PROBLEM SET 5, 18.155
DUE 27 OCTOBER, 2017

This is partly to check that you know a bit about Hilbert spaces.

Q5.1

Show that a Hilbert space which has a compact unit ball is finite dimensional.

Q5.2

Suppose $u \in L^1(\mathbb{R}^n)$ vanishes outside some ball. Using the uniform convergence of the power series for the exponential function show that

$$\begin{gathered}\hat{u}(\zeta) = \int e^{-ix\cdot\zeta} u(x)dx \text{ is defined for all } \zeta \in \mathbb{C}^n \text{ and} \\ \hat{u}(\zeta) = \sum_\alpha c_\alpha(-i\zeta)^\alpha,\ c_\alpha = \frac{1}{\alpha!}\int x^\alpha u(x)\end{gathered} \tag{1}$$

converges uniformly on any compact set.

Q5.3

Show that a subset of $L^2(\mathbb{R}^n)$ is compact if and only if it is closed, bounded, equi-small at infinity and with Fourier transform equi-small at infinity. That is the conditions (which *do* hold for any one element $u \in L^2(\mathbb{R}^n)$)

$$\begin{gathered}\text{Given } \epsilon > 0\ \exists\ R \text{ s.t. } \int_{|x|>R} |u|^2 < \epsilon^2 \\ \text{Given } \epsilon > 0\ \exists\ R \text{ s.t. } \int_{|\xi|>R} |\hat{u}|^2 < \epsilon^2\end{gathered} \tag{2}$$

hold uniformly – for each $\epsilon > 0$ there is one R which gives the estimate simultaneously for all u in the set.

Hint. There are actually several options, here is one, maybe not the simplest.

Then show necessity: That these conditions hold for one element follows from LDC. If u satisfies these conditions for $\epsilon > 0$ and R then all elements in a small ball around u satisfy them for this R and 2ϵ. Use this to show that the image (all the functions forming the elements of) a convergent sequence satisfy the uniformity conditions – use a small ball around the limit and the fact that there are only finitely many points outside this. Deduce that if the uniformity conditions do not hold then there is sequence without a convergent subsequence in the set which therefore cannot be compact.

To show sufficiency first check that a closed subset of a metric space is compact iff there is a compact which is 'ϵ-close' to it for each $\epsilon > 0$. Find compact sets close by in the finite-dimensional subspace to be polynomials of degree at most N cut off on a big ball (or the inverse Fourier transform of this). Use the first condition to throw away what is outside a big ball, then the preceding problem to show equi-convergence on a big ball for the Fourier transforms; together with throwing away what is outside a big ball this gives a close approximation.

Q5.4

Show that for any $s > 0$ the unit ball (with respect to the H^s norm of course) in the space

$$\{u \in H^s(\mathbb{R}^n); u = 0 \text{ in } |x| > R\} \tag{3}$$

is pre-compact (has compact closure) in $L^2(\mathbb{R}^n)$.

Q5.5

Let $P(D)$ be an elliptic operator of order m. Show that the map, for any fixed $R > 0$,

$$P(D) : \{u \in H^m(\mathbb{R}^n); u = 0 \text{ in } |x| > R\} \longrightarrow L^2(\mathbb{R}^n) \tag{4}$$

is injective with closed range.

Hint: The injectivity we know – there can be no compactly supported distributions satisfying $P(D)u = 0$. For the closedness of the range use the parameterix $b*$ we constructed. Everything is compactly supported so we see that if $P(D)u = f$ then $b * f = u + \theta * u$ where $\theta \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$. So, suppose f_n is a convergent sequence in the range, $f_n = P(D)u_n$ so $\operatorname{supp}(f_n)$ is contained in the ball or radius R. There are two possibilities, $\|u_n\|_{L^2}$ is bounded or not. In the latter case we can pass to a subsequence where $\|u_n\|_{L^2} \to \infty$ and consider $v_n = u_n/\|u_n\|$ which satisfies $v_n = f_n/\|u_n\| - \theta * v_n$. Now the last term here is bounded in H^m and has support in a fixed ball, so has a convergent subsequence in L^2 by the compactness results above. It follows that v_n has a convergent subsequence in H^m but the limit satisfies $P(D)v = 0$ which means $v = 0$ but since $\|v_n\|_{L^2} = 1$ this is not possible. Sooo, the sequence is bounded in L^2. Repeat the argument without dividing and you will find $u_n \to u$ in H^m.

Q5.6-Optional

Use the last result to show that if $P(D)$ is elliptic and $f \in L^2$ then for any R there exists $u \in \mathcal{C}^{-\infty}(B_R)$, $B_R = \{x; |x| < R\}$ satisfying $P(D)u = f$ in B_R.

Q5.7-Optional

Suppose that $a : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}$ is smooth and (positively) homogeneous of degree $z \in \mathbb{C}$ with $\operatorname{Re} z > -n$. Show that there is a unique element $b \in \mathcal{S}'(\mathbb{R}^n)$ which is equal to a in $\mathbb{R}^n \setminus \{0\}$ and in $H^s_{\text{loc}}(\mathbb{R}^n)$ for some $s > -n/2$.