BRIEF NOTES FOR 18.155 LECTURE 5
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Abstract.

Sobolev embedding theorem recalled and proved – see Lecture 4.

Corollary 1. *An element* $u \in \mathcal{S}'(\mathbb{R}^n)$ *is in* $\mathcal{S}(\mathbb{R})$ *if and only if*

$$(1 + |x|^2)^k(1 + \Delta)^k u \in L^2(\mathbb{R}^n)\ \forall\ k,\ (\Delta = -\partial_1^2 - \cdots - \partial_n^2) \tag{1}$$

and the countably many norms

$$\|(1 + |x|^2)^k(1 + \Delta)^k u\|_{L^2} \tag{2}$$

give the topology on $\mathcal{S}(\mathbb{R}^n)$.

The norms 'the other way around' $\|(1 + \Delta)^k(1 + |x|^2)^k u\|_{L^2}$ are equivalent.

[Maybe it isn't a corollary if it requires a proof!]

Proof. We know that $(1 + \Delta)^k u \in L^2(\mathbb{R}^n)$ implies that $u \in H^{2k}(\mathbb{R}^n)$. So from (2) by taking the Fourier transform we find that

$$(1 + \Delta_\xi)^k(1 + |\xi|^2)^k\hat{u} \in L^2 \Longrightarrow (1 + |\xi|^2)^k\hat{u} \in H^{2k}(\mathbb{R}^n) \Longrightarrow \\ (1 + |\xi|^2)^k\hat{u} \in \mathcal{C}^k_\infty(\mathbb{R}^n),\ k > n/2. \tag{3}$$

This gives the finiteness of the norms $\sup|\partial^\alpha x^\beta \hat{u}|$ so $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$. □

These are all Hilbert norms and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a 'projective limit of Hilbert spaces' in this sense is quite significant.

What more do we want to know about Sobolev spaces. One important property is duality:-

Proposition 1. *For any* $s \in \mathbb{R}$ *the integral pairing*

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \longmapsto \int \phi\psi \in \mathbb{C} \tag{4}$$

extends by continuity to a perfect pairing

$$H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{5}$$

i.e. such that every continuous linear functional on $H^s(\mathbb{R}^n)$ *is represented by a unique element of* $H^{-s}(\mathbb{R}^n)$ *(and the other way around of course!)*

Also we have 'representability':- For any $k \in \mathbb{N}_0$ each element of $H^s(\mathbb{R}^n)$ is of the form

$$u = \sum_{|\alpha| \le k} \partial^\alpha u_\alpha,\ u_\alpha \in H^{s+k}(\mathbb{R}^n). \tag{6}$$

Restriction to a hypersurface (also higher codimension subspaces).

Lemma 1. *If $s > \frac{1}{2}$ the restriction map*

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi(0, \cdot) \in \mathcal{S}(\mathbb{R}^n) \tag{7}$$

extends by continuity to a surjective map

$$\Big|_{x_1=0} : H^s(\mathbb{R}^n) \longrightarrow H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}). \tag{8}$$

The surjectivity is not quite so easy to see.
What about the fractional-order spaces, what do they really represent?
Notes Prop 5.7

Proposition 2. *For any $0 < s < 1$, if $u \in L^2(\mathbb{R}^n)$ then $u \in H^s(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy < \infty. \tag{9}$$

Schwartz struture theorem:

Proposition 3. *Any element of $\mathcal{S}'(\mathbb{R}^n)$ can be written uniquely in the form*

$$u = (1 + |x|^2)^k (\varDelta + 1)^p v, \ v \in L^2(\mathbb{R}^n) \tag{10}$$

for some integers k and p (and any larger ones).

Proof. Proof in Problem set 3. Using the Sobolev embedding theorem it follows that for $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\sup |\partial^\alpha \langle x\rangle^k \phi| \le C_s \|\langle x\rangle^k \phi\|_{H^s}, \ s > |\alpha| + n/2. \tag{11}$$

In particular each of the norms on $\mathcal{S}(\mathbb{R}^n)$ has a bound

$$\|\phi\|_N \le C \|\langle x\rangle^k \phi\|_{H^s} \tag{12}$$

where C and s only depend on N. This is discussed above.

It follows that any element $u \in \mathcal{S}'(\mathbb{R}^n)$, which has to be bounded by some norm, satisfies

$$|u(\phi)| \le C \|\langle x\rangle^k \phi\|_{H^s} \tag{13}$$

for some k and s (which can be taken equal in fact). This means

$$|(\langle x\rangle^{-k} u)(\phi)| \le C\|\phi\|_{H^s} \Longrightarrow \langle x\rangle^{-k} u \in H^{-s} \Longrightarrow u = \langle x\rangle^k (1 + \varDelta)^p v, \ v \in L^2(\mathbb{R}^n) \tag{14}$$

for some k and p. $\square$

So 'what we have done' in defining $L^2(\mathbb{R}^n)$ is to give a consistent description of differentiation, and multiplcation by functions of polynomial growth, of L^2 functions.

Support of a tempered distribution.
Distributions supported at a point.

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