormal basis to is the construction of approximate identities which give uniform, i.e. norm convergent, finite rank approximations to smoothing operators.

PROPOSITION 12.12. *Having chosen a smooth orthonormal basis $e_i \in \mathcal{C}^\infty(M; E)$ for a fibration $M \longrightarrow B$ (corresponding to a choice of Hermitian structure and smooth fibre densities) the orthogonal projections $\pi_{(N)}$ onto the span of the first N elements are such that*

$$\|\pi_{(N)} A \pi_{(N)} - A\|_{L^2(M/B;E)} \to 0 \textit{ as } N \to \infty\ \forall\ A \in \Psi^{-\infty}(M/B; E). \tag{12.50}$$

PROOF. This follows from the compactness of smoothing operators on L^2. $\square$

12.7. Adiabatic limit

The main content of the K-theory version of the families index theorem of Atiyah and Singer is that there really is only one way to define an index map, essentially because this is a push-forward map in K-theory. We start the proof by showing this in one particular case. Namely we have shown above **** that if M is a compact manifold which is fibred over B then the K-theory of B can be realized as the homotopy classes of sections of the bundle of groups $G^{-\infty}(M/B;E)$ for any bundle E over M. That is, instead of maps from B into $G^{-\infty}(Z)$ it is fine to consider the twisted case where Z is the varying fibre of $\phi : M \longrightarrow B$. The proof above is by deformation to finite rank, i.e. in both cases sections can be replaced by maps into $\mathrm{GL}(N;\mathbb{C})$ for some appropriately large N depending on the section.

Now, suppose that the total space M of a fibration is itself the base of another fibration

$$\begin{tikzcd} \tilde{Z} \arrow[r, dash] & \tilde{M} \arrow[d, "\tilde{\phi}"'] \arrow[dd, bend left, "\psi"] \\ Z \arrow[r, dash] & M \arrow[d, "\phi"'] \\ & B. \end{tikzcd} \tag{12.51}$$

In this setting we will show that the 'adiabatic calculus' of smoothing operators gives a quantization map

$$K^1(T^*(M/B)) \longrightarrow K^1(B) \tag{12.52}$$

which is defined in terms of smoothing operators on the fibres of $\tilde{\phi} : \tilde{M} \longrightarrow M$.

To define this we need to investigate the adiabatic algebra of smoothing operators for a fibration and then for an iterated fibration. First we start with the case that the overall base, B is a point. Thus M may be replaced by Z and we consider a fibration, with compact fibres

$$\begin{tikzcd} \tilde{Z} \arrow[r, dash] & Y \arrow[d, "\tilde{\phi}"'] \\ & Z \end{tikzcd} \tag{12.53}$$

Definition 12.1. *A smooth family of smoothing operators, $A \in \mathcal{C}^\infty((0,1]_\delta;\Psi^{-\infty}(Y;E))$ (for a vector bundle over Y) is an* adiabatic family *for the fibration $\tilde{\phi}$ in* (12.53) *if and only if its Schwartz kernel (also denoted A) has the following properties as $\delta \downarrow 0$:*

(1) *If $\tilde{\chi} \in \mathcal{C}^\infty(Y^2)$ has support disjoint from the fibre diagonal $\{(p,p') \in Y^2; \tilde{\phi}(p) = \tilde{\phi}(p')\}$ then*

$$\tilde{\chi}A \in \delta^\infty\mathcal{C}^\infty([0,1];\Psi^{-\infty}(Y;E)) \tag{12.54}$$

i.e. is smooth down to $\delta = 0$ where it vanishes to infinite order.

(2) *If $\chi \in \mathcal{C}^\infty(Z)$ has support in a coordinate patch over which $\tilde{\phi}$ is trivial (and E reduces to the pull-back of a bundle $\tilde{E}$ over $\tilde{Z}$) with coordinate z*

then

(12.55)
$$\chi(z)A\chi(z') = \delta^{-m}A'(z, \frac{z-z'}{\delta}; \Psi^{-\infty}(\tilde{Z};\tilde{E})),$$
$$A \in \mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^m \times \tilde{Z} \times \tilde{Z}; \mathrm{Hom}(\tilde{E}) \otimes \Omega(\tilde{Z})_R)$$

having compact support in the first variable and being Schwartz in the second.

The set space of such operators will be denoted $\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(Y;E)$.

