PROBLEM SET 4, 18.157
'DUE' MONDAY 10 MARCH, 2014, 7AM

Geodesics and the wave equation (eventually).
Okay, this is probably both too long and too demanding. Do what you can and ask if you need help.

Recall that a Riemannian metric on a manifold, M, – which will be compact here although this is not essential – is a smooth positive definite inner product on the fibres of TM varying smoothly with the point. So it is actually a smooth symmetric 2-cotensor, which can be written in any local coordinates x^i as

$$(1) \qquad g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j$$

where g_{ij} is smooth and positive-definite as matrix. The inverse matrix defines and 'energy function' on the cotantent space

$$(2) \qquad |\xi|^2 = E(x,\xi) = \sum_{i,j=1}^{n} g^{ij}(x)\xi^i\xi^j$$

which is independent of the dual coordinates used to define it (this should be in Tuesday's lecture) – it is the length squared of the dual metric on the cotangent space.

Many of these statements have one-line proofs.

(1) Show that the Hamilton vector field H_E is non-vanishing except at $\xi = 0$.

(2) Show that H_E is tangent to each energy surface $E(x,\xi) = e$ and that these are all smooth (quote the IFT).

(3) Recall that the integral curve of a vector field on a compact manifold starting at each point can be extended maximally until it is parameterized by $\mathbb{R}$; deduce that this is true of the integral curves of H_E. The projections of these curves into M are called *geodesics*.

(4) Show that the Laplacian-Beltrami operator on funcions (defined as d^*d) has symbol $|\xi|^2$.

(5) Observe that $\int((\Delta + 1)u)\bar{v}\nu_g$ is a Hilbert inner product on $H^2(M)$ and higher powers give inner products on $H^{2k}(M)$.

(6) Conclude that if ϕ_i, $i \geq 0$ is listing of orthonormal eigenfunctions in $L^2(M)$ so that the eigenvalues are weakly increasing (so

ϕ_0 is constant) and each finite eigenspace is spanned in order then for $u \in \mathcal{C}^\infty(M)$

$$(3) \qquad u(x) = \sum_i \langle u, \phi_i \rangle \phi_i$$

converges to u in $\mathcal{C}^\infty$. (Hint – show that the Fourier coefficients here are such that $\lambda_i^k |\langle u, \phi_i \rangle|$ is bounded (or in l^2) for each k by looking at $\Delta^k u$ and hence prove convergence of the series in each of the even Sobolev norms above).

(7) Now, show that the series

$$(4) \quad u(t, x) = \sum_i \langle u_0, \phi_i \rangle \phi_i(x) \cos \lambda_i t$$

$$+ \sum_{i>0} \langle u_1, \phi_i \rangle \phi_i(x) \frac{\sin \lambda_i t}{\lambda_i} + t \langle u_1, \phi_0 \rangle \phi_0$$

converges in $\mathcal{C}^\infty(\mathbb{R} \times M)$ if u_0, $u_1 \in \mathcal{C}^\infty(M)$.

(8) Show that

$$(5) \qquad (D_t^2 - \Delta)u(t, x) = 0, \ u(0, x) = u_0, \ \partial_t u(0, x) = u_1.$$

(9) Can you come up with an argument to show that $u(t, x)$ is the unique (smooth but actually it doesn't matter) solution to (5)?

(10) Show that even if u_0 and u_1 are distributions, $u(t, x)$ defines a distribution on $\mathbb{R} \times M$ which is 'smooth in t' in the sense that $D_t^k \in \mathcal{C}(\mathbb{R}; H^{N-k}(M))$ where N depends on the regularity of u_0 and u_1.

(11) Now, use Hörmander's theorem to say something about the wavefront set of u – the integral curves of the Hamilton vector field of the symbol $\tau^2 - |\xi|^2$ of the wave operator can be expressed in terms of the geodesics (they are the light rays).

(12) Guess at a relationship between the singularities of $u(t, x)$ and those of u_0 and u_1.