18.155 LECTURE 19
16 NOVEMBER, 2017

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Let me quickly recall the approach to elliptic regularity I introduced last time. We have a differential operator

$$P(x,D_x)=\sum_{|\alpha|\le m}p_\alpha(x)D_x^\alpha,\ p_\alpha(x)=\tilde p_\alpha+q_\alpha(x),\ q_\alpha\in\mathcal{S}(\mathbb{R}^n) \tag{1}$$

where the $\tilde p_\alpha$ are constants. We also assume that P is elliptic at each point which, because of the assumed decomposition, means that it is uniformly elliptic. So there is a smooth cutoff $\chi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, only in the ξ variable, such that

$$a(x,\xi)=\frac{1-\chi}{p(x,\xi)}=\tilde a(\xi)+c(x,\xi)\in S^{-m}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{R}^n) \tag{2}$$

where I will soon recall what this means.

The idea is that we want to 'quantize' $a(x,\xi)$ to an operator in much the same way that $P(x,D_x)$ comes from $p(x,\xi)$. That is

$$P(x,D)\phi(x)=(2\pi)^{-n}\int e^{-i(x-y)\cdot\xi}p(x,\xi)\hat\phi(\xi)d\xi. \tag{3}$$

Now, for P this is easy to understand what is happening, since we can expand out the polynomial and move the coefficient out of the integral and see that the Schwartz kernel of $P(x,D)$ is

$$\begin{aligned}P(x,y)&=(2\pi)^{-n}\int e^{-i(x-y)\cdot\xi}p(x,\xi)d\xi\\&=\sum_{|\alpha|\le m}p_\alpha(x)(2\pi)^{-n}\int e^{-i(x-y)\cdot\xi}\xi^\alpha d\xi=\sum_{|\alpha|\le m}p_\alpha(x)D_x^\alpha\delta(x-y).\end{aligned} \tag{4}$$

The idea is to construct an operator from $a(x,\xi)$ by the same formula (3) for the Schwartz kernel

$$A(x,y)=(2\pi)^{-n}\int e^{-i(x-y)\cdot\xi}a(x,\xi)d\xi=q_L(a). \tag{5}$$

As I showed last time, there is no problem making this formal integral rigorous in terms of the partial (inverse) Fourier transform

$$A(x,y)=\mathcal{G}_{\xi\longrightarrow z}a(x,\xi)\big|_{z=x-y}\in\mathcal{S}'(\mathbb{R}^{2n}) \tag{6}$$

followed by an invertible linear change of variabels. The problem is to understand the properties of the resulting operators, which by definition are pseudodifferential operators. Still we set

$$\begin{aligned}\Psi^M_{\mathcal{S}}(\mathbb{R}^n)=\{&A:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^n);\\&A\text{ is given by (6) with }a\in S^M(\mathbb{R}^n)+\mathcal{S}(\mathbb{R}^n_x;S^M(\mathbb{R}^n)).\end{aligned} \tag{7}$$

These two spaces are defined by estimates

$$
\begin{aligned}
&a_\infty \in S^M(\mathbb{R}^n) \Longleftrightarrow a \in \mathcal{C}^\infty(\mathbb{R}^n), \sup_{\xi\in\mathbb{R}^n,\ |\alpha|\le p} \langle\xi\rangle^{-M+|\alpha|}|D^\alpha_\xi a(\xi)| < \infty\ \forall\ p \\
&\tilde a \in \mathcal{S}(\mathbb{R}^n; S^M(\mathbb{R}^n)) \Longleftrightarrow \tilde a \in \mathcal{C}^\infty(\mathbb{R}^{2n}), \\
&\sup_{x,\xi\in\mathbb{R}^n,\ |\alpha|+\beta|+N\le p} \langle x\rangle^N \langle\xi\rangle^{-M+|\alpha|}|D^\beta_x D^\alpha_\xi \tilde a(x,\xi)| < \infty\ \forall\ p
\end{aligned}
\tag{8}
$$

To prove that these operators are actually bounded on Sobolev spaces we expand in Hermite series.

Lemma 1. *If $\tilde a \in \mathcal{S}(\mathbb{R}^n; S^M(\mathbb{R}^n))$ then the Hermite coefficients*

$$
\tilde a_\alpha(\xi) = \int \tilde a(x,\xi) e_\alpha dx \in S^M(\mathbb{R}^n) \tag{9}
$$

form a rapidly decreasing sequence with respect to each continuous seminorm on $S^M(\mathbb{R}^n)$ and the series

$$
\tilde a(x,\xi) = \sum_{\alpha\in\mathbb{N}_0^n} e_\alpha(x)\tilde a_\alpha(\xi) \tag{10}
$$

converges with respect to each seminorm on $\mathcal{S}(\mathbb{R}^n; S^M(\mathbb{R}^n))$.

The quantization map (5) maps the individual terms in (10) into the product of two operators,

$$
q_L(\phi(x)c(\xi)) = \phi(x) b*,\ \hat b = c \in S^M(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{11}
$$

We know that the convolution operator and the multiplication operator are both bounded on Sobolev spaces, $b*$ decreases order by M and multiplcation by a Schartz function leaves it unchanged. In fact

$$
\|e_\alpha(x) b *\|_{\mathcal{B}(H^{s+M};H^s)} \le C_s \|\phi\|_{\mathcal{S},p(s)} \|a\|_{S^M,p(s)} \tag{12}
$$

where the constant and norms – continuous on $\mathcal{S}$ and S^M – only depend on s (in fact the second norm can be taken as the supremum of $\langle\xi\rangle^{-M}|a|$.

