18.155 LECTURE 6
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ABSTRACT. Notes before and after lecture – if you have questions, ask!

Read: I have changed the order of material from the section 'Convolution and density' and some of the proofs but it would still be appropriate to read that!

Separation – if $K \Subset \Omega \subset \mathbb{R}^n$ is a compact subset of an open set then there exists $0 \le \chi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ such that $\chi \equiv 1$ on K.
Partition of unity. If $\Omega_i \subset \mathbb{R}^n$ are a collection of open sets and $K \Subset \bigcup_i \Omega_i$ is compact then there exist finitely non-zero $0 \le \chi_i \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_i)$ such that

$$\sum_i \chi_i = 1 \text{ on } K. \tag{1}$$

Can make the equality on a small neighbourhood of K.
We defined condition that $u \in \mathcal{S}'(\mathbb{R}^n)$ 'vanishes on an open set Ω' (written $u = 0$ on Ω to mean

$$u(\phi) = 0 \ \forall \ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega). \tag{2}$$

This only make sense because:

Lemma 1. *If $u = 0$ on open sets Ω_i then $u = 0$ on $\Omega = \bigcup_i \Omega_i$.*

Proof. if $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ then take a partition of unity satisfying (1) for $K = \operatorname{supp}(\mu)$. Then $\chi_i\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_i)$ so

$$u(\mu) = u(\sum_i \chi_i\mu) = \sum_i u(\chi_i\mu) = 0 \tag{3}$$

where the sums are finite. □

So now we can define for any element $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\operatorname{supp}(u) = \mathbb{R}^n \setminus \bigcup\{\Omega \subset \mathbb{R}^n \text{ open such that } u = 0 \text{ on } \Omega\}. \tag{4}$$

So the support is the complement of the largest open set on which the distribution vanishes.
As part of the homework this week I ask you to give a different characterization (as a dual space) of

$$\mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) = \{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\} \tag{5}$$

(note that the supports are required to be compact).

I should also mention here the notion of *exhaustion* of an open set $\Omega \subset \mathbb{R}^n$ by compact subsets $K_i \Subset \Omega$. We can choose these so that K_j is the closure of its interior, so that K_j is contained in the interior of K_{j+1} for all j and such that for any compact subset $K \Subset \Omega$, $K \subset K_j$ for some j. In particular this implies that

$$\bigcup_j K_j = \Omega.$$

How to get such an exhaustion? Try

$$K_j = \{x \in \Omega; |x| \le j \text{ and } d(x, \Omega^{\complement}) = \inf_{y \notin \Omega} d(x,y) \ge 1/j\} \tag{6}$$

starting at j large enough so that these have interiors. Note that the distance to a closed set in this sense is continuous.

Proposition 1. *A distribution* $u \in \mathcal{S}'(\mathbb{R}^n)$ *with* $\operatorname{supp}(u) \subset \{0\}$ *is a finite sum*

$$u = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha \delta_0 \tag{7}$$

of the Dirac mass at 0, $\delta_0(\phi) = \phi(0)$.

Proof. We know two things about u. From the argument for density of $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ we see that

$$u(\phi) = 0 \text{ if } \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \phi = 0 \text{ in } \{|x| < \epsilon, \epsilon > 0\} \tag{8}$$

where of course ϵ can vary with ϕ. Secondly we know that u is continuous so for some N,

$$|u(\phi)| \le C \sup_{|\alpha|+|\beta| \le N,\ x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)|. \tag{9}$$

As usual, we need a result about test functions:

Lemma 2. *The closure of the subset of* $\mathcal{S}(\mathbb{R}^n)$ *consisting of the elements with* $0 \notin \operatorname{supp}(\phi)$ *with respect to the norm in* (9) *contains*

$$\{\phi \in \mathcal{S}(\mathbb{R}^n); \partial^\beta \phi(0) = 0 \ \forall\ |\beta| \le N\}. \tag{10}$$

Certainly this is necessary since convergence in this norm implies uniform convergence of all derivatives up to order N in a neighbourhood of 0 and since these vanish at 0 for all ϕ with $0 \notin \operatorname{supp}(\phi)$ they must vanish on the closure. Conversely we just try a cut-off. Choose $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\chi(x) = 1$ in some neighbourhood of 0 and for $\phi \in \mathcal{S}(\mathbb{R}^n)$ look at

$$\phi_n = (1 - \chi(xn))\phi \in \mathcal{S}(\mathbb{R}^n). \tag{11}$$

Then, provided $\partial^\beta \phi(0) = 0$ for $|\beta| \le N$, $\phi_n \to \phi$ with respect to the norm in (9). This proves the lemma.

Now, using the same χ for a general $\psi \in \mathcal{S}(\mathbb{R}^n)$ set

$$\phi = \psi - \sum_{|\beta| \le N} \frac{\partial^\beta \phi(0)}{\beta!} x^\beta \chi. \tag{12}$$

It follows that $u(\phi) = 0$ so

$$u(\psi) = \sum_{|\beta| \le N} c_\alpha \partial^\beta \psi(0) \tag{13}$$

proves the result. $\square$

Other things to try at this stage:- Show in dimension one that

$$\frac{du}{dx} = 0 \Longrightarrow u = c \text{ a constant function.} \tag{14}$$

Also,

$$\frac{d}{dx} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{15}$$

is surjective.
Defining distributions on a general open set $\Omega \subset \mathbb{R}^n$.
The sheaf properties
Convolution and supports, extension to distributions.

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