18.155 LECTURE 7
28 SEPTEMBER, 2017

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Read:

(1) Today I will finish the proof of the characterization of distributions with support in 0 and start talking about convolution of distributions:

(2) Supports

$$
\begin{tikzcd}
\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) \arrow[r, hook] & \mathcal{S}'(\mathbb{R}^n) \arrow[r, hook] & \mathcal{C}^{-\infty}(\mathbb{R}^n) \\
\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \arrow[r, hook] \arrow[u, hook] & \mathcal{S}(\mathbb{R}^n) \arrow[r, hook] \arrow[u, hook] & \mathcal{C}^{\infty}(\mathbb{R}^n) \arrow[u, hook]
\end{tikzcd}
\tag{1}
$$

(3) Convolution. We extend from the smooth case to the case with one distributional factor and then to both factors being distributions (with one always compactly supported) as follows:-

$$
\begin{gathered}
\phi * \psi(x) = \int \phi(y)\psi(x-y)dy \text{ both smooth} \Longrightarrow \\
\int (\phi*\psi)(x)\mu(x)dx = \int \phi(y)\psi(x-y)\mu(x)dydx = \int \phi(y)(\check{\psi}*\mu)(y),\ \check{\psi}(z) = \psi(-z). \\
u * \psi(x) = u(\phi(x-\cdot) \text{ one smooth} \\
u * v(\mu) = u(\check{v}*\mu) \text{ both distributions.}
\end{gathered}
\tag{2}
$$

(4) This means we get

$$
\begin{gathered}
\mathcal{C}^{\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \subset \mathcal{C}^{\infty}(\mathbb{R}^n) \\
\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \subset \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \\
\mathcal{C}^{-\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \subset \mathcal{C}^{\infty}(\mathbb{R}^n) \\
\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \subset \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n) \\
\mathcal{C}^{-\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n) \\
\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) \subset \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n).
\end{gathered}
\tag{3}
$$

(5) δ_0 gives the identity $\delta_0 * u = u$.

(6) (Next week) The concept of a fundamental solution $P(D)E = \delta_0$.

(7) For a non-constant polynomial $P(D)$ *cannot* have a fundamental solution of compact support.

(8) Examples: Δ.

(9) Ellipticity

So I went off the rails a little on Tuesday, talking about things I meant to leave until a little later. Here is a brief version of what I talked about.

Suppose $\Omega \subset \mathbb{R}^n$ is open. We have associated two spaces of smooth 'test' functions with Ω, namely

$$\begin{gathered}\mathcal{C}^\infty(\Omega) = \{\phi : \Omega \longrightarrow \mathbb{C} \text{ infinitely differentiable}\} \text{ and} \\ \mathcal{C}^\infty_{\mathrm{c}}(\Omega) = \{\phi \in \mathcal{C}^\infty(\Omega); \operatorname{supp}(\phi) \Subset \Omega\}\end{gathered} \tag{4}$$

So the smaller space consists of the smooth functions on Ω which vanish outside a compact subset of Ω. Because this, if we just extend and element $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ to be zero outside Ω the result is a smooth function. So it is natural to ignore the difference and to regard

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \tag{5}$$

as the subspace with (compact) supports which happen to be in Ω.

In the homework this week you are supposed to examine the dual space to $\mathcal{C}^\infty(\Omega)$ which is a Fréchet space. Let me think about the dual of $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ which is *not* a Fréchet space.

We can decompose $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ according to where the supports lie:-

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) = \bigcup_{K \Subset \Omega} \mathcal{C}^\infty_{\mathrm{c}}(K),\ \mathcal{C}^\infty_{\mathrm{c}}(K) = \{\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n); \operatorname{supp}(\phi) \subset K\}. \tag{6}$$

Each of these spaces, $\mathcal{C}^\infty_{\mathrm{c}}(K)$ (which consists of the smooth functions on $\mathbb{R}^n$ which vanish on $\mathbb{R}^n \setminus K$) is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$ and hence a Fréchet space in its own right. This is clear from the fact that convergence in $\mathcal{S}(\mathbb{R}^n)$ certainly implies convergence of each derivative uniformly on compact sets – so the limit of a convergent sequence in $\mathcal{C}^\infty_{\mathrm{c}}(K)$ vanishes outside K. In fact the same topology is obtained just by using the countably many '$\mathcal{C}^k$ norms', the sum of the supremum norms of the derivatives up to order k for each k.

