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We start with what for the moment is a formal expression:

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)dyd\xi,$$
$$\tilde{a} = (1+|x-y|^2)^{-w} a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n). \tag{1}$$

We showed last time that if $m < -n$ and $u \in \mathcal{S}(\mathbb{R}^n)$ then $Au(x)$ is continuous and

$$\sup_{x\in\mathbb{R}^n} |(1+|x|)^{-w} Au(x)| \le C\|\tilde{a}\|\|u\| \tag{2}$$

where the seminorms on $\tilde{a}$ and u are continuous for the symbol topology with $m < -n$ and the Schwartz topologies respectively. Note that this gives decay when $w < 0$.

We proceed to use the identities arising from differentiating the exponential – for instance

$$(1+|x-y|^2)e^{i(x-y)\cdot\xi} = (1+\Delta_\xi)e^{i(x-y)\cdot\xi},$$
$$(1+|\xi|^2)e^{i(x-y)\cdot\xi} = (1+\Delta_y)e^{i(x-y)\cdot\xi} \text{ and}$$
$$(1+|\xi|^2)e^{i(x-y)\cdot\xi} = (1+\Delta_x)e^{i(x-y)\cdot\xi}. \tag{3}$$

Assuming that $m = -\infty$, so the integrals in (1) are very strongly convergent, using the first identity (and dividing by the factor on the left) integration by parts allows us to rewrite (1) as

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(1+|x-y|)^{-N}(1+\Delta_\xi)^N a(x,y,\xi)u(y)dyd\xi \tag{4}$$

From the symbol estimates $(1+\Delta_\xi)^N a = (1+|x-y|^2)^w(1+\Delta_\xi)^N\tilde{a}$ and $(1+\Delta_\xi)^N\tilde{a} \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. So we conclude for any N,

$$\sup_{x\in\mathbb{R}^n} |(1+|x|)^N Au(x)| \le C\|\tilde{a}\|_N\|u\| \tag{5}$$

where again seminorms on $\tilde{a}$ and u are continuous for the symbol topology with $m < -n$ and the Schwartz topologies.

Now apply the second identity, again dividing by the factor on the left – integration by parts is again fine (assuming $m = -\infty$) and we see that

(6)
$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(1+|\xi|^2)^k(1+|x-y|)^{-N}(1+|\xi|^2)^{-k}(1+\Delta_\xi)^N(1+\Delta_y)^k\left(a(x,y,\xi)u(y)\right) d$$

for any k. The action of the Laplacian can be expanded – derivatives in y falling on the symbol do not change its order, and falling on u map still into the Schwartz space. However, the factor of $(1+|\xi|^2)^{-k}$ lowers the order of the symbol by $2k$. Given $m \in \mathbb{R}$ we can choose $2k > m+n$ and the amplitude after expanding out will be of order less than $-n$ if a is initially of order m. Thus, we get the same estimates (5) for any chosen m with the (larger) seminorms on the right now continuous on S^m and $\mathcal{S}$.

Finally we use the last identity in (3) by applying $(1+\Delta_x)^l$ to Au. Provided the integral converges absolutely after differentiation under the integral sign this is admissible. Having chosen l we choose $2k > m+n+2l$ and the integral remains convergent so finally (5) can be replaced by

$$|(1+|x|)^N(1+\Delta_x)^l Au| \le C_{m,N,l}\|\tilde{a}\|_{N,m,l}\|u\|_{N,m,l} \tag{7}$$

where the norms on the right are continuous on S^m and $\mathcal{S}$ but depend on N, m and l. This shows that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \tag{8}$$

is continous, and is actually jointly continous with respect to the symbol topology.

This means we can compose these operators with each other.

Now, the next step is to show that any operator of the form (1) can be written in the same form, but with an amplitude which does not depend on y at all – so also of course can have no factors of $1+|x-y|^2$. For this we first use the integration by parts identity and Taylor series:

$$\begin{gathered} D^\alpha_\xi e^{i(x-y)\cdot\xi} = (x-y)^\alpha e^{i(x-y)\cdot\xi}, \\ a(x,y,\xi) = \sum_{|\alpha|\le N}(y-x)^\alpha b_\alpha(x,\xi) + \sum_{|\beta|=N+1}(y-x)^\beta R_\beta(x,y,\xi). \end{gathered} \tag{9}$$

The coefficients in the Taylor series are the y-derivatives at $y=x$ and are therefore elements of S^m if $\tilde{a} \in S^m$. We need to choose the form of the remainder terms and examine their properties, but clearly the remainder *sum* $R \in (1+|x-y|^2)^w S^m$, (if $w \ge 0$) since all the other terms are. Assuming as usual that $m = -\infty$ (and eventually using density) we can use the identity to integratate by parts and see that

$$A = \sum_{|\alpha|\le N} A_\alpha + B \tag{10}$$

where A_α is given by (1) with amplitude

$$b_\alpha = i^{|\alpha|} D_\xi^\alpha D_y^\alpha a(x,x,\xi) \in S^{m-|\alpha|}. \tag{11}$$

So all these terms are in the desired 'left-reduced' form.

Now, we need to go back to symbols and use *asymptotic summation* to see that there exists $b \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$b - \sum_{|\alpha|\le N} b_\alpha \in S^{m-N}(\mathbb{R}^n;\mathbb{R}^n)\ \forall\ N. \tag{12}$$

See the notes.

We also check, using Lagrange's (I think) form of the remainder that (9) holds with $(1+|x-y|^2)^{-N} R_\alpha \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$. Integration by parts using the first part of (9) shows that the operators D_β corresponding to the R_β are each of the form (1) with an amplitude (depending on all variables and with a weight factor $(1+|x-y|^2)^N$) of order $m-N-1$. Integration by parts removes the extra weight factors, so if since B is the operator with left-reduced symbol b what (12) and (9) show is that for any N,

$$A - B = G_N,\ G_N \in \Psi^{m-N-1}(\mathbb{R}^n) \tag{13}$$

meaning G_N is of the form (1) for some amplitude of order $m-N-1$ and fixed weight w.

The important point is that the operator on the left is fixed, independent of N. Then we proceed to show, by going back through the statements above, that the Schwartz kernel of $A-B$, $G(x,y)$ satisfies

$$|D_x^\alpha D_y^\beta G(x,y)| \le C_{\alpha,\beta,N}(1+|x-y|)^{-N}\ \forall\ \alpha,\ \beta \text{ and } N. \tag{14}$$

From this in turn it follows that $A-B$ can be written as a left-reduced operator with amplitude

$$g(x,\xi) = \int e^{iz\cdot\xi} G(x,x-z)dz \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{15}$$