# PROBLEM SET 4, 18.155
# DUE 13 OCTOBER, 2017

## Q4.1

Recall from L9 that the space of symbols of order $m \in \mathbb{R}$ is

(1)
$$S^m(\mathbb{R}^n) = \left\{ a \in \mathcal{C}^\infty(\mathbb{R}^n); \|a\|_{m,k} = \sup_{\xi \in \mathbb{R}^n,\ |\alpha| \le k} \langle \xi \rangle^{-m+|\alpha|} |\partial_\xi^\alpha a(\xi)| < \infty \right\}.$$

Show that

(1) $S^m(\mathbb{R}^n) \subset S^{m'}(\mathbb{R}^n)$, $m' \ge m$.
(2) $\langle \xi \rangle^s \in S^s(\mathbb{R}^n)$.
(3) The $S^*$ form a filtered algebra under multiplication

$$a \in S^m(\mathbb{R}^n),\ a' \in S^{m'}(\mathbb{R}^n) \Longrightarrow aa' \in S^{m+m'}(\mathbb{R}^n).$$

## Q4.2

Suppose $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$ in $|\xi| < 1$. Show that if $a \in S^m(\mathbb{R}^n)$ then $a_n(\xi) = a(\xi)\chi(\xi/n)$ is bounded with respect to each of the semi-norms in (1) and that $a_n \to a$ with respect to each of the norms $\|\cdot\|_{m',k}$ if $m' > m$.

## Q4.3

Using the density shown in Q4.2, or otherwise, show that the operators (discussed in L10 but you should be able to see this before then)

$$(\beta_a)* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \beta_a = \mathcal{G}a \in \mathcal{S}'(\mathbb{R}^n),\ a \in S^m(\mathbb{R}^n) \tag{2}$$

form a graded algebra. That is, if $a \in S^m(\mathbb{R}^n)$, $b \in S^{m'}(\mathbb{R}^n)$ then there exists $c \in S^{m+m'}(\mathbb{R}^n)$ such that

$$(\beta_b) * (\beta_a * \phi) = \beta_c * \phi \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{3}$$

[This is the algebra of pseudodifferential operators with constant coefficients on $\mathbb{R}^n$. In fact $c = ab$. In class I will show that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{\phi * \psi} = \widehat{\phi}\widehat{\psi}.$$

This still works for $u * \phi$ where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ but you would need to prove it – you are interested in this with $u = \beta_a$. You can do that here by using the approximation result above.]

Q4.4

Assuming that $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ have disjoint supports and that $E \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ has $\operatorname{singsupp}(E) \subset \{0\}$ show that

$$\mathcal{C}^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \phi(E * (\psi u)) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n).$$

Q4.5

Explain the precise meaning of the formula

$$\Delta |x|^{-n+2} = c_n \delta_0,\ n > 2, \tag{4}$$

and compute the constant.

Q4.6 (Optional)

Given any (relatively of course) closed subset of an open set $\Omega \subset \mathbb{R}^n$ show that there is a distribution $u \in \mathcal{C}^{-\infty}(\Omega)$ with this as singular support. The usual argument is called 'condensation of singularities' if you want to look it up.

Hint: I described this very quickly in Lecture 9.

Q4.7 (Optional)

A differential operator $P(D)$ with constant coefficients is said to be *hypoelliptic* if for every (equivalently any one non-empty) open set $\Omega$

$$\operatorname{singsupp}(P(D)u) = \operatorname{singsupp}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\Omega).$$

Show that this condition is equivalent to the existence of a function of slow growith $v$ such that $P(\xi)v(\xi) = 1 + e(\xi)$, $e \in \mathcal{S}(\mathbb{R}^n)$. [It is actually equivalent to the existence of any smooth function with this property but that involves more work.]