PROBLEM SET 2, 18.157
'DUE' MONDAY 24 FEBRUARY, 2014, 7AM

Here due means that I will look at whatever has come in by then and grade them.

This is particularly intended for undergraduates, since they need a serious grade. Of course it would not hurt ...

No doubt you did not miss the problem set for last week!

(1) Compute the wavefront set of the distribution

$$u(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \in \mathcal{S}(\mathbb{R}^n). \tag{1}$$

(2) (Reminder) Show that if $f \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ and $\lambda \notin [0, \infty]$ then

$$u \in \mathcal{S}'(\mathbb{R}^n),\ (\Delta - \lambda)u = f \Longrightarrow u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n). \tag{2}$$

(3) Consider a metric on $\mathbb{R}^n$ which is a compactly supported perturbation of the Euclidean metric

$$g_{ij}(x) = \delta_{ij} \text{ in } |x| > R,\ \sum_{ij} g_{ij}(x)\xi^i\xi^j \ge c|\xi|^2,\ c > 0. \tag{3}$$

Show, for $\lambda \notin [0, \infty)$ that

$$u \in \mathcal{S}'(\mathbb{R}^n),\ (\Delta_g - \lambda)u \in \langle x\rangle^k L^2(\mathbb{R}^n) \Longrightarrow u \in \langle x\rangle^k H^2(\mathbb{R}^n). \tag{4}$$

Hint: Use the previous question and then elliptic regularity.

d not work this out carefully ... Same set up as previous question. We know that $\Delta_g - \lambda$ has a bounded inverse on $L^2(\mathbb{R}^n)$ for $\lambda \notin [0, \infty)$. We would like to prove that it is a 'standard' pseudodifferential operator. What can you deduce from the identity

$$(\Delta - \lambda)(\Delta_g - \lambda)^{-1} + P(\Delta_g - \lambda)^{-1} = \mathrm{Id} \tag{5}$$

where Δ is the flat Laplacian? What about using the adjoint of this as well. Here $P = \Delta_g - \Delta$ has compactly-supported coefficients.

(4) Show that if $A \in \Psi^m(U)$, the pseudodifferential operators on an open subset of $\mathbb{R}^n$ is properly supported, meaning in terms of the Schwartz' kernel the two projections from $U \times U$ restrict to proper maps

$$\pi_L,\ \pi_R : \mathrm{supp}(A) \longrightarrow U \tag{6}$$

then

$$A : H^s_{\mathrm{loc}}(U) \longrightarrow H^{s-m}_{\mathrm{loc}}(U)\ \forall\ s. \tag{7}$$

Hint: Freely use uniqueness of Schwartz' kernels on open sets, maybe prove the adjoint preserves the spaces of compact support?