PROBLEM SET 8, 18.155
DUE 1 DECEMBER, 2017

One thing that I have not been able to describe is the *wavefront set* of a distribution, so I ask you to assimilate the definition and deduce some basic properties. This notion involves cones in $\mathbb{R}^n \setminus \{0\}$ so let me define 'the open cone of aperture $\epsilon > 0$ around a point' to be

$$\Gamma(\bar{\xi}, \epsilon) = \left\{ \xi \in \mathbb{R}^n \setminus \{0\}; |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \epsilon \right\}. \tag{1}$$

Make sure you see that this is just a ball around the point in the sphere $\bar{\xi}/|\bar{\xi}| \in \mathbb{S}^{n-1}$ extended radially.

If $u \in \mathcal{C}^{-\infty}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, the wave front set of u is the subset

$$\mathrm{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\}) \tag{2}$$

defined in terms of its complement

$$\begin{aligned}(3)\quad \Omega \times (\mathbb{R}^n \setminus \{0\}) \ni (\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u) \Longleftrightarrow \\ \exists\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \phi(\bar{x}) \neq 0 \text{ and } \epsilon > 0 \text{ such that} \\ \sup_\Gamma |\xi|^N |\mathcal{F}(\phi u)(\xi)| < \infty\ \forall\ N,\ \Gamma = \Gamma(\bar{\xi}, \epsilon).\end{aligned}$$

The idea is that the wavefront set gives information about the (co-) direction of singularities, not just their position.

Q8.1

For $u \in \mathcal{C}^{-\infty}(\Omega)$ show that

(1) $\mathrm{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ is closed (as a subset of course)
(2) $\mathrm{WF}(u)$ is 'conic' i.e.

$$(x, \xi) \in \mathrm{WF}(u) \Longrightarrow (x, t\xi) \in \mathrm{WF}(u),\ (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}),\ t > 0. \tag{4}$$

(3)

$$\mathrm{WF}(u) \subset \operatorname{singsupp}(u) \times (\mathbb{R}^n \setminus \{0\}). \tag{5}$$

Q8.2

Given $\bar{\xi} \in \mathbb{R}^n \setminus \{0\}$ and $\epsilon_1 > \epsilon_2 > 0$ small construct a(n almost) conic cut-off $0 \leq \psi \in S^0(\mathbb{R}^n)$ (the symbol space) such that

$$\operatorname{supp} \psi \subset \Gamma(\bar{\xi}, \epsilon_1),\ \psi = 1 \text{ on } \Gamma(\bar{\xi}, \epsilon_2) \cap \{|\xi| > 2\}. \tag{6}$$

Show that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$ is equivalent to

$$\psi \mathcal{F}(\phi u) \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow b_\psi * (\phi u) \in \mathcal{S}(\mathbb{R}^n),\ \hat{b}_\psi = \psi, \tag{7}$$

for some $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\phi(\bar{x}) \neq 0$, $\epsilon_1 > \epsilon_2 > 0$.

Hint:- One way is easy here. The other way the issue is that the definition of $\mathrm{WF}(u)$ only gives directly the condition that $b_\psi * \phi u \in H^\infty(\mathbb{R}^n)$ (the intersection of the Sobolev spaces). You should recall that b_ψ is the sum of a compactly supported distribution and an element of $\mathcal{S}(\mathbb{R}^n)$.

Q8.3

(1) Now show that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$ implies that for some $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\phi(\bar{x}) \neq 0$ and some cone $\Gamma(\bar{x}, \epsilon)$, $\epsilon > 0$

$$b * (\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \hat{b} \in S^m(\mathbb{R}^n),\ \mathrm{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon). \tag{8}$$

(2) Recall (you do not have to prove this, I will do it, and more, in class in L20) that if $b \in S^m(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ then there exist $\phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$ and $b_\alpha \in S^{m-j}$ such that given k there exists $N = N_k$ such that the operator

$$E_N : u \longmapsto b * (\phi u) - \sum_{|\alpha| \le N} \phi_j (b_j * u) \tag{9}$$

has Schwartz kernel in $\mathcal{C}^k(\mathbb{R}^{2n})$.

(3) Conclude that if (8) holds then for any $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$

$$b * (\mu \phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \hat{b} \in S^m(\mathbb{R}^n),\ \mathrm{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon).$$

Hint: A kernel in $\mathcal{C}^k(\mathbb{R}^{2n})$ defines a map from $H^{-k}_{\mathrm{c}}(\mathbb{R}^n)$ to $H^k_{\mathrm{loc}}(\mathbb{R}^n)$ so as k increases this becomes 'increasingly a smoothing operator'. If you know something about the support properties as well (from its definition) you get more.

