PROBLEM SET 1, 18.155
DUE FRIDAY 15 SEPTEMBER, 2017

The first two of these problems only use the material from Lecture 1; the next two refer to continuity of linear maps from $\mathcal{S}(\mathbb{R}^n)$ to itself which will be in Lecture 2. Only solutions to the first 5 problems need to be submitted. If you do some of the optional questions they will be graded too but they are there mainly for interest.

As a general rule for the problem sets, you may talk to anyone and consult anything you wish concerning these questions but you must absorb and write out the solutions yourself. Anything close to copying is not permitted.

You may send me a message via Stellar, or directly via email, asking for hints and I will try to respond reasonably quickly. Please tell me if you see something that you think is wrong!

Homework can be submitted via Stella (I hope it works). It is not essential that you work with TeX although it is preferable. If you write out the solutions and scan them to pdf please check that they are readable. Solutions are due on Fridays, but if they arrive by early on Saturday morning that should be okay. Late homework will be graded but you will lose marks, how many depends on circumstances.

Q1.1 (L1)

Prove (probably by induction) the multi-variable form of Leibniz' formula for the derivatives of the product of two (sufficiently differentiable) functions:-

$$\partial^\alpha(fg) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} g.$$

[Obviously you need to work out what the combinatorial coefficients are, or define them by induction at least].

Brief solution: Applying ∂_j to the left, assuming sufficient differentiability, gives the same formula for α replaced by $\alpha + e_j$ where

$$\binom{\alpha+e_j}{\beta} = \binom{\alpha}{\beta} + \binom{\alpha}{\beta+e_j} \tag{1}$$

(where the binomial coefficients are taken as zero if the variables are out-of-range). These identities certainly determine the coefficients and have the solution

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!} \tag{2}$$

where $\alpha! = \alpha_1! \times \dots \alpha_n!$.

Q1.2 (L1)

Consider the norms, for each $N \in \mathbb{N}$, on $\mathcal{S}(\mathbb{R}^n)$

$$\|f\|_N = \sum_{|\beta|+|\alpha|\le N} \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha f(x)|.$$

Show that

$$\|f\|'_N = \sum_{|\beta|+|\alpha|\le N} \sup_{x\in\mathbb{R}^n} |\partial^\alpha (x^\beta f(x))|$$

are equivalent norms, $\|f\|_N \le C_N \|f\|'_N$ and $\|f\|'_N \le C'_N \|f\|_N$ for some constants C_N and C'_N.

Brief solution: Use the product formula to see that

$$\partial^\alpha(x^\beta f(x)) = x^\beta \partial^\alpha f(x) + \sum_{\alpha'<\alpha,\beta'<\beta} c_{\alpha',\beta'} x^{\beta'} \partial^{\alpha'} f. \tag{3}$$

Proceed by induction over N, assuming that $\|f\|_M \le C_M \|f\|'_M$ and $\|f\|'_M \le C'_M \|f\|_M$ for all $M < N$. Then (3) shows that if $|\alpha| + |\beta| = N$ then

$$\sup|\partial^\alpha(x^\beta f(x))| \le \sup|x^\beta \partial^\alpha f(x)| + C\|f\|_{N-1} \text{ and}$$
$$\sup|x^\beta \partial^\alpha f(x)| \le \sup|\partial^\alpha(x^\beta f(x))| + C\|f\|'_{N-1}$$

This gives the inductive step.

Q1.3 (L2)

Consider $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ which is an infinitely differentiable function of polynomial growth, in the sense that for each α there exists $N(\alpha) \in \mathbb{N}$ and $C(\alpha) > 0$ such that

$$|\partial^\alpha F(x)| \le C(\alpha)(1+|x|)^{N(\alpha)}.$$

(1) Show that multiplication by F gives a continuous linear map $\times F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$.
(2) Generalize this to show that a linear differential operator with coefficients which are infinitely differentiable and of slow growth:

$$Pu(x) = \sum_{|\alpha|\le m} F_\alpha(x) \partial_x^\alpha u(x) \tag{4}$$

defines a continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ to itself.