*** This is just a quick redefintion of the semiclassical cotangent bundle.

We may define a bundle $\pi_{\mathrm{sl}} : {}^{\mathrm{sl}}T^*Z \longrightarrow Z$ as the restriction to $\delta = 0$ of the bundle over $[0,1]_\delta \times Z$ which has global sections of the form $\alpha(\delta,z)/\delta$ where $\alpha \in \mathcal{C}^\infty([0,1];T^*Z)$ is a smooth 1-form on Z (depending smoothly on δ.) This bundle is bundle isomorphic to T^*Z (since it is so over $(0,1)\times Z$) but not naturally so, however there is a well-defined homotopy class of bundle isomorphisms between ${}^{\mathrm{sl}}T^*Z$ and T^*Z. We may then pull the fibration $Y \longrightarrow Z$ back to a fibration $\pi^*Y \longrightarrow Z$ which has the same fibre, $\tilde{Z}$, but now has base ${}^{\mathrm{sl}}T^*Z$. This allows us to define the smoothing operators on the fibres and also to see that Schwartz sections are well-defined, giving the algebra $\mathcal{S}({}^{\mathrm{sl}}T^*Z; \Psi^{-\infty}(\pi^*_{\mathrm{sl}}Y/{}^{\mathrm{sl}}T^*Z;E))$.

Proposition 12.13. *The adiabatic smoothing operators for a fibration form an algebra of operators on* $\mathcal{C}^\infty([0,1]\times Y;E)$ *with a multiplicative short exact symbol sequence*

(12.56)
$$0 \longrightarrow \delta\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(Y;E) \hookrightarrow \Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(Y;E) \xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{S}({}^{sl}T^*Z; \Psi^{-\infty}(\pi^*_{\mathrm{sl}}Y/{}^{sl}T^*Z;E)) \longrightarrow 0.$$

Proof. The usual. □

Now, we can extend this construction to the iterated fibration (12.51), to define a similar algebra $\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E)$ for any bundle E over $\tilde{M}$. These are just smooth families with respect to the variables in B with each operator being an adiabatic family on the fibre above $b \in B$ – so for the fibration of $\tilde{\phi}^{-1}(Z_b) \subset \tilde{M}$ over $Z_b = \phi^{-1}(b)$. Thus when $\delta \downarrow 0$ additional commutative variables appear in the fibre ${}^{\mathrm{sl}}T^*(Z_b)$; combined with the variables in B this means that the adiabatic symbol has parameters in ${}^{\mathrm{sl}}T^*(M/B)$ as a bundle over M. So the multiplicative short exact sequence (12.56) becomes

(12.57)
$$\delta\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E) \hookrightarrow \Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E) \xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{S}({}^{\mathrm{sl}}T^*(M/B); \Psi^{-\infty}(\pi^*_{\mathrm{sl}}\tilde{M}/{}^{\mathrm{sl}}T^*(M/B);E))$$

where I dropped off the zeros to save space.

Proposition 12.14. *If* $a \in \mathcal{C}^\infty_c({}^{sl}T^*(M/B); G^{-\infty}(\pi^*_{\mathrm{sl}}\tilde{M}/{}^{sl}T^*(M/B);E))$, *i.e. is the sum of the identity and a family in the image of the symbol map in* (12.57) *which has compact support such that the result is always invertible, then any adiabatic family* $A \in \Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E)$ *with* $\sigma_{\mathrm{ad}}(A) = a$ *is such that* $\mathrm{Id} + A(\delta) \in G^{-\infty}(\tilde{M}/B;E)$ *for* $0 : \delta < \delta_0$ *for* $\delta_0 > 0$ *small enough, and this defines unambiguously an index map*

$$\mathrm{Ind}_{\mathrm{sl}} : K^1(T^*(M/B)) \longrightarrow K^1(B) \tag{12.58}$$

which (as the notation indicates) is equal to the semiclassical index map as previously defined.

Proof. The first step is to show homotopy invariance and stability as before. Then, use the (*** currently non-existent) result above showing that smoothing operators on a fibration can be uniformly approximated by finite rank families to deform the symbol a to a finite rank operator i.e. acting on a trivial finite dimensional bundle of $\mathcal{C}^\infty(\tilde{M}/M; E)$ as a bundle over M, and then just observe that a semiclassical quantization of this gives an adiabatic quantization. Hence the maps are the same – the adiabatic quantization is just a more general construction which is 'retractible' to the semiclassical case. $\square$

12.8. Multiplicativity

One of the crucial properties of the semiclassical index, defined in (12.19) is that it gives a commutative diagramme under iteration of fibrations. Thus suppose we are again in the set-up of (12.51). The composite map is then a fibration and we wish to prove that the same (semiclassical) index map arises by quantization in 'one step' and in 'two steps'.