(4) Hence deduce that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$ is equivalent to the apparently stronger statement that for some $\epsilon > 0$

$$\begin{aligned} b * (\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \mathrm{supp}\, \phi \subset B(\bar{x}, \epsilon),& \\ \hat{b} \in S^m(\mathbb{R}^n),\ \mathrm{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon).& \end{aligned} \tag{10}$$

Q8.4

Prove a complement to the last part of Q8.1 in the sense that for any $u \in \mathcal{C}^{-\infty}(\Omega)$ the wavefront set is a refinement of the singular support:-

$$\pi(\mathrm{WF}(u)) = \mathrm{singsupp}(u),\ \pi(x, \xi) = x \tag{11}$$

Q8.5

This is a somwhat unfair, open-ended, question but I could not resist! Make of it what you will.

We now have three 'support' sets for distributions in an open set U – support itself, singular support and this notion of wavefront set – the first two are subsets of U but the third is a subset of $U \times (\mathbb{R}^n \setminus \{0\})$. Since it is conic we can also think of $\mathrm{WF}(v)$ as a (closed) subset of $U \times \mathbb{S}^{n-1}$. Try to explain how these three correspond to sheaves, in the three cases

(1) The support corresponds to the sheaf of linear spaces $\mathcal{C}^{-\infty}(U)$ over $\mathbb{R}^n$.
(2) The second corresponds to the sheaf of linear spaces

$$\mathcal{C}^{-\infty}(U)/\mathcal{C}^{\infty}(U)$$

over $\mathbb{R}^n$.
(3) The new notion corresponds to a sheaf over $\mathbb{R}^n \times \mathbb{S}^{n-1}$ where the linear space over an open set $V \subset \mathbb{R}^n \times \mathbb{S}^{n-1}$ is the quotient

$$\mathcal{C}^{-\infty}(U)/\{v \in \mathcal{C}^{-\infty}(U) \text{ s.t. } \mathrm{WF}(v) \cap V = \emptyset\},\ U = \pi_1(V) \tag{12}$$

being projection onto the first factor.
Hint-Questions
(a) If $U_1,\ U_2 \subset \mathbb{R}^n$ are open and $u \in \mathcal{C}^{\infty}(U_1 \cap U_2)$ do there exists functions $u_i \in \mathcal{C}^{\infty}(U_i)$ such that $u_1 - u_2 = u$? It is enough to do this for the constant function 1 on $U_1 \cap U_2$.
(b) If $U_1,\ U_2 \subset \mathbb{R}^n$ are open, $V_i \subset U_i \times \mathbb{S}^{n-1}$ open and $u \in \mathcal{C}^{-\infty}(U_1 \cap U_2)$ is such that $\mathrm{WF}(u) \cap (V_1 \cap V_2) = \emptyset$ do there exists $u_i \in \mathcal{C}^{-\infty}(U_i)$ with $\mathrm{WF}(u_i) \cap V_i = \emptyset$ and $u_1 - u_2 = u$? You could try first writing u as the difference of two distributions on $U_1 \cap U_2$ where one has WF not meeting V_1 and the other has WF not meeting V_2.

Q8.6-Opt.

Show the 'microellipticity of elliptic operators': If

$$P(x, D) = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha$$

has coefficients $p_\alpha \in \mathcal{C}^{\infty}(\Omega)$ and is elliptic in Ω then

$$\mathrm{WF}(P(x, D)u) = \mathrm{WF}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\Omega). \tag{13}$$

Pseudodifferential operators are also microlocal! You can use the properties of these operators in the notes to show this and deduce *microlocal* *elliptic regularity* of differential operators:

$$\text{if } P(x,D_x) = \sum_{|\alpha|\le m} p_\alpha(x) D_x^\alpha,\ p_\alpha \in \mathcal{C}^\infty(U),$$
$$p_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha, \quad \text{then} \quad \forall\ u \in \mathcal{C}^{-\infty}(U)$$
$$p_m(\bar{x},\bar{\xi}) \neq 0 \Longrightarrow (\bar{x},\bar{\xi}) \in \mathrm{WF}(P(x,D)u) \text{ iff } (\bar{x},\bar{\xi}) \in \mathrm{WF}(u). \tag{14}$$

Q8.7-opt.

Show that if $u,\ v \in \mathcal{C}^{-\infty}(\Omega)$ and there is no point $(x,\xi) \in \mathrm{WF}(u)$ such that $(x,-\xi) \in \mathrm{WF}(v)$ then it is possible to define the product $uv \in \mathcal{C}^{-\infty}(\Omega)$ consistently with multiplication when one element is smooth.

