PROBLEM SET 5, 18.157
'DUE' THURSDAY 20 MARCH, 2014

Calculus of wavefrontsets.

As usual a bit open-ended and wordy. I will likely not have time to look at these on Monday, but will try to get to it at the end of next week – hence the deadline. If you send it in later I will still look at it eventually.

Let's consider distributions on open subsets of Euclidean spaces. What we do here can be easily transferred to manifolds but I don't want to overwhelm the notation straight off. For smooth functions there are two operations we want to think about – there are more of course.

(1) Pull back under a smooth map $F : U \longrightarrow V$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$:

$$F^* : \mathcal{C}^\infty(V) \longrightarrow \mathcal{C}^\infty(U). \tag{1}$$

(2) Products

$$\mathcal{C}^\infty(U) \times \mathcal{C}^\infty(U) \ni (u, v) \longrightarrow uv \in \mathcal{C}^\infty(U). \tag{2}$$

The question is: To what extent can these be extended to distributions? Of course, we know that if F is a diffeomorphism then (1) extends to distributions. Now, if F is a *submersion* meaning its differential is surjective $F_* : T_xU \longrightarrow T_{F(x)}V$ for each $x \in U$, it follows that the image is open and that

$$F^* : \mathcal{C}^{-\infty}(V) \longrightarrow \mathcal{C}^{-\infty}(U) \tag{3}$$

extends. You can see this by using the Implicit Function Theorem to show that a submersion is, locally near each point in U, the composite of a diffeomorphism and a projection onto a subset of the variables. Under the second map the distribution lifts to be 'independent of the extra variables' and then it is transformed by the diffeomorphism. What about showing that in this case,

$$\mathrm{WF}(F^*u) \subset F^* \mathrm{WF}(u). \tag{4}$$

One way to do this is to work locally and use the decomposition of the map – I claim we know it for diffeomorphisms already, so it is enough to check it for projections.

P1. Show in a couple of lines that a distribution independent of the some subset of the variables has wavefront set in the span of the differentials of the others. [Write down the differential equations it satisfies!].

What about the general case? Well, let's approach it indirectly.

Instead, try extending the product to distributions. It cannot work in general to give a distribution – try multiplying two delta functions at the same point. On the other hand there are many special cases where it is possible. One is the 'exterior product' to define

$$\mathcal{C}^{-\infty}(U) \times \mathcal{C}^{-\infty}(V) \longrightarrow \mathcal{C}^{-\infty}(U \times V). \tag{5}$$

P2. Write this out carefully – again only a few lines – by showing for instance that if $\phi \in \mathcal{C}^\infty_{\text{c}}(U \times V)$ and $v \in \mathcal{C}^{-\infty}(V)$ then pairing in the second set of variables gives a smooth function of compact support which can be paired with a distribution in the first variables.

P3. Compute the wavefront set of the distribution $u(x)v(y)$ just discussed. [You can use the Fourier transform characterization effectively here.]

Note that you need to be a bit careful about the points $(x, y, \xi, 0)$ and $(x, y, 0, \eta)$.

P4. In case $U = V$ apply the restriction theorem from class on Thursday to get a condition on $\mathrm{WF}(u)$ and $\mathrm{WF}(v)$ which allows the product to be defined by

$$u(x)v(x) = (u(x)v(y)|\big|_{x=y}. \tag{6}$$

[It is that $(x, \xi) \in \mathrm{WF}(u)$ implies $(x, -\xi) \notin \mathrm{WF}(v)$.]

P5. What is the Schwartz kernel of pull-back by F (on smooth functions). [In words, it is a delta function on the graph.] Give a bound on its wavefront set.

P6+. You can apply P4 to the kernel in P5 to see when the product $K(x, y)u(y)$ is defined for a distribution u (lifted to be independent of x). If this is defined, the integral over y is always defined (provided the support is compact) and you have defined F^*u. What condition do you get so that F^*u exists?