Proposition 12.15. *For an iterated fibration as in* (12.51)*, the semiclassical index map for* ψ*, the overall fibration, is the composite*

$$\operatorname{Ind}_{\mathrm{sl}(\psi)} = \operatorname{Ind}_{\mathrm{sl}(\psi)} \circ \operatorname{Ind}_{\mathrm{sl}(\tilde{\phi}^*)}\,. \tag{12.59}$$

Notice that the map on the rightmost here is *not* quite the usual semiclassical index map for $\tilde{\phi}$ as a fibration from $\tilde{M}$ to M (with fibre $\tilde{Z}$), rather it is the semiclassical index for the pull-back of this fibration to ${}^{\mathrm{sl}}T^*(M/B)$ as a bundle over M to give a fibration

$$\begin{tikzcd} \tilde{Z} \arrow[r, dash] & {}^{\mathrm{sl}}T^*(M/B) \times_M \tilde{M} \arrow[d, "\tilde{\phi}^*"] \\ & {}^{\mathrm{sl}}T^*(M/B). \end{tikzcd} \tag{12.60}$$

It is precisely for this purpose that the adiabatic-semiclassical algebra was discussed earlier. Unfortunately, at the time of writing, the notation in the earlier discussion has not been reversed – it is here but this is potentially confusing.

So consider the very special case where the iterated fibration (12.60) reverts to a single fibration, namely $B = \{\mathrm{pt}\}$. We can then declare $M = Z$ and so write the single fibration as

$$\begin{tikzcd} \tilde{Z} \arrow[r, dash] & M \arrow[d, "\tilde{\phi}"] \\ & Z. \end{tikzcd} \tag{12.61}$$

So, we proceed to construct the algebra $\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E)$ of adiabatic-semiclassical smoothing operators associated to this fibration. Going back to Section 2.20 we consider 2-parameter families, where the parameters are $\epsilon > 0$ and $\delta > 0$ of smoothing operators on $\tilde{M}$ (acting on sections of E). Thus

$$\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \subset \mathcal{C}^\infty((0,1)_\epsilon \times (0,1)_\delta; \Psi^{-\infty}(\tilde{M}; E)) \tag{12.62}$$

and we only need describe exactly the admissible behaviour of the kernels as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, separately and jointly. Of course we do this in terms of the local families discussed in Section 2.20.

Let z_i, $\tilde{z}_j$ be local coordinates in $\tilde{M}$ with the z_i coordinates in the base and $\tilde{z}_j$ coordinates in the fibre. In M^2 we take two copies of such local coordinates with primed versions in the right factors. The fibre diagonal is the globally well-defined manifold given locally by $z_i = z_i'$. The assumptions we make on the kernels are

- As $\delta \downarrow 0$ the kernels vanish rapidly with all derivatives in $z \neq z'$
- As $\epsilon \downarrow 0$ the kernels vanish rapidly with all derivatives in $z \neq z'$ or $\tilde{z} \neq \tilde{z}'$.
- As $\delta \downarrow 0$ but $\epsilon \geq \epsilon_0 > 0$, the kernels a of the form

$$\delta^{-k} A(\delta, \epsilon, \tilde{z}, \tilde{z}', z, \frac{z - z'}{\delta}), \ A \in \mathcal{C}^\infty([0,1] \times [\epsilon_0, 1] \times \tilde{U}, \tilde{U}', U; \mathbb{S}(\mathbb{R}^k)) \tag{12.63}$$

near $z = z'$ (for possibly different coordinate patches U, U' in the fibres.)
- In $\epsilon < \epsilon_0$ for $\epsilon_0 > 0$ sufficiently small, the kernels are uniformly of the form

$$\delta^{-n-k} A(\delta, \epsilon, \tilde{z}, \frac{\tilde{z} - \tilde{z}'}{\epsilon}, z, \frac{z - z'}{\epsilon\delta}), \ A \in \mathcal{C}^\infty([\delta_0, 1] \times [0,1] \times \tilde{U}, U; \mathbb{S}(\mathbb{R}^n)) \tag{12.64}$$

near $z = z'$, $\tilde{z} = \tilde{z}'$.

