PROBLEM SET 1, 18.157
WITH VERY BRIEF SOLUTIONS

Here due means that I will look at whatever has come in by then and grade them – but that is it until the following Monday.

This is mostly intended for undergraduates, since they need a serious grade. Of course it would not hurt ...

(1) Consider the ring of differential operators on $\mathbb{R}^n$ with smooth coefficients. These can be written in standard ‘left-reduced’ form:

$$P : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n),\ Pu(x) = \sum_{|\alpha|\le m} p_\alpha(x) D^\alpha,\ D = \frac{1}{i}\partial. \tag{1}$$

The polynomial in ξ wth smooth coefficients (you might want to assume that the coefficients and all their derivatives are bounded to fit in with the lectures but it does not matter for the moment)

$$p(x,\xi) = \sum_{|\alpha|\le m} p_\alpha(x)\xi^\alpha \tag{2}$$

determines the operator and is its ‘left-reduced full symbol’ or left symbol. Derive a formula for the left symbol of $P \circ Q$ in terms of the left symbols of P and Q in the form

$$\sigma_L(P\circ Q) = \sum_\beta c_\beta D_\xi^\beta \sigma_L(P) D_x^\beta \sigma_L(Q). \tag{3}$$

My short soluton: Look at the differential operator $D^\alpha q(x)$ with coefficient on the right:

$$(D^\alpha q(x))u(x) = \sum_{\gamma\le\alpha} \binom{\alpha,\gamma}{(} D_x^\gamma q) D_x^{\alpha-\gamma} u \tag{4}$$

Since $\frac{\alpha!}{(\alpha\gamma)!}\xi^{\alpha-\gamma} = \partial_\xi \xi^\alpha$ when $\gamma \le \alpha$ and vanishes otherwise, this give the left-reduced form of this differential operator as

$$\sigma_L(D^\alpha q(x)) = \sum_\gamma \frac{1}{\gamma!}(\partial_\xi^\gamma p(x,\xi))(D_x^\gamma q(x,\xi)) \tag{5}$$

where $p(x,\xi) = \xi^\alpha$ and $q(x,\xi) = q(x)$. However, adding a coefficient to the left and differentiation to the right gives the same formula which therefore holds in general by superposition.

(2) Recall that $\mathcal{C}^\infty(\mathbb{S})$, $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ can be identified with smooth functions on the line which are 2π-periodic and that there Fourier series converge absolutely with all derivatives

$$u(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} u_k e^{ikx},$$

$$u_k = \int_{\mathbb{S}} e^{-iky} u(y)dy \tag{6}$$

$$\mathcal{C}^\infty(\mathbb{S}) \longleftrightarrow \{(u_k)_{k=-\infty}^{\infty}; \sum_k |u_k| k^N < \infty \ \forall\ N\}.$$

Consider the Hardy space

$$\mathcal{C}^\infty_H(\mathbb{S}) = \{u \in \mathcal{C}^\infty(\mathbb{S}); u_k = 0,\ k < 0\} \tag{7}$$

and show that

(8)
$$\mathcal{C}^\infty_H(\mathbb{S}) = \{u \in \mathcal{C}^\infty(\mathbb{S});\ \exists\ \tilde{u} \in \mathcal{C}^\infty(\mathbb{D}),\ \bar{\partial}\tilde{u} = 0 \text{ in } |z| < 1,\ u = \tilde{u}\big|_{|z|=1}\},$$
$$\mathbb{D} = \{z \in \mathbb{C}; |z| \le 1\}.$$

My short solution: All the derivatives are in L^2 and it follows that $|u_k| \le C_N(1+|k|)^{-N}$ for any N if $u \in \mathcal{C}^\infty(\mathbb{S})$. Then the series

$$\tilde{u} = \sum_{k\ge 0} u_k z^k \tag{9}$$

converges uniformly with all derivatives in $|z| \le 1$ and hence defines $\tilde{u} \in \mathcal{C}^\infty(\mathbb{D})$ restricting to u on $|z| = 1$. Conversely, if $\tilde{u} \in \mathcal{C}^\infty(\mathbb{D})$ then the Fourier series $\tilde{u}(re^{i\theta})$ converges for each $r \le 1$ and the coefficients are $\mathcal{C}^1$ in $r \in [0,1]$ and are rapidly decaying. If $\tilde{u}$ is holonomrphic then $du_k(r)/dr = iku_k(r)$ so in fact (9) holds with $u_k = u_k(1)$.

(3) (Easy) Define the projection

$$Hu = \frac{1}{2\pi}\sum_{k\ge 0} u_k e^{ikx} \tag{10}$$

and show that it extends to a bounded operator on $L^2(\mathbb{S})$.

My short solution: $\|Hu\|_{L^2} = c\sum_{k\le 0} |u_k|^2 \le \|u\|^2_{L^2}$ is bounded.

(4) A smoothing operator on $\mathcal{C}^\infty(\mathbb{S})$ can be defined in terms of Fourier series by

$$(Au)_k = \sum_{j\in\mathbb{Z}} a_{kj} u_j \text{ where } |a_{jk}| \le C_N(1+|j|+|k|)^{-N}\ \forall\ N. \tag{11}$$

Show that this is the same as an integral operator with smooth Schwartz kernel on $\mathbb{S} \times \mathbb{S}$.

My short solution: For a smooth function on $\mathbb{S}\times\mathbb{S}$ the double Fourier series

$$a(x,y) = \sum_{k,j} a_{k,j} e^{ikx+ily} \tag{12}$$

converges rapidly in both variables – $|a_{j,k}| \le C_N(1+|j|+|k|)^{-N}$ (since the expansion in one variable is smooth in the other with rapidly decaying derivatives). This allows the integral

$$Au(x) = \int_{\mathbb{S}} a(x,y)u(y)dy \tag{13}$$

to be expanded and the argument can be reversed.

(5) If $a \in \mathcal{C}^\infty(\mathbb{S})$ is a smooth function then multiplication by a is a differential operator of order 0 on $\mathcal{C}^\infty(\mathbb{S})$. Show that the commutator $[H,a]$ is a smoothing operator.

My short solution: Multiplication by a smooth function is a convolution operator on the Fourier coefficients

$$(a(x)u(x))_k = \sum_j a_{k-j}u_j,\ \ |a_p| \le C_N(1+|p|)^{-N} \tag{14}$$

The commutator with H on the Fourier side is therefore

$$([H,a]u)_k = \sum_{k\ge 0, j<0} a_{k-j}u_j - \sum_{k<0, j\ge 0} a_j \tag{15}$$

since the 'diagonal' parts cancel. In this region $|k-j| = |k|+|j|$ so the coefficients are rapidly decreasing in both variables.

(6) If P and Q are differential operators on $\mathbb{S}$ (just differential operators on $\mathbb{R}$ with periodic coefficients) show that there exists a differential operator R on $\mathbb{S}$ such that

$$HRH = HPHQH + A \tag{16}$$

with A a smoothing operator.

My short solution: $H^2 = H$ so

$$HPQH - HPHQH = HPQH^2 - HPHQH = HP[Q,H]H = A \tag{17}$$

is smoothing since, the commutator $[Q,H]$ is smoothing – expand the differential operator and use previous question plus the fact (I should have mentioned) that composition of a differential and a smoothing operator on right or left is smoothing, as is composition with H.