Brief solution: Again apply the product formula for derivatives above to see that

$$\sup |x^\beta \partial^\alpha (F\phi)| \le \sum_{\gamma \le \alpha} C(\gamma) \|\phi\|_{N+M}, \ M = \max_{\gamma \le \alpha} C(\gamma). \tag{5}$$

This proves that $\times F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ and gives the desired continuity estimates

$$\|F\phi\|_N \le C\|\phi\|_{N+M}$$

where M is large enough, depending on N.

The second part follows from the fact that the composite of continuous (linear) maps is continuous as is the sum.

Q1.4 (L2)

Show that if $s \in \mathbb{R}$ then $F_s(x) = (1 + |x|^2)^{s/2}$ is a smooth function of polynomial growth in the sense discussed above and that multiplication by F_s is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$.

Q1.5

Brief solution: That $\langle x\rangle^s$ is of slow growth follows from a suitable formula for the derivatives such as

$$\partial^\alpha \langle x\rangle^s = p_\alpha(x)\langle x\rangle^{s-2|\alpha|} \tag{6}$$

where p_α is a polynomial of degree at most $|\alpha|$. This in turn follows from the fact that

$$\partial_j \langle x\rangle^2 = 2x_j \langle \partial_j \langle x\rangle^s = s x_j \langle x\rangle^{s-2}. \tag{7}$$

Thus in fact

$$|\partial^\alpha \langle x\rangle^s| \le C_s \langle x\rangle^{s-|\alpha|}, \ \forall \ \alpha \tag{8}$$

which is something we will explore further later ('symbol estimates'). The inverse is given by multiplication by $\langle x\rangle^{-s}$ so both are continuous by the preceding problem.

(1) Consider one-point, or stereographic, compactification of $\mathbb{R}^n$. This is the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ obtained by sending x first to the point $z = (1, x)$ in the hyperplane $z_0 = 1$ where $(z_0, \dots, z_n)$ are the coordinates in $\mathbb{R}^{n+1}$ and then mapping it to the point $Z \in \mathbb{R}^{n+1}$ with $|Z| = 1$ which is also on the line from the 'South Pole' $(-1, 0)$ to $(1, x)$.

(2) Derive a formula for T and use it to find a formula for the inversion map $I : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ which satisfies $I(x) = x'$ if $T(x) = (z_0, z)$ and $T(x') = (-z_0, z)$. That is, it correspond to reflection across the equator in the unit sphere.

(3) Show that if $f \in \mathcal{S}(\mathbb{R}^n)$ then $I^*f(x) = f(x')$, defined for $x \neq 0$, extends by continuity with all its derivatives across the origin where they all vanish.
(4) Conversely show that if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a function with all derivatives continuous, then $f \in \mathcal{S}(\mathbb{R}^n)$ if I^*f has this property, that all derivatives extend continuously across the origin and vanish there – i.e. they all have limit zero at the origin.

Brief Solution: Draw a picture! The points on the straight line between $(1, x)$ and $(-1, 0)$ are of the form

$$t(1, x) + (1 - t)(-1, 0) = (2t - 1, tx) \tag{9}$$

which hits the unit sphere when

$$(2t - 1)^2 + t^2|x|^2 = 1. \tag{10}$$

This has solutions $t = 0$ and $(4 + |x|^2)t = 4$ so the solution we want is

$$Tx = \left(\frac{4 - |x|^2}{4 + |x|^2}, \frac{4x}{4 + |x|^2}\right) \tag{11}$$

which is on the plane $z_0 = 0$ if $|x|^2 = 4$. So we can expect the inversion map, determined by $Ix = x'$, to fix this spehere (as a check ...). It must satisy

$$\left(\frac{4 - |x'|^2}{4 + |x'|^2}, \frac{4x'}{4 + |x'|^2}\right) = Tx' = \left(\frac{|x|^2 - 4}{4 + |x|^2}, \frac{4x}{4 + |x|^2}\right) \tag{12}$$