*** Describe the two tangent and cotangent bundles, adiabatic and semiclassical-adiabatic.

The adiabatic-semiclassical calculus corresponds to a modified cotangent bundle ${}^{\mathrm{sl\,ad}}T^*\tilde{M}$, over $\tilde{M} \times [0,1]_\epsilon \times [0,1]_\delta$. Namely the fibre at any point is the quotient of the space of smooth linear combinations

$$\frac{df}{\epsilon}, \ \frac{dg}{\epsilon\delta}, \ f \in \mathcal{C}^\infty(\tilde{M}), \ g \in \mathcal{C}^\infty(Z) \tag{12.65}$$

by the corresponding product with the ideal of functions vanishing at that point. There are canonical isomorphisms

$$\begin{gathered} {}^{\mathrm{sl\,ad}}T^*\tilde{M}\big|_{\delta>0} \equiv {}^{\mathrm{sl}}T^*\tilde{M}, \\ {}^{\mathrm{sl\,ad}}T^*\tilde{M}\big|_{\epsilon>0} \equiv {}^{\mathrm{ad}}T^*\tilde{M} \text{ and} \\ {}^{\mathrm{sl\,ad}}T^*\tilde{M}\big|_{\epsilon>0,\delta>0} \equiv T^*\tilde{M}. \end{gathered} \tag{12.66}$$

Proposition 12.16. *For a fibration* (12.61) *the space of operators* $\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E)$ *forms an alegbra under composition with two multiplicative exact symbol sequences* (12.67)

$$\delta\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{\sigma_{\mathrm{ad}}\infty}_{\mathrm{sl}}({}^{sl}T^*(M/B) \times_M \tilde{M}/{}^{sl}T^*(M/B))$$

$$\epsilon\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \xrightarrow{\ \sigma_{\mathrm{sl}}\ } \mathcal{C}^\infty\left([0,1]_\delta; \mathcal{S}({}^{\mathrm{sl\,ad}}T^*\tilde{M})\right).$$

Furthermore the joint symbol map gives a short exact sequence

$$\epsilon\delta\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M}; E) \xrightarrow{\sigma_{\mathrm{sl}} \oplus \sigma_{\mathrm{ad}}} \oplus \tag{12.68}$$

12.9. Analytic index

The analytic index map of Atiyah and Singer is defined for any fibration with compact fibres

$$\mathrm{Ind}_{\mathrm{a}} : \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^0(B). \tag{12.69}$$

It starts with the realization of the K-theory as equivalence classes of pairs $(\mathbb{E}, a)$ of a superbundle, i.e. pair or complex vector bundles $\mathbb{E} = (E^+, E^-)$, over M, the total space of the fibration, and an isomorphism $a \in \mathcal{C}^\infty(S^*(M/B); \mathbb{E})$ between the pull-backs to $S^*(M/B)$, the boundary of the radial compactification of $T^*(M/B)$. From the surjectivity of the symbol map for the algebra of pseudodifferential operators on the fibres of $\phi : M \longrightarrow B$,

$$\Psi^0(M/B; \mathbb{E}) \longrightarrow \mathcal{C}^\infty(S^*(M/B); \mathbb{E}) \tag{12.70}$$

we know there exists a family $A \in \Psi^0(M/B; \mathbb{E})$ with $\sigma_0(A) = a$.

Since a is assumed to be invertible, A is, by definition, elliptic. Again from the properties of the calculus we can choose $B \in \Psi^0(M/B; \mathbb{E}^-)$, $\mathbb{E}^- = (E^-, E^+)$, which is a two-sided parameterix for A, so

$$BA = \mathrm{Id} - R_+, \ AB = \mathrm{Id} - R^-, \ R^\pm \in \Psi^{-\infty}(M/B; E^\pm). \tag{12.71}$$

The existence of finite rank exhaustions $\pi^\pm_N \in \mathcal{C}^\infty(M/B; E^\pm)$, for which $\pi^\pm_N R^\pm \longmapsto R$ in $\Psi^{-\infty}(M/B; E^\pm)$ for any element R of this space, allows A to be stabilized to have finite rank. Namely, for N large enough, $\mathrm{Id} - R^+(\mathrm{Id} - \pi^+_N)$ is invertible, with inverse necessarily of the form $\mathrm{Id} - S$, $S \in \Psi^{-\infty}(M/B; E^+)$ and then

$$(\mathrm{Id} - R^+)(\mathrm{Id} - \pi^+_N) = (\mathrm{Id} - R^+(\mathrm{Id} - \pi^+_N))(\mathrm{Id} - \pi^+_N) \Longrightarrow (\mathrm{Id} - S)BA(\mathrm{Id} - \pi_N) = \mathrm{Id} - \pi_N. \tag{12.72}$$

