PROBLEM SET 1, 18.157
DUE FRIDAY 10 FEBRUARY, 2014, 7AM

Here due means that I will look at whatever has come in by then and grade them – but that is it until the following Monday.

This is mostly intended for undergraduates, since they need a serious grade. Of course it would not hurt ...

(1) Consider the ring of differential operators on $\mathbb{R}^n$ with smooth coefficients. These can be written in standard 'left-reduced' form:

$$P : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n),\ Pu(x) = \sum_{|\alpha|\le m} p_\alpha(x) D^\alpha,\ D = \frac{1}{i}\partial. \tag{1}$$

The polynomial in ξ wth smooth coefficients (you might want to assume that the coefficients and all their derivatives are bounded to fit in with the lectures but it does not matter for the moment)

$$p(x,\xi) = \sum_{|\alpha|\le m} p_\alpha(x)\xi^\alpha \tag{2}$$

determines the operator and is its 'left-reduced full symbol' or left symbol. Derive a formula for the left symbol of $P \circ Q$ in terms of the left symbols of P and Q in the form

$$\sigma_L(P \circ Q) = \sum_\beta c_\beta D^\beta_\xi \sigma_L(P) D^\beta_x \sigma_L(Q). \tag{3}$$

(2) Recall that $\mathcal{C}^\infty(\mathbb{S})$, $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ can be identified with smooth functions on the line which are 2π-periodic and that there Fourier series converge absolutely with all derivatives

$$u(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} u_k e^{ikx},$$
$$u_k = \int_{\mathbb{S}} e^{-iky} u(y)dy \tag{4}$$
$$\mathcal{C}^\infty(\mathbb{S}) \longleftrightarrow \{(u_k)_{k=-\infty}^{\infty}; \sum_k |u_k| k^N < \infty\ \forall\ N\}.$$

Consider the Hardy space

$$\mathcal{C}^\infty_H(\mathbb{S}) = \{u \in \mathcal{C}^\infty(\mathbb{S}); u_k = 0,\ k < 0\} \tag{5}$$

and show that

(6)
$$\mathcal{C}^\infty_H(\mathbb{S}) = \{u \in \mathcal{C}^\infty(\mathbb{S});\ \exists\ \tilde u \in \mathcal{C}^\infty(\mathbb{D}),\ \bar\partial \tilde u = 0 \text{ in } |z|<1,\ u = \tilde u\big|_{|z|=1}\},$$
$$\mathbb{D} = \{z \in \mathbb{C}; |z| \le 1\}.$$

(3) (Easy) Define the projection

(7) $$Hu = \frac{1}{2\pi}\sum_{k\ge 0} u_k e^{ikx}$$

and show that it extends to a bounded operator on $L^2(\mathbb{S})$.

(4) A smoothing operator on $\mathcal{C}^\infty(\mathbb{S})$ can be defined in terms of Fourier series by

(8) $$(Au)_k = \sum_{j\in\mathbb{Z}} a_{kj}u_j \text{ where } |a_{jk}| \le C_N(1+|j|+|k|)^{-N}\ \forall\ N.$$

Show that this is the same as an integral operator with smooth Schwartz kernel on $\mathbb{S}\times\mathbb{S}$.

(5) If $a \in \mathcal{C}^\infty(\mathbb{S})$ is a smooth function then multiplication by a is a differential operator of order 0 on $\mathcal{C}^\infty(\mathbb{S})$. Show that the commutator $[H, a]$ is a smoothing operator.

(6) If P and Q are differential operators on $\mathbb{S}$ (just differential operators on $\mathbb{R}$ with periodic coefficients) show that there exists a differential operator R on $\mathbb{S}$ such that

(9) $$HRH = HPHQH + A$$

with A a smoothing operator.