BRIEF NOTES FOR 18.155 LECTURE 3
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- The Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$

  We know that the Fourier transform, defined by

$$\hat{\phi}(\xi) = \int e^{-ix\cdot\xi} u(x) \tag{1}$$

maps $\mathcal{S}(\mathbb{R}^n)$ continuously to $\mathcal{S}(\mathbb{R}^n)$ and that

$$\partial_{\xi_j}\hat{\phi}(\xi) = -i\widehat{x_j\phi(x)}(\xi),\ \xi_j\hat{\phi}(\xi) = -i\widehat{\partial_j\phi}(\xi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{2}$$

If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi(0) = 0$ then

$$\phi(x) = \sum_{i=1}^{n} x_i\psi_i(x),\ \psi_i \in \mathcal{S}(\mathbb{R}^n). \tag{3}$$

Proof directly for $n = 1$ and by induction over dimension.

Any $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be written

$$\phi = \phi(0)\Psi + \sum_{i=1}^{n} x_i\psi_i,\ \psi_i \in \mathcal{S}(\mathbb{R}^n),\ \Psi = \exp(-|x|^2/2). \tag{4}$$

This is true for any $\Psi \in \mathcal{S}(\mathbb{R}^n)$ with $\Psi(0) = 1$.

Weakened version of inversion formula

$$\int \hat{\phi} = C\phi(0),\ C = \int \hat{\Psi}. \tag{5}$$

Proof: Integration by parts.

Standard formulæ and computation shows that

$$\int_{\mathbb{R}^n} \Psi(x) = (2\pi)^{n/2},\ \hat{\Psi} = c\Psi,\ \hat{\Psi}(0) = c = \int \Psi,\ \int \hat{\Psi} = C \Longrightarrow C = (2\pi)^n \tag{6}$$

So

$$\phi(0) = (2\pi)^{-n} \int \hat{\phi}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{7}$$

Translation:

$$\widehat{\phi(\cdot + y)}(\xi) = e^{iy\cdot\xi}\hat{\phi}(\xi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{8}$$

Inversion formula

$$\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F} = \mathrm{Id},\ \mathcal{G}\phi(\xi) = (2\pi)^{-n}(\hat{\phi})(-\xi) \tag{9}$$

Proof:$\int \hat{\phi}(\cdot + y) = (2\pi)^n\phi(y) = \int e^{iy\cdot\xi}\hat{\phi}(\xi)d\xi$.

- The Fourier transform defines an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$.
  For $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\int \phi\hat{\psi} = \int \hat{\phi}\psi. \tag{10}$$

  We can defined the Fourier transform of $U \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\hat{U}(\phi) = u(\hat{\phi}). \tag{11}$$

  Then $\mathcal{S}'(\mathbb{R}^n) \ni U \longmapsto \hat{U} \in \mathcal{S}'(\mathbb{R}^n)$ is a bijection, consistent with our inclusion map $\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto U_\phi \in \mathcal{S}'(\mathbb{R}^n)$, i.e.

$$\hat{(}U_\phi) = U_{\hat{\phi}}.$$

- For ϕ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ Parseval's identity holds:-

$$\int \hat{\phi}\overline{\hat{\psi}} = (2\pi)^n \int \phi\overline{\psi}. \tag{12}$$

- Bump functions. I described the construction of a function for given $0 < \epsilon < 1$ such that

$$\chi \in \mathcal{S}(\mathbb{R}^n),\ 1 \geq \chi(x) \geq 0,\ \chi(x) = 0 \text{ in } |x| > 1,\ \chi(x) = 1 \text{ in } |x| < 1 - \epsilon. \tag{13}$$

Coming up:

- Density of test functions in square-integrable functions
- Fourier transform of square-integrable functions
- Sobolev spaces

References

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