From this it follows that $A(\mathrm{Id} - \pi_N)$ has null space precisely the range of π_N on each fibre. In particular its null spaces form a smooth bundle over B and since it has the same symbol we can replace A with $A(\mathrm{Id} - \pi_N)$. Since the numerical index of a Fredholm family, such as A, is constant, the range of this new choice of A has a finite dimensional complement of constant rank, which can be identified with the null space of A^* for choices of smooth inner products and a smooth family of fibre densities. Let $\pi^- \in \Psi^{-\infty}(M/B; E^-)$ be the family of projections onto this finite dimensional bundle. Then

$$(\mathrm{Id} - \pi^-)A = A \tag{12.73}$$

and B can be replaced by the generalized inverse, which is the inverse of A as a map from the range of $\mathrm{Id} - \pi^-$ to the range of $\mathrm{Id} - \pi_N$ extended as zero to the range of π^-. With this choice (12.71) is replaced by

$$BA = \mathrm{Id} - \pi^+_N, \ AB = \mathrm{Id} - \pi^-. \tag{12.74}$$

Proposition 12.17 (Analytic index). *The class $[\pi^+_N, \pi^-] \in \mathrm{K}^0(B)$ constructed above for N large enough depends only on $[a] \in \mathrm{K}^0_c(T^*(M/B))$ and not on the choices made and gives a well-defined homomorphism* (12.69)*.*

Proof. Choices:

(1) a as representative of $[a]$.
(2) A with symbol a
(3) N
(4) Adjoints, densities

Increasing N to N' adds the bundle $\pi^+_{N'} - \pi^+_N$ to the null space of the quantization of the symbol, changing $A(\mathrm{Id}^+_N)$ to $A(\mathrm{Id}^+_{N'})$ and adds the image of this bundle under A to the complement of the range – i.e. subtracts it from the range. Thus it leaves the index class unchanged. Stabilizing a by adding the identity on some bundle F which is added to both E^+ and E^- also does not change the index and bundle isomorphisms of $E^\pm$ do not change the index either. All the other equivalences can be done by smooth homotopies and this corresponds to adding an interval (or if you want to be very careful, a circle – by reversing the homotopy on the other side) to the base. Then the null and conull bundles are defined over $B \times [0, 1]$ and it follows that their restrictions to the ends are bundle isomorphic. Thus the analytic index is well-defined. □

The existence of an invertible family of pseudodifferential operators of any real order shows that the definition of the index can be extended to elliptic families of any (fixed) order.

12.10. Analytic and semiclassical index

We have now defined the analytic index in the form given by Atiyah and Singer and also a similar map using semiclassical quantization of projections. The latter has also been reduced to the odd semiclassical index by suspension. So the main remaining step in the proof of the index theorem, in K-theory, of Atiyah and Singer is the equality of the analytic and semiclassical index maps.

THEOREM 12.1 (Analytic=Semiclassical index). *The maps* (12.69) *and* (12.20) *are equal.*

Obviously we need to 'put the semiclassical and standard quantizations together.' Once again we do this by developing (yet) another calculus of operators! Fortunately in this case it is the semiclassical calculus for symbols of finite order, rather than the smoothing operators used in the semiclassical calculus, and we have been carrying this along for some time.

The main difference between the two index maps is the realizations of the compactly supported K-theory of the fibrewise cotangent bundle on which they are based. To remove irrelevancies, consider the more general case of a real vector bundle V over a compact manifold B. The index may of Atiyah and Singer is based on the identification of the K-theory with compact supports of V as the K-theory of the radial compactification $\bar{V}$, relative to its boundary $\mathbb{S}V = (V \setminus 0/\mathbb{R}^+$, together with the fact that any vector bundle over $\bar{V}$ is isomorphic to the pull-back of a bundle over B.

Forgetting the latter fact we consider the more general 'chain space' for K-theory consisting of triples (π^+, π^-, a) where $\pi^\pm \in \mathcal{C}^\infty(\bar{V}; M(N, \mathbb{C}))$ are smooth families of projections, $(\pi^\pm)^2 = \pi^\pm$, over the radial compactification and $a \in \mathcal{C}^\infty(\mathbb{S}V; M(N, \mathbb{C}))$ is an identificaiton of their ranges over $\mathbb{S}V$, namely we demand that

$$a\pi^+ = a,\ \pi^- a = a,\ \mathrm{Nul}(a) = (\pi^+),\ \mathrm{Ran}(a) = \mathrm{Ran}(\pi^-). \tag{12.75}$$

That is, a is precisely an isomorphism of the range of π^+ to the range of π^- over the boundary of the radial compactication.

This could be said better!

