# PROBLEM SET 2, 18.157
# SOLUTIONS TO REWORDED QUESTIONS!

This was a bit of a disaster, my fault I think – in 1) I should have given you a pointer, 2) was grossly misstated 3) I did not ask the question directly enough

(1) Recall that the wavefront set of a distribution can be written as the intersections of the characteristic varieties of pseudodifferenial operators which map it to a smooth function. Maybe using differential operators show that the wavefront set of the distribution

$$u(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \in \mathcal{S}(\mathbb{R}^n) \tag{1}$$

is contained in a smooth conic manifold of dimension $n$ (which is in fact a Lagrangian submanifold of $T^*\mathbb{R}^n$).

Solution: Using coordinate independence to replace the sphere by a plane works – although I have not proved coordinate independence for the wavefrontset yet – then one can use the Fourier tranform.

A direct way is to observe that the singular support is $|x| = 1$ so $\mathrm{WF}(u)$ must lie above it. Now, the definition is

$$\mathrm{WF}(u) = \bigcap \{\Sigma(A); Au \in \mathcal{C}^\infty\}.$$

We can find some differential operators, namely the generators of the rotation group, which annihilate $u$ :

$$(x_i D_j - x_j D_i)u = 0.$$

So $x_i\xi_j - x_j\xi_i = 0$ on $\mathrm{WF}(u)$. Fixing a point, $x = (1, 0, \dots, 0)$ we see that this means $\xi_j = 0$ for $j \neq 1$ and in general that $(x, \xi) \notin \mathrm{WF}(u)$ if $\xi \neq a \pm x$, $a > 0$. So as a conic set the wave front set is contained in the conormal variety to the sphere

$$\mathrm{WF} \subset N^*S \subset T^*\mathbb{R}^n,$$

consisting of the (non-vanishing) multiples of $d|x|^2$ at $|x| = 1$. This is clearly a smooth manifold, since it is just $\mathbb{S}^{n-1} \times (\mathbb{R}\backslash\{0\})$.

What about the converse, that the wave front set is no smaller than this?

(2) (Reminder) Show that if $f \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ and $\lambda \notin [0, \infty]$ then

$$u \in \mathcal{S}'(\mathbb{R}^n),\ (\Delta - \lambda)u = f \Longrightarrow u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n). \tag{2}$$

Solution: I really messed this up. By elliptic regularity, the singular support of $u$ must be compact, contained in that of $f$, and if we cut off

$$u = u_1 + u_2,\ u_1 = \phi u,\ u_2 = (1 - \phi)u$$

where $\phi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ is identically equal to 1 in a neighbourhood of the support of $f$ then $(\Delta - \lambda)u_2 \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$. We can simply look at the Fourier transform

$$\hat{u}_2(\xi) = \hat{f}/(|\xi|^2 - \lambda) \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow u_2 \in \mathcal{S}(\mathbb{R}^n).$$

(3) Consider a metric on $\mathbb{R}^n$ which is a compactly supported perturbation of the Euclidean metric

$$g_{ij}(x) = \delta_{ij} \text{ in } |x| > R,\ \sum_{ij} g_{ij}(x)\xi^i\xi^j \geq c|\xi|^2,\ c > 0. \tag{3}$$

Show, for $\lambda \notin [0, \infty)$ that

$$u \in \mathcal{S}'(\mathbb{R}^n),\ (\Delta_g - \lambda)u \in \langle x\rangle^k L^2(\mathbb{R}^n) \Longrightarrow u \in \langle x\rangle^k H^2(\mathbb{R}^n). \tag{4}$$

Hint: Use the previous question and then elliptic regularity.

Solution:

d not work this out carefully ... Same set up as previous question. We know that $\Delta_g - \lambda$ has a bounded inverse on $L^2(\mathbb{R}^n)$ for $\lambda \notin [0, \infty)$. We would like to prove that it is a 'standard' pseudodifferential operator. What can you deduce from the identity

$$(\Delta - \lambda)(\Delta_g - \lambda)^{-1} + P(\Delta_g - \lambda)^{-1} = \mathrm{Id} \tag{5}$$

where $\Delta$ is the flat Laplacian? What about using the adjoint of this as well. Here $P = \Delta_g - \Delta$ has compactly-supported coefficients.

Solution: What follows from (5) is that

$$(\Delta_g - \lambda)^{-1} = -(\Delta - \lambda)^{-1}P(\Delta_g - \lambda)^{-1} + (\Delta - \lambda)^{-1}.$$

There is a similar identity for composition the other way:

$$(\Delta_g - \lambda)^{-1} = -(\Delta_g - \lambda)^{-1}P(\Delta - \lambda)^{-1} + (\Delta - \lambda)^{-1}.$$

Inserting the second one in the first gives

$$(\Delta_g-\lambda)^{-1} = (\Delta-\lambda)^{-1}P(\Delta_g-\lambda)^{-1}P(\Delta-\lambda)^{-1} - (\Delta-\lambda)^{-1}P(\Delta-\lambda)^{-1} + (\Delta-\lambda)^{-1} \tag{6}$$

Now, we do know from the parametrix construction that

$$(\Delta_g - \lambda)^{-1} = Q + R,\ Q \in \Psi_{\infty}^{-1}(\mathbb{R}^n),$$

where $R$ is a smoothing operator, which maps $H^k(\mathbb{R}^n)$ to $H^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{R}$ and with adjoint which does the same – what we don't know is that $R \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$.

In fact it follows from (6) that indeed

$$(\Delta_g - \lambda)^{-1} \in \Psi^{-2}_\infty(\mathbb{R}^n)$$

since the second two terms are in the space and the middle part of the first term $P(\Delta_g - \lambda)^{-1}P$ is the sum of a element of this space and a compactly supported smoothing operator – which is also in the space.

(4) Show that if $A \in \Psi^m(U)$, the pseudodifferential operators on an open subset of $\mathbb{R}^n$ is properly supported, meaning in terms of the Schwartz' kernel the two projections from $U \times U$ restrict to proper maps

$$\pi_L,\ \pi_R : \operatorname{supp}(A) \longrightarrow U \tag{7}$$

then

$$A : H^s_{\text{loc}}(U) \longrightarrow H^{s-m}_{\text{loc}}(U)\ \forall\ s. \tag{8}$$

Hint: Freely use uniqueness of Schwartz' kernels on open sets, maybe prove the adjoint preserves the spaces of compact support?

Solution: We know that $A^* \in \Psi^m(U)$ is a continuous linear map from $H^{-s+m}_{\text{c}}(U)$ to $H^{-s}_{\text{loc}}(U)$. The properness of the support of $A$, and hence of $A^*$ means that

$$A^* : H^{-s+m}_{\text{c}}(U) \longleftrightarrow H^{-s}_{\text{c}}(U)$$

and then by duality $A$ is a map as suggested.