PROBLEM SET 3, 18.155
DUE EARLY SEPTEMBER 30, 2017

In Lecture 6 I will talk about distributions on open sets and supports. I want you to go through the basics of distribution theory 'yet again' to find the distributions of compact support another way. As usual the questions starting at Q6 are mostly for you amusement.

Q3.1 (L6)

Suppose $\Omega \subset \mathbb{R}^n$ is open, consider

$$\mathcal{C}^\infty(\Omega) = \{\phi : \Omega \longrightarrow \mathbb{C}; \phi \text{ infinitely differentiable}\}. \tag{1}$$

Take an exhaustion of Ω by an increasing sequence K_j of compact subsets (meaning every compact subset of Ω is contained in one of the K_j and each K_j is contained in the interior of K_{j+1}), and show that $\mathcal{C}^\infty(\Omega)$ is complete as a metric space where

$$\begin{gathered} d(\phi, \psi) = \sum_k 2^{-k} \frac{\|\phi - \psi\|_{(k)}}{1 + \|\phi - \psi\|_{(k)}}, \\ \|\phi\|_{(k)} = \sup_{|\alpha| \le k,\ x \in K_k} |D^\alpha \phi(x)|. \end{gathered} \tag{2}$$

[Note that these are only seminorms, so make sure you understand why the vanishing of the distance between two points implies that they are equal.]

Q3.2

Conclude that an element of the dual space, denoted $\mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega)$, (this is an apparently weird notation for the continuous linear maps $u : \mathcal{C}^\infty(\Omega) \longrightarrow \mathbb{C}$ but there is a method here) is precisely a linear map such that for some C and k

$$|u(\phi)| \le C\|\phi\|_{(k)}\ \forall\ \phi \in \mathcal{C}^\infty(\Omega). \tag{3}$$

Q3.3 (L6)

Observe that for every $\Omega \subset \mathbb{R}^n$ open there is a restriction map $\big|_\Omega : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\Omega)$ and use this to show that

$$u'(f) = u(f\big|_\Omega)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n) \text{ defines } u' \in \mathcal{S}'(\mathbb{R}^n). \tag{4}$$

and gives an injection identifying $\mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ with the tempered distributions with compact support contained in Ω as defined in L6.

Q3.4 (L5)

Recall the Sobolev embedding theorem and show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there exist k and N such that

$$|((1+|x|^2)^{-k}u)(\phi)| \le C_N \|\phi\|_{H^{2N}} \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Q3.5 (L4)

Conclude that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there exist $f \in L^2(\mathbb{R}^n)$, and k, $N \in \mathbb{N}$ such that

$$u = (1+|x|^2)^k(1+\varDelta)^N f, \ \varDelta = -\sum_{i=1}^{n} \partial_i^2.$$

Q3.6 (Optional)

The formula above has a certain lack of symmetry between 'real space' (multiplication by functions) and 'dual space' (differentiation). First show that there is a similar expression with the order reversed. Suppose you knew (as is indeed the case) that the harmonic oscillator $\varDelta + |x|^2$ is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$ and hence on $\mathcal{S}'(\mathbb{R}^n)$ which has the property

(5)
$$(\varDelta+|x|^2)^{-1}[(1+|x|^2)^{k/2}H^{-k}(\mathbb{R}^n)] \subset (1+|x|^2)^{(k-1)/2}H^{1-k}(\mathbb{R}^n) \ \forall \ k \in \mathbb{N}.$$

Show that for each $u \in \mathcal{S}'(\mathbb{R}^n)$ there exists $N \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^n)$ such that

(6)
$$u = (\varDelta + |x|^2)^N f.$$

Q3.7 (Optional)

From the results above, observe that $\mathcal{S}'(\mathbb{R}^n)$ is a union of Hilbert subspaces. Give it the inductive limit topology in which a set is open if it intersects each of these subspaces in an open set. Show that a linear functional on $\mathcal{S}'(\mathbb{R}^n)$ which is continuous in this topology is given by pairing with an element of $\mathcal{S}(\mathbb{R}^n)$.