PROPOSITION 12.18. *The triples (π^+, π^-, a) above give $\mathrm{K}^1_c(V)$ under the equivalence relations of bundle isomorphism over $\bar{V}$ for $\pi^\pm$, homotopy for a and stability, in the sense of taking direct sum with (p, p, Id_p) where $p \in \mathcal{C}^\infty(\bar{V}; M(N', \mathbb{C}))$ is some family of projections. Moreover the inclusion of $(\pi, \pi_\infty, \pi_\infty)$, where $\pi \in \mathcal{C}^\infty(\bar{V}, M(N, \mathbb{C}))$ is a family of projections constant near infinity (with constant value π_∞) and of $(\mathbb{E}, a)$ by complementing E^- to a trivial bundle, induce retractions of the chain spaces.*

PROOF. Deformation. $\square$

Now the idea is that if we can define a 'big' index map from these general triples (π^+, π^-, a) which reduces to the semiclassical and the analytic index maps under the inclusions of these chain spaces, then we prove the desired equality. In fact we will simplify the 'big' chain space by arranging that

$$\pi^+ \in M(N, \mathbb{C}) \tag{12.76}$$

is actually constant. We can do this by stabilizing to a trivial bundle. **** Do more

To 'quantize' a general triple subject to (12.76), we first use Proposition **** to choose a semiclassical family of projections $\Pi^- \in \Psi^0_{\mathrm{sl}}(M/B; \mathbb{C}^N)$ with

$$\sigma_{\mathrm{sl}}(\Pi^-) = \pi^- \text{ and } \sigma_0(\Pi^-) = \pi^-\big|_{S^*(M/B)} \ \forall\ \epsilon > 0. \tag{12.77}$$

Then we choose a standard quantization of the family of matrices a, namely $A' \in \Psi^0(M/B; \mathbb{C}^N)$ with $\sigma_0(A') = a$. For $\epsilon > 0$ but sufficiently small consider the family of 'Toeplitz' operators

$$A = \Pi^- A' \Pi^+ \in \Psi^0(M/B; \mathbb{C}^N),\ \Pi^+ = \pi^+ \in M(N, \mathbb{C}). \tag{12.78}$$

Now, let π_N be our usualy family of finite rank smooth projecitons approximation the identity on the fibres of M/B. As in the standard case, we shall check that

$$A(\mathrm{Id} - \pi_N) \text{ has null space } \mathrm{Ran}(\pi_N \pi^+) \tag{12.79}$$

where by arrangement, $\pi_N \pi^+$ is a family of projections, so defines a smooth bundle over B. Now, (12.79) is just the usual parametrix argument. Let b be the inverse of a as a map from $\mathrm{Ran}(\pi^-)$ to $\mathrm{Ran}(\pi^+)$. Thus it has the same properties as a in (12.75) but with the signs reversed. Then quantize it to $B' \in \Psi^0(M/B; \mathbb{C}^N)$ and replace this by $B = \Pi^+ B' \Pi^-$. From the symbol calculus it follows that

$$BA = \Pi^+(\mathrm{Id} + R^+)\Pi^+,\ \Pi^+ R^+ \Pi^+ = R^+ \tag{12.80}$$

where initially $R^+ \in \Psi^{-1}(M/B; \mathbb{C}^N)$. Then taking an asymptotic sum $(\mathrm{Id} + R)$, with $R \in \Psi^{-1}(M/B; \mathbb{C}^N)$ and $\Pi^+ R \Pi^+ = R$ of the Neumann series for $(\mathrm{Id} + R)$ and composing on the left with $\Pi^+(\mathrm{Id} + R)\Pi^+$ gives (12.80) with error $R^+ \in \Psi^{-\infty}(M/B; \mathbb{C}^N)$. Then (12.77) follows since $R^+(\mathrm{Id} - \Pi_N) \to 0$ in $\Psi^{-\infty}(M/B; \mathbb{C}^N)$.

Once the null space of A has been stabilized to a bundle, i.e. it is replaced by $A(\mathrm{Id} - \pi_N)$ for N sufficiently large, it follows that its range inside the range of the family Π^- has finite dimensional complement, given by a smooth family of projections $\tilde{\pi} \in \Psi^{-\infty}(M/B; \mathbb{C}^N$ with $\Pi^- \tilde{\pi} \Pi^- = \tilde{\pi}$. Then the index in this more general setting is

$$\mathrm{Ind}(\pi^+, \pi^-, A) = [\Pi^+ \pi_N \ominus \tilde{\pi}] \in \mathrm{K}^0(B). \tag{12.81}$$

So, it remains to check that this is independent of the choices made in its definition and that it reduces to the semiclassical index and the analytic index in the corresponding special cases. ****

