① Microlocal refers to more local than local, corresponding to the fact that we will work with concepts on the cotangent space, for Euclidean space this means $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

② We see this first in the representation of linear differential operators $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ of the form

$$Pu = P(x,D)u = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha u$$

$\alpha \in \mathbb{N}_0^n$ a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_j = \frac{1}{i}\partial_{x_j}, \quad i = \sqrt{-1}.$$

We can suppose $p_\alpha(x)$ has all derivatives bounded for notation, so $|D_x^\beta p_\alpha(x)| \le C_{\alpha\beta}$ on $\mathbb{R}^n$ $\forall \beta, \alpha$.

Ex if $g_{ij}(x)$ is a positive definite $n \times n$ matrix of the type: $\sum_{ij} g_{ij}(x) \eta_i \eta_j \ge c|\eta|^2$, $c > 0$,

$$|D^\beta g_{ij}(x)| \le C_\beta \quad \forall x \in \mathbb{R}^n$$

then

$$\Delta_g = \sum_{ij} \frac{1}{\sqrt{\det g}} D_i g^{ij} \sqrt{\det g}\, D_j$$

is the Laplacian.

__Claim__ If $\lambda \notin [0, \infty)$ $(\Delta_g - \lambda)^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ __de.__

③ Thus $(\Delta_g - 1)^{-1}$ is an example of a pseudodifferential operator: $(\Delta_g - 1)^{-1} \in \Psi_\infty^{-2}(\mathbb{R}^n)$; which I will proceed to discuss and prove.

④ $\mathrm{Diff}_\infty^{*}(\mathbb{R}^n) = \{P \text{ as above}\}$ is an algebra. In Problems I ask you to compute $P \cdot Q$. We will define a bigger algebra $\Psi_\infty^{\mathbb{Z}}(\mathbb{R}^n)$ which contains $(\Delta_g - 1)^{-1}$ above for instance – and more importantly its properties provide tools for use to aid in proofs of constructions.

⑤ We would start by noting that $D^\alpha u$ can be written as the inverse Fourier transform of $\xi^\alpha \hat{u}$ so

$$P(x,D)u = (2\pi)^{-n} \int e^{ix\cdot\xi} P(x,\xi)\, \hat{u}(\xi)\, d\xi$$

$$= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} P(x,\xi)\, u(y)\, d\xi\, dy.$$

Here we really have to brush up the background with manipulations to make sense of this.

– Now see Chapter 2 of the notes.