So we know a suitable topology on $\mathcal{C}^\infty_{\mathrm{c}}(K)$. The union in (6) is topologized as the 'inductive limit'. In concrete terms this just means that

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) \supset \mathcal{O} \text{ is open iff } \mathcal{O} \cap \mathcal{C}^\infty_{\mathrm{c}}(K) \text{ is open in the norm topology for each } K \Subset \Omega. \tag{7}$$

We can replace this by the countably many conditions that $\mathcal{O} \cap \mathcal{C}^\infty_{\mathrm{c}}(K_j)$ be open for an exhaustion K_j of Ω.

Then a (general) distribution on Ω is an element of the dual space

$$\mathcal{C}^{-\infty}(\Omega) = (\mathcal{C}^\infty_{\mathrm{c}}(\Omega))' = \{u : \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \longrightarrow \mathbb{C} \text{ linear and continuous.}\}. \tag{8}$$

The left equality here is defining the notation.

Now from the definition of the topology, continuity means that the inverse image of the open unit ball in $\mathbb{C}$ is open, so

$$\{\phi \in \mathcal{C}^\infty_{\mathrm{c}}(K); |u(\phi)| < 1\} \subset \mathcal{C}^\infty_{\mathrm{c}}(K)$$

must be open. We know that this implies there must be an open ball in $\mathcal{C}^\infty_{\mathrm{c}}(K)$ with respect to one of the $\mathcal{C}^k$-norms on which $|u(\phi)| < 1$. This just means, watching the quantifiers carefully, that

(9) For each $K \Subset \Omega$ there exist $C,\ k$ such that

$$|u(\phi)| \le C\|\phi\|_{\mathcal{C}^k}\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \operatorname{supp}(\phi) \subset K.$$

The converse follows as well. So that is continuity, and hence the definition of $\mathcal{C}^{-\infty}(\Omega)$.

Now, I did this now because we are talking about supports and I wanted to mention that

Proposition 1. *The spaces* $\mathcal{C}^0(\Omega)$, $\mathcal{C}^\infty(\Omega)$, $\mathcal{C}^{-\infty}(\Omega)$, $L^1_{\mathrm{loc}}(\Omega)$, $L^2_{\mathrm{loc}}(\Omega)$ *form sheaves over* $\mathbb{R}^n$.

A *pre-sheaf* of vector spaces over a topological space $\mathcal{T}$ is an 'assigment' (i.e. there is a functor doing the assigning) of a vector space $P(U)$ to each open set $U \subset \mathcal{T}$ together with linear 'restriction maps'

$$R_{U,V} : P(V) \longrightarrow P(U) \text{ whenever } V \supset U \tag{10}$$

satisfying the two conditions

$$R_{U,U} = \mathrm{Id},\ R_{U,V} \circ R_{V,W} = R_{U,W} \text{ if } U \subset V \subset W. \tag{11}$$

Note that as usual this is completely abstract – you do not know what the vector spaces are or how these 'so-called restriction maps' are defined, just that they do exist.

All the spaces liste in the Proposition satisfy this where restriction for functions is the obvious notion and restriction for a distribution from Ω to a smaller open set $\Omega' \subset \Omega$ just corresponds to limiting the domain, the space of test functions on which it acts, from $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ to $\mathcal{C}^\infty_{\mathrm{c}}(\Omega')$. Note that when the function spaces are mapped into $\mathcal{C}^{-\infty}(\Omega)$, via the usual 'weak realization' by pairing with test functions, the restriction maps agree with the distributional definition.