12.11. Atiyah-Singer index theorem in K-theory

In Theorem 12.1 the two variants of the index map introduced above, have been shown to be equal. The index theorem of Atiyah and Singer therefore reduces to the equality of either of these and the topological index

$$\mathrm{Ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^0(B) \tag{12.82}$$

which we proceed to define. As the name indicates this map does not involve any 'analytic constructions', except that Bott periodicity is involved which we proved analytically. This third map (12.82) for a fibration (12.61) is defined using an embedding into a trivial fibration as in Proposition 12.1. The Collar Neighbourhood Theorem shows that for each point in the base $b \in B$ the corresponding fibre has a neighbourhood $\Omega_b \subset \mathbb{R}^M$ which is a bundle over Z_b which is diffeomorphic (with Z_b mapped to the zero section) to the normal bundle of $Z_b \subset \mathbb{R}^M$. Moverover this is all smooth in b so that in

$$\begin{tikzcd} N \arrow[r, leftrightarrow] \arrow[dr] & \Omega \arrow[r, hook] & B \times \mathbb{S}^M \arrow[ddl] \\ & M \arrow[u] \arrow[d] & \\ & B. & \end{tikzcd} \tag{12.83}$$

the bundle maps are consistent.

Since Ω is smoothly (although by no means naturally) identified with a bundle over M it follows that the relative cotangent bundle of Ω as a bundle over B is smoothly identified as

$$T^*(\Omega/B) \simeq T^*(M/B) \oplus N \oplus N^*. \tag{12.84}$$

Since $N \oplus N^*$ is a symplectic bundle over M we know from the Thom isomorphism that

$$\mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \simeq \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)). \tag{12.85}$$

On the other hand $\Omega \hookrightarrow B \times \mathbb{R}^M$ is an open embedding of fibrations, so there is a pull-back map for compactly supported K-theory:

$$\mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^M) \times B) \tag{12.86}$$

where on the right the relative cotangent bundle of $\mathbb{R}^M \times B$ as a bundle over B is written out. Finally $T^*(\mathbb{R}^M) = \mathbb{R}^{2M}$ so using Bott periodicity combined with the previous maps we define the topological index as the composite

$$\mathrm{Ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \xrightarrow{\iota^*} \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^M) \times B)) \equiv \mathrm{K}^0(B). \tag{12.87}$$

It is not immediately clear that this map is independent of the embedding of the fibration which is used to define it. This is not difficult to show directly but we will instead show that it is equal to something which we already know to be independent of choices.

Theorem 12.2 (Index theorem in K-theory). *The topological index map* (12.87) *is equal to the analytic index map.*

Proof. We will follow the proof in [**2**] at least in outline. That is, we follow the semiclassical index through the diagramme (12.83) and check that the analytic index factors through each step. In view of Theorem 12.1 we can use either the analytic or the semiclassical index map at each stage.

The first stage is to consider the iterated fibration

$$N \longrightarrow M \longrightarrow B. \tag{12.88}$$

Here, N is the normal bundle to the embedding of M. Theorem 12.15 applies to this iterated fibration and gives the commutativity of the three maps on the right, corresponding to (12.59)

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(T^*(M/B) \oplus (N \oplus N^*)) \arrow[d, shift left, "\mathrm{Ind_a}"] \arrow[dd, bend left=60, "\mathrm{Ind_{sl}}"] \\ \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \arrow[u, shift left, "\mathrm{Thom}"] \arrow[d, "\mathrm{Ind_a}"] \\ \mathrm{K}^0(B). \end{tikzcd} \tag{12.89}$$

Since we know that the analytic index is the inverse of the Thom isomorphism we conclude that

$$\mathrm{Ind_a} = \mathrm{Ind_{sl}} \circ \mathrm{Thom}\,. \tag{12.90}$$

The space on the top in (12.89) is diffeomorphic to Ω as a bundle over B so the same identity (12.90) holds with the semiclassical index map for Ω. Thus we have passed through the first map in (12.87) to start the commutative digramme

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \arrow[r] \arrow[dr, "\mathrm{Ind_a}"'] \arrow[d, phantom] & \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \arrow[r, "\iota^*"] \arrow[d, "\mathrm{Ind_{sl}}"] & \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^M) \times B)) \arrow[dl, "\mathrm{Ind_{sl}}"] \\ & \mathrm{K}^0(B). \arrow[from=1-1, to=2-2, bend left=80, "\mathrm{Ind_t}"] & \end{tikzcd} \tag{12.91}$$

Since we already know excision for the semiclassical index, the right triangle in (12.91) also commutes. Then again Bott perioditicity is the same as the index map (now thought of as 'analytic') so in fact the (12.91) shows that the topological index in (12.87) also defines the analytic, or semiclassical, index. $\square$

12.12. Chern character of the index bundle

In the case of the isotropic or Toeplitz index, which is to say the Thom isomorphism, we have already obtained a formula for the Chern character of the index in Proposition 10.24. Starting form this and following the proof above of the index theorem in K-theory, the computation of the Chern character of the index bundle is fairly straightforward. The main complication is that we have a plethora of index maps and we have to keep them a little separated (even though the are all the same). The simplest, and from the current perspective the most fundamental, is the semiclassical odd index.

