PROBLEM SET 3, 18.157
'DUE' MONDAY 3 MARCH, 2014, 7AM

The resolvent. See what you can make of this! If it is confusing please ask. Mainly intended to get you thinking about these constructions.

Let $A \in \Psi^m(M;V)$, $m > 0$, an integer be a (formally) self-adjoint and elliptic pseudodifferential operator acting on sections of a Hermitian vector bundle over a compact manifold M, equipped with a smooth density $0 < \nu \in \mathcal{C}^\infty(M;\Omega)$. If the bundle is a problem, drop it and work with the scalar case – the difference is mainly in the notation anyway.

The formal self-adjointness means just that in terms of the pairing on the fibres of V and the volume form

$$\int_M \langle Au, v\rangle_V \nu = \int_M \langle u, Av\rangle_V \nu \ \forall\ u, v\ in \mathcal{C}^\infty(M;V). \tag{1}$$

For $\lambda \notin \mathbb{R}$ show that

$$R(A,\lambda) = (A-\lambda)^{-1} \in \Psi^{-m}(M;V) \tag{2}$$

exists as the unique 2-sided inverse.

Do it as cleanly as possible!

(1) Use elliptic regularity to show that A really is self-adjoint with domain the Sobolev space $H^m(M;V)$ and that

$$A - \lambda : H^m(M;V) \longrightarrow L^2(M;V) \tag{3}$$

is an isomorphism.

(2) Observe that $R(A,\lambda)$ should have homogeneous principal symbol $\sigma_m(A)^{-1}$ and that an operator $R_0 \in \Psi^{-m}(M;V)$ with this symbol exists.

(3) Show that $(A-\lambda)R_0 = \operatorname{Id} - E_1(\lambda)$ with $E_1(\lambda) \in \Psi^{-1}(M;V)$.

(4) Use asymptotic completeness to construct $B(\lambda) \in \Psi^{-1}(M;V)$ such that

$$(\operatorname{Id} - E_1(\lambda))(\operatorname{Id} - B(\lambda) = \operatorname{Id} - E_\infty(\lambda),\ E_\infty(\lambda) \in \Psi^{-\infty}(M;V).$$

(5) Use this to modify $R_0(\lambda)$ to $R_\infty(\lambda)$ to get an inverse modulo smoothing errors – first on the right, then on the left and then show that these differ by a smoothing operator

(6) Use the fact that the smoothing operators form a 'corner':

$$\Psi^{-\infty}(M;V)\mathcal{B}(L^2(M;V)\Psi^{-\infty}(M;V) \subset \Psi^{-\infty}(M;V) \tag{4}$$

and the 2-sided inverse/parameterix identities for $R(\lambda) = (A_\lambda)^{-1}$ and $R_\infty(\lambda)$ to show that the differ by a smoothing operator and hence

$$R(\lambda) \in \Psi^{-m}(M; V). \tag{5}$$

(7) Check that
(8) Maybe a little more thought required here! Show that $R(\lambda)$ extends to a meromorphic function on $\mathbb{C}$ with poles only at a discrete set of point on the real axis near which it looks like

$$R(\lambda) = (\lambda - \lambda_i)^{-1} P_i + E(\lambda) \tag{6}$$

with P_i a finite rank smoothing operator which is a projection and $E(\lambda)$ holomorphic near $\lambda_i \ni \mathbb{R}$. Holomorphy here and above can be strong – the Taylor series in bounded operators on L^2 (or any Sobolev space) converges in norm (or even with values in the pseudodifferential operators if you can work out what this means).