Now, a pre-sheaf is a *sheaf* if it satisfies two more (related) conditions for any pair of open sets U and V. Namely

$$\begin{aligned} &1) \text{ If } u \in P(U \cup V) \text{ and } R_{U,U\cup V}u = 0 = R_{V,U\cup V}u \text{ then } u = 0 \text{ in } P(U \cup V).\\ &2) \text{ If } u \in P(U) \text{ and } v \in P(V) \text{ are such that } R_{U\cap V,U}u = R_{U\cap V,V}v\\ &\qquad \text{then } \exists\ w \in P(U \cup V) \text{ satisfying } u = R_{U,U\cup V}w,\ v = R_{V,U\cup V}w. \end{aligned} \tag{12}$$

The first condition implies that the w in the second condition is unique. The idea is that the elements of the $P(U)$ for a sheaf are 'function-like' objects.

So, I leave it to you to check the details that the function spaces listed above satisfy these conditions – and that something like $L^1(\Omega)$ (without the 'loc') does not [Try to say clearly why it does not].

Now, let's prove that the $\mathcal{C}^{-\infty}(\Omega)$ do form a sheaf. So, suppose u is as in condition 1). Then, take $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(U \cup V)$. The compactness of $\mathrm{supp}(\phi) \Subset U \cup V$ allows the result from last lecture to be applied to show that there exist functions $\chi_U \in \mathcal{C}^\infty_{\mathrm{c}}(U)$ and $\chi_V \in \mathcal{C}^\infty_{\mathrm{c}}(V)$ such that $\chi_U + \chi_V = 1$ on $\mathrm{supp}(\phi)$. This means

$$\phi = (\chi_U + \chi_V)\phi = (\chi_U\phi) + (\chi_V\phi) = \phi_U + \phi_V,\ \phi_U \in \mathcal{C}^\infty_{\mathrm{c}}(U),\ \phi_V \in \mathcal{C}^\infty_{\mathrm{c}}(V). \tag{13}$$

From this we see that

$$u(\phi) = u(\phi_U) + u(\phi_V) = 0 \longrightarrow u = 0 \text{ on } U \cup V. \tag{14}$$

So we have 1).

Now, suppose u and v are as in 2). For $\phi \in \mathcal{C}^\infty(U \cup V)$ we can use (13) to define w by

$$w(\phi) = u(\phi_U) + v(\phi_V) \tag{15}$$

since the right side makes sense. We need to check that w is an element of $\mathcal{C}^{-\infty}(U \cup V)$ which means in particular that the definition does not depend on the choices made in defining ϕ_U and ϕ_V. A different choice comes from different choices of

$\chi'_U \in \mathcal{C}^\infty_{\mathrm{c}}(U)$ and $\chi'_V \in \mathcal{C}^\infty_{\mathrm{c}}(V)$ such that $\chi'_U + \chi'_V = 1$ on $\operatorname{supp}(\phi)$. The difference in the two versions of the right side in (15) is

$$u(\chi_U\phi) + v(\chi_V\phi) - u(\chi'_U\phi) - v(\chi'_V\phi) = u((\chi_U - \chi'_U)\phi) + v((\chi_V - \chi'_V)\phi). \tag{16}$$

Notice however that $\chi'_U + \chi'_V = \chi_U + \chi_V$ so $(\chi'_U - \chi_U) = -(\chi'_V - \chi_V)$. One side has compact support in U and the other has compact support in V so both, being equal, must have compact support in $U \cap V$. Since $u = v$ on $U \cap V$ by hypothesis,

$$u((\chi_U - \chi'_U)\phi) + v((\chi_V - \chi'_V)\phi) = 0. \tag{17}$$

The definition of w does not depend on which χ_U, χ_V we use.

I leave it to you to check from (15) that w is linear and satisfies the continuity condition to imply that $w \in \mathcal{C}^{-\infty}(U \cup V)$; to do so note that the same χ_U and χ_V can be used when $\operatorname{supp}(\phi) \subset K$ for a fixed $K \Subset U \cup V$, and that it restricts to be u and v on U and V.

Okay, so that is what I was talking about on Tuesday.

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