THEOREM 12.3. *For any fibration the Chern character of the semiclassical odd index map is given by*

$$\mathrm{Ch}^{\mathrm{odd}}(\mathrm{Ind}_{\mathrm{sl}}(a)) = \int_{T^*(M/B)} \mathrm{Ch}^{\mathrm{odd}}(a) \wedge \mathrm{Td}(\phi), \tag{12.92}$$

where the Todd class of the fibration is fibre integral of the Chern class of the Bott element on the normal bundle to an embedding of the fibration in a trivial bundle over the base.

PROOF. The index map for a trivial bundle has been shown in §10.13 to be given by the integration of the Chern character over the fibres, in either the odd or even cases. The index map itself is shown above to factor through this Thom case by embedding

$$\mathrm{Ind}(a) = \mathrm{Ind}(\iota(a \otimes \beta_N)) \tag{12.93}$$

where β_N is the Bott element on the fibres of the normal bundle to the embedding and ι represents the inclusion ('excision') map for K-theory with compact supports in a neighbourhood of the embedding of the total space of the fibration. Thus, applying the bundle case and then the consistency properties of the Chern character,

$$\begin{aligned} \mathrm{Ch}^{\mathrm{odd}}(\mathrm{Ind}(a)) &= \mathrm{Ch}^{\mathrm{odd}}(\mathrm{Ind}(\iota(a \otimes \beta_N)) \\ = \int_{\mathbb{R}^{2N}} \mathrm{Ch}^{\mathrm{odd}}(\iota(a \otimes \beta_N)) &= \int_{N \oplus N*} \mathrm{Ch}^{\mathrm{odd}}(a) \wedge \mathrm{Ch}(\beta_N) \\ &= \int_{T^*(M/B)} \mathrm{Ch}^{\mathrm{odd}}(a) \wedge \mathrm{Td}(\phi). \end{aligned} \tag{12.94}$$

Here the Todd class of the bundle is the integral over the fibres of the Bott element on normal bundle to the embedding. □

Since $\mathrm{Td}(\phi)$ is an absolute cohomology class on $T^*(M/B)$ it can also be identified, via the 'easy' Thom isomorphism, with the pull-back of a cohomology class on M; this is the usual interpretation. Of course one would like to know that $\mathrm{Td}(\phi)$ is determined by ϕ and not by the chosen embedding of M in a trivial bundle. However, the Todd class, being the Chern character of the Bott element, or harmonic oscillator, is stable under the addition of trivial bundles – this again follows from the discussion in §10.13. Under duality it switches sign, since this is just reversing the order of N and N^*. Since the embedding is into a trivial space we see that the normal bundle is a summand of the tangent bundle to a trivial bundle. It follows that its the Todd class is independent of choices. It can of course be identified with a characteristic class but I will not do this here.

Other cases, even semiclassical and Atiyah-Singer now follow from the previous identifications.

12.13. Dirac families

The most commonly encountered families of non-self-adjoint elliptic differential operators, at least in a geometric setting, are Dirac operators. So we discuss these briefly and derive the index formula in cohomology in that case. Indeed, computations based on the special properties of Dirac operators can be used to derive the index formula in general.

12.14. Spectral sections

Problems

PROBLEM 12.1. Proof of Proposition 12.1.

PROBLEM 12.2. Embedding manifolds.

APPENDIX A

Bounded operators on Hilbert space

Some of the main properties of bounded operators on a complex Hilbert space, H, are recalled here; they are assumed at various points in the text.

(1) Boundedness equals continuity, $\mathcal{B}(H)$.
(2) $\|AB\| \le \|A\|\|B\|$
(3) $(A-\lambda)^{-1} \in \mathcal{B}(H)$ if $|\lambda| \ge \|A\|$.
(4) $\|A^*A\| = \|AA^*\| = \|A\|^2$.
(5) Compact operators, defined by requiring the closure of the image of the unit ball to be compact, form the norm closure of the operators of finite rank.
(6) Fredholm operators have parametrices up to compact errors.
(7) Fredholm operators have generalized inverses.
(8) Fredholm operators for an open subalgebra.
(9) Hilbert-Schmidt operators?
(10) Operators of trace class?
(11) General Schatten class?

[1]

[1] Known as Gerard, my PhD advisor

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