These equations imply that $8(4 + |x'|^2)^{-1} = 2|x|^2(4 + |x|^2)^{-1}$ so

$$|x'|^2 = 4(4 + |x|^2)|x|^{-2} - 4 = 16|x|^{-2}.$$

Thus

$$Ix = x' = x\frac{4 + |x'|^2}{4 + |x|^2} = \frac{4x}{|x|^2}. \tag{13}$$

Now if $f \in \mathcal{S}(\mathbb{R}^n)$ then $g(x) = I^*f(x) = f(Ix) = f(\frac{4x}{|x|^2})$ is defined for $|x| > 0$. We wish to show that it extends smoothly across the origin, where it vanishes with all its derivatives. Certainly by the chain rule it is infinitely differentiable in $x \neq 0$ and we can see by induction that

$$\partial_x^\alpha g(x) = \sum_{\gamma \le \alpha} \frac{p_{\alpha,\gamma}(x)}{|x|^{2|\alpha|}}(\partial^\gamma g)(\frac{4x}{|x|^2}) \tag{14}$$

with the $p_{\alpha,\gamma}$ polynomials. Indeed, this is true for $\alpha = 0$ and the inductive step, over $|\alpha|$, follows by applying each of the ∂_j. From (14) and the rapid decay of the derivatives of f it follows that for any M, taking $N >> M + |\alpha|$,

$$|x|^{-M}|\partial^\alpha g(x)| \le C_{\alpha,N}|x|^{-2|\alpha|-M}(1 + |\frac{4x}{|x|^2}|)^{-N} \to 0 \text{ as } x \to 0. \tag{15}$$

Thus g and all its derivatives extend continuously across 0 where they all vanish rapidley. The difference quotients at 0 for g all tend to 0 so these conditions imply that g *is* differentiable with derivative equal to the continuous extension from $x \neq 0$. It follows by induction that g is infinitely differentiable near zero with all derivatives vanishing there.

It remains to prove the converse, that if f is infinitely differentiable on $\mathbb{R}^n$ and $g = I^*f$ is infinitely differentiable across 0 with all derivatives vanising at 0 then $f \in \mathcal{S}(\mathbb{R}^n)$. This is a matter of showing that the derivatives of f all vanish rapidly at infinity. By definition, or inspection, $I^2 = \mathrm{Id}$ so we can use the same formula (14) with f and g switched to see that

$$x^\beta \partial_x^\alpha g(x) = \sum_{\gamma \le \alpha} \frac{x^\beta p_{\alpha,\gamma}(x)}{|x|^{2|\alpha|}}(\partial^\gamma f)(\frac{4x}{|x|^2}) \text{ is bounded.} \tag{16}$$

Q1.6 (Optional)

Prove that $\mathcal{S}(\mathbb{R}^n)$ is a *Montel space* which means that it has an analogue of the Heine-Borel property. Namely, (you have to show that) if $D \subset \mathcal{S}(\mathbb{R}^n)$ is closed and 'bounded' in the sense that for each N there exists C_N such that $\|\phi\|_N \le C_N$ for all $\phi \in D$, then D is compact.

Q1.7 (Optional)

A) Show that if $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is measurable and

$$(1 + |x|)^{-N} u \in L^1(\mathbb{R}^n) \tag{17}$$

for some N then $I(u)(\phi) = \int u\phi$, for $\phi \in \mathcal{S}(\mathbb{R}^n)$ defines an element $I(u) \in \mathcal{S}'(\mathbb{R}^n)$.

B) Now, refute the idea that these are the 'most general' functions which define distributions – this is a dangerously vague statement anyway and I'm sure you would not say such a thing. NAMELY observe that

$$u(x) = \exp(i \exp(x))$$

defines an element of $\mathcal{S}'(\mathbb{R})$ and hence conclude that, in a sense you should make clear, so does

$$\exp(x) \exp(i \exp(x)) \tag{18}$$

but that this does NOT satisfy (4) above.