Hint: First think about the corresponding result for singular supports, which is just that $\mathrm{singsupp}(u) \cap \mathrm{singsupp}(v)$ allows you to define uv and try to do something similar.

Proof of the statement in Q8.3(2)

In response to a request:-

We are given a symbol $a \in S^m(\mathbb{R}^n)$ and we consider the convolution operator $b*$ where $\hat{b} = a$. We know that we can write $b = b_0 + b_1$ with $b_1 \in \mathcal{S}(\mathbb{R}^n)$ and with b_0 having compact support – so $\hat{b}_0 - a \in \mathcal{S}(\mathbb{R}^n)$. Now we are interested in the composite operator

$$u \longmapsto b * (\phi u)$$

where $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$. We are allowing ourselves to make 'errors' which have very smooth Schwartz kernels, in particular the map given by replacing b by b_1 has kernel $b_1(x-y)\phi(y) \in \mathcal{S}(\mathbb{R}^{2n})$, which is indeed very smoothing. This means we can replace b by b_0, which is to say we can assume it has compact support in the first place. The kernel of this operator is

$$b(x-y)\phi(y)$$

However, we can multiply on the left by a cut-off $\chi(x)$ and on the right by $\chi(y)$ where $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to one on a large enough set and it will not change anything, so the kernel is also

$$\chi(x)b(x-y)\phi(y)\chi(y).$$

Now, think of $\phi(y)$ as a smooth function on $\mathbb{R}^{2n}_{x,y}$ and look at it Taylor series around the diagonal:

$$\phi(y) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}(y-x)^\alpha \partial^\alpha \phi(x) + \sum_{|\alpha|=N+1} (y-x)^\alpha e_\alpha(x,y),\ e_\alpha \in \mathcal{C}^\infty(\mathbb{R}^{2n}).$$

This decomposes the kernel into two sums where the first part is the sum of

$$\chi(x)\partial^\alpha\phi(x)(y-x)^\alpha b(x-y)\frac{1}{\alpha!}\chi(y).$$

Now the we can easily compute the Fourier transform

$$\mathcal{F}((-z)^\alpha b(z))(\xi) = i^{-|\alpha|}\partial_\xi^\alpha a(\xi) \in S^{m-|\alpha|}(\mathbb{R}^n). \tag{15}$$

Thus each of these terms is of the form

$$\chi(x)\partial^\alpha\phi(x)b_\alpha(x-y)\frac{1}{\alpha!}\chi(y),\ \hat{b}_\alpha \in S^{m=|\alpha|}. \tag{16}$$

A similar argument applies to the 'remainder terms' which are a sum over $|\alpha| = N+1$. Namely, the kernel is similar but with a more complicated coefficient

$$\chi(x)e_\alpha(x,y)\phi(x)b_\alpha(x-y)\frac{1}{\alpha!}\chi(y)$$

with the same b_α's (for bigger $|\alpha|$) so they each have Fourier transform in $S^{m-N-1}(\mathbb{R}^n)$. We know by Sobolev embedding that for large N

$$b_\alpha(z) \in \mathcal{C}^k(\mathbb{R}^n),\ k < N+1-m-n/2.$$

This means that each of these remainder terms does indeed have very smooth Schwartz kernel if N is large enough and they all have compact support as well.

We are pretty much where we want to be as far as the kernels in (16) are concerned, except for the niggling factor of $\chi(y)$ still on the right! However we chose things so that $\chi = 1$ in a neighbourhood of $\operatorname{supp}\phi$ – which means the factor of χ on the *left* is actually irrelevant. Now, recall the fact that $b_\alpha(z)$ is singular only at $z = 0$ means that if we look at each of the kernels

$$\chi(x)\partial^\alpha\phi(x)b_\alpha(x-y)\frac{1}{\alpha!}(1-\chi(y)) \in \mathcal{S}(\mathbb{R}^{2n}) \tag{17}$$

they $\operatorname{supp}(\phi(x)) \cap \operatorname{supp}(1-\chi)(y)$ does not meet the diagonal so we can remove the singularity of b_α without changing anything.

The upshot is that for $N \geq k+m+n/2$ the kernel of the difference

$$u \longmapsto b*(\phi u) - \sum_{|\alpha|\leq N}\frac{1}{\alpha!}\partial^\alpha\phi(b_\alpha * u)$$

is in $\mathcal{C}^k(\mathbb{R}^{2n})$ and is equal to an element of $\mathcal{S}(\mathbb{R}^{2n})$ outside a compact set. This means it maps $H^{-k}(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$.