PROBLEM SET 2, 18.155
DUE SEPTEMBER 22, 2017

These questions assume some of the material from Lecture 4, Tuesday Sept 19. Here is enough to get you going if you want to work on them before then. You may assume shown that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and that in turn the map $L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ (sending u to $U_u(\psi) = \int u\psi$) is an injection. We interpret this as meaning that '$L^2(\mathbb{R}^n)$ is a well-defined subspace of $\mathcal{S}'(\mathbb{R}^n)$' (consisting precisely of the elements in the image of this injection).

By analogy with what we did for the Fourier transform in L3, define the action of ∂_j and x_i on $\mathcal{S}'(\mathbb{R}^n)$ by

(1)
$$(\partial_j U)(\psi) = U(-\partial_j \psi),\ x_j U(\psi) = U(x_j\psi),\ U \in \mathcal{S}'(\mathbb{R}^n),\ \psi \in \mathcal{S}(\mathbb{R}^n)$$

on the grounds that these extend the formulæ that hold for $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned} U_{\partial_j\phi}(\psi) &= \int (\partial_j\phi)\psi = -\int \phi\partial_j\psi = U_\phi(-\partial_j\psi) \\ U_{x_j\phi}(\psi) &= \int (x_j\phi)\psi = \int \phi x_j\psi = U_\phi(x_j\psi). \end{aligned} \tag{2}$$

The sobolev spaces are then defined by the condition

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\},\ s \in \mathbb{R}, \tag{3}$$

and you should make sure you understand precisely why this makes sense.

Question 2.1

Define the positive Laplacian acting on $\mathcal{S}(\mathbb{R}^n)$ by

$$\Delta\phi = \sum_{j=1}^n -\partial_j^2\phi$$

Show that $\Delta + 1$ is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$.

Question 2.2

From the definitions above, Δ is defined as a map from $\mathcal{S}'(\mathbb{R}^n)$ to itself. Show that

$$\begin{aligned} &(\Delta + 1) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is a bijection} \\ &\Delta : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is not a bijection.} \end{aligned} \tag{4}$$

Hint: The constant function 1 is annihilated by each ∂_j.

Question 2.3

The Dirac delta 'function' $\delta \in \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\delta(\phi) = \phi(0) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n)$$

is amongst the most important distributions (it is a measure).

A) Find explicit formulae for the derivatives $\partial^\alpha \delta$ evaluated on test functions
B) Compute the Fourier transform of $\partial^\alpha \delta$.
C) Show that

$$\partial^\alpha \delta \in H^{-|\alpha|-n/2-\epsilon}(\mathbb{R}^n)$$

for $\epsilon > 0$ but not for $\epsilon = 0$.

Question 2.4

Go through the convolution discussion for L^1. That is, show that if $u \in L^1(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then the convolution integral

$$u * \psi(x) = \int u(y)\psi(x-y)$$

is a well-defined infinitely differentiable function with all derivatives in L^1. Show (you might want to look up and are allowed to use continuity-in-the-mean of L^1 functions) that if $\phi_k(x) = k^n\phi(kx)$ is an approximate identity (as in class) then

$$u * \phi_k \to u \text{ in } L^1(\mathbb{R}^n).$$

Use this to show (trivially, i.e. in one line) that $L^1(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ (meaning an injection) by the usual definition $u \longrightarrow U_u$, $U_u(\phi) = \int u(x)\phi(x)$.

Question 2.5

A function $u \in L^2(\mathbb{R}^n)$ is said to have weak derivatives in L^2 if for each j there exists $v_j \in L^2(\mathbb{R}^n)$ such that

$$\int u\partial_j\psi = -\int v_j\psi \ \forall \ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{5}$$

Show that this is true if and only if $\langle\xi\rangle\hat{u} \in L^2(\mathbb{R}^n)$ i.e. $u \in H^1(\mathbb{R}^n)$.

Question 2.6 (Optional)

Discuss the weak topology on $\mathcal{S}'(\mathbb{R}^n)$. This is the weakest topology such that all the linear maps

$$L_\phi : \mathcal{S}'(\mathbb{R}^n) \ni u \longmapsto u(\phi) \in \mathbb{C}, \ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{6}$$

are continuous. Show that a set is open in this topology if and only if it is a (n arbitrary) union of finite intersections of the (semi-norm) balls

$$B(u,\phi,\epsilon)=\{v\in\mathcal{S}'(\mathbb{R}^n);|(v-u)(\phi)|\le\epsilon\}.$$

Show that this topology is *not* metrizable (you will need some topology for this).