# 18.155 LECTURE 9
# 5 OCTOBER, 2017

RICHARD MELROSE

Abstract. Notes before and after lecture – if you have questions, ask!

## Singular support and symbols

- Singular support:- Since $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$ is a well-defined subset the definition

$$u \in \mathcal{C}^\infty(U) \text{ is smooth on } \Omega \subset U \text{ both open } \text{ if } u\big|_\Omega \in \mathcal{C}^\infty(\omega) \tag{1}$$

makes sense.

**Lemma 1.** *If* $u \in \mathcal{C}^{-\infty}(\Omega)$ *and*

$$\mathcal{O} = \bigcup\{U \subset \Omega; U \text{ is open and } u\big|_U \in \mathcal{C}^\infty(U)\} \tag{2}$$

*then* $u\big|_{\mathcal{O}} \in \mathcal{C}^\infty(\mathcal{O})$.

Thus there is a largest open set to which $u \in \mathcal{C}^{-\infty}(\Omega)$ restricts to be smooth and we may unambiguously define the (relatively) closed set

$$\operatorname{singsupp}(u) = \Omega \setminus \mathcal{O}. \tag{3}$$

Obviously $\operatorname{singsupp}(u) \subset \operatorname{supp}(u)$.

*Proof.* By assumption there is an open covering $U_a$ of $\mathcal{O}$ such that for each $a$, $u\big|_{U_a} = v_a \in \mathcal{C}^\infty(U_a)$. The fact that $\mathcal{C}^\infty(U) \subset \mathcal{C}^{-\infty}(U)$ is well-defined and the (obvious) pre-sheaf property of the $\mathcal{C}^{-\infty}(U)$ means that for any $a$, $b$

$$v_a\big|_{U_a \cap U_b} = v_b\big|_{U_a \cap U_b} = u\big|_{U_a \cap U_b} \tag{4}$$

so by the (again obvious) sheaf property ('locality') of the $\mathcal{C}^\infty(U)$s there exists one function $v \in \mathcal{C}^\infty(\mathcal{O})$ such that $v\big|_{U_a} = v_a$ for all $a$. Now, we just have to show that $u\big|_{\mathcal{O}} = v$. This just means that $u(\psi) = v(\psi) = \int v\psi$ for all $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{O})$. Since $\operatorname{supp}(\psi)$ is compact it has a finite cover by the $U_{a_i} = U_i$ and we know that we can then decompose using a partition of unity to get $\psi = \sum_i \psi_i$, $\psi_i \in \mathcal{C}^\infty_{\mathrm{c}}(U_i)$. So

$$u(\psi) = \sum_i u(\psi_i) = \sum_i v(\psi_i) = v(\psi). \tag{5}$$

$\square$

Check some of the basic properties of singular support, in particular that if $u \in \mathcal{C}^{-\infty}(\Omega)$ and $\psi \in \mathcal{C}^\infty(\Omega)$ then

$$\operatorname{singsupp}(\psi u) \subset \operatorname{singsupp}(u). \tag{6}$$

What is more important for us is that

**Proposition 1.** *If* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *and* $v \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ *then*

$$\operatorname{singsupp}(u * v) \subset \operatorname{singsupp}(u) + \operatorname{singsupp}(v). \tag{7}$$

This follows from the smoothness of the convolution if either factor is smooth and the same inclusion, (7), for supporst.

Exercise: If you are so inclinded you might like to check that the quotient spaces $\mathcal{C}^{-\infty}(U)/CI(U)$ for open sets $U$ form a sheaf over $\mathbb{R}^n$. This is called the sheaf of *microfunctions*. Note that this is *not* a general fact, the quotient of sheaf by a subsheaf is always a presheaf but not in general a sheaf; here no 'sheafification' is required.

- Symbols:- Ellipticity of $P(D)$. Last time I showed that ellipticity of $P(D)$ is equivalent to the fact that there is a smooth function of compact support $\psi$ such that

$$a(\xi) = \frac{1-\psi(\xi)}{P(\xi)} \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } |a(\xi)| \le C\langle\xi\rangle^{-m}. \tag{8}$$

In fact $a$ has the special properties of a *symbol of order* $-m$.

- A symbol of order $s$ (for any real $s$) is a function $a \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfying the estimates

$$|\partial^\alpha a(\xi)| \le C_\alpha \langle\xi\rangle^{s-|\alpha|} \Longleftrightarrow \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-s+|\alpha|}|\partial^\alpha a(\xi)| \le \infty \ \forall \ \alpha \in \mathbb{N}_0^n. \tag{9}$$

We write $S^s(\mathbb{R}^n)$ for the linear space of such symbols. It is a Fréchet space with topology given by the seminorms implicit in (9). That, for an elliptic $P(\xi)$, $a \in S^{-m}(\mathbb{R}^n)$ follows by differentiating (8) – by induction we find

$$\partial^\alpha a(\xi) = \frac{Q_\alpha}{P(\xi)^{1+|\alpha|}} \tag{10}$$

where $Q_\alpha$ is a polynomial of degree $(m-1)|\alpha|$ – proof as usual by differentiating again. This gives (9).

- Symbols and the Fourier transform:- The (inverse) Fourier transform of a symbol is a distribution $\mathcal{G}a \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\operatorname{singsupp}(\mathcal{G}a) \subset \{0\},\ \mathcal{G}a \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n),\ P(D)\mathcal{G}a = \delta + R,\ R \in \mathcal{S}(\mathbb{R}^n). \tag{11}$$

- Parametrices for constant coefficient elliptic operators.
- Local Sobolev spaces
- Local elliptic regularity.

Department of Mathematics, Massachusetts Institute of Technology
*E-mail address*: rbm@math.mit.edu