If we apply this to each term in the series (9) we get a convergent series

$$
A = q_L(\tilde a(x,\xi)) = \sum_{\alpha\in\mathbb{N}_0^n} q_L(e_\alpha(x)\tilde a_\alpha(\xi)) \text{ converges in } \mathcal{B}(H^{s+M};H^s)\ \forall\ s \tag{13}
$$

since a fixed norm applied to e_α gives a slowly increasing sequence while a norm on S^M applied to $\tilde a_\alpha(\xi)$ gives a rapidly decreasing sequence. Thus

Proposition 1. *Each element of $\Psi^M_{\mathcal{S}}(\mathbb{R}^n)$ gives a bounded operator*

$$
a(x,D_x) : H^{s+M}(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n)\ \forall\ s. \tag{14}
$$

An operator which satisfies (14) is said to have *Sobolev order M*.

We can also show that

$$
a(x,D_x) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ a(x,D_x) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{15}
$$

I will not use these for the moment but it may be informative to briefly look at a proof. We know boundedness on Sobolev spaces, as in (14), so what we need to handle is 'decay'. Consider the identity for the partial inverse Fourier transform

$$
z_i \mathcal{G}_{\xi\longrightarrow z} a(x,\xi) = i\mathcal{G}_{\xi\longrightarrow z}\partial_{\xi_i} a(x,\xi). \tag{16}
$$

Introducing $z = x - y$, as we do in (6), shows that the commutator

$$[x_i, a(x, D_x)] = c_i(x, D),\ c_i = i\partial_{\xi_i} a(x, \xi) \in S^{M-1}_{\mathcal{S}}(\mathbb{R}^n; \mathbb{R}^n) \tag{17}$$

since its Schwartz kernel is precisely $x_i A(x, y) - A(x, y) y_i$. Applying this we find

$$x_i u \in H^{s+M}(\mathbb{R}^n) \Longrightarrow x_i a(x, D_x) u \in H^s(\mathbb{R}^n). \tag{18}$$

Iterating this shows that $a(x, D_x) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous.

The next thing we want to check is actually easy:-

$$\begin{gathered}\Psi^M_{\mathcal{S}}(\mathbb{R}^n) \ni a(x, D) \longmapsto P(x, D) \circ a(x, D) = g(x, D) \in \Psi^{m+M}_{\mathcal{S}}(\mathbb{R}^n),\\ g(x, D) = q_L(g),\ g(x, \xi) = p(x, \xi) a(x, \xi) + e(x, \xi),\\ e \in S^{m+M-1}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n; S^{m+M-1}(\mathbb{R}^n)).\end{gathered} \tag{19}$$

The important point here is that the 'error term' is one order lower than the main term which is just the 'commutative product' of the symbols.

The final property we need to show is that these pseudodifferential operators are pseudolocal:

Lemma 2. *If $a(x, D_x) \in \Psi^M_{\mathcal{S}}(\mathbb{R}^n)$ then*

$$\operatorname{singsupp}(a(x, D_x) u) \subset \operatorname{singsupp}(u),\ \forall\ u \in H^{-\infty}(\mathbb{R}^n) = \bigcup_s H^s(\mathbb{R}^n). \tag{20}$$

In fact follows from (17) that this holds for all tempered distributions, but (20) suffices for the moment.

Proof. From the properties of singular support it is enough to show that if $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $\phi \equiv 1$ on a neighbourhood of the support of $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ then

$$\psi A(x, D_x)(1 - \phi) u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)\ \forall\ u. \tag{21}$$

Indeed, then if $\psi u \in \mathcal{C}^\infty(\mathbb{R}^n)$ it follows that $\phi A(x, D_x) u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

We already know (21) for the constant part $A_\infty(D_x)$ – we did precisely this in the proof of elliptic regularity for constant coefficient operators, so we can suppose that $a(x, \xi) \in \mathcal{S}(\mathbb{R}^n; S^M(\mathbb{R}^n))$. Then the idea is that kernel of the operator in (21) is

$$\psi(x) b(x, x - y)(1 - \phi(y)) \in \mathcal{S}(\mathbb{R}^{2n}),\ \mathcal{F}_{z \longrightarrow \xi} b(x, z) = a(x, \xi).$$

To see this, observe that the supports of the functions on the left and right are disjoint on $\mathbb{R}^n$ – meaning that $x \neq y$ on $\operatorname{supp}(\psi(x)(1 - \phi(y)))$. It follows that we can choose a cut-off $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, equal to one near 0 such that

$$\begin{gathered}\chi(x - y)\psi(x)(1 - \phi(y)) \equiv 0 \Longrightarrow\\ \psi(x) b(x, x - y)(1 - \phi(y)) = \psi(x)(1 - \chi(x - y)) b(x, x - y)(1 - \phi(y)) \in \mathcal{S}(\mathbb{R}^{2n}).\end{gathered} \tag{22}$$

This last part because of the properties of the inverse Fourier transforms of symbols. Since it has kernel in $\mathcal{S}(\mathbb{R}^{2n})$,

$$\psi(x) A(x, D_x)(1 - \phi) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

is a 'smoothing operator' and (21) follows. $\square$

Now, to start to put all this together. First observe that the product of an operator of Sobolev order M_1 and one of Sobolev order M_2 is of Sobolev order $M_1 + M_2$. Similarly, the adjoint (always with respect to the L^2 pairing) of the transpose (with respect to the real L^2 pairing) of an operator of Sobolev order M are also of Sobolev order M because of the duality of Sobolev spaces.

Secondly the argument leading to () works just as well the other way around, for $(1-\phi(y))(x)b(x,x-y)\psi$. Since the kernel of the adjoint of an operator with kernel $A(x,y)$ is $\overline{A(y,x)}$ this shows that the adjoint, or transpose, is also pseudolocal. Further more the composite of two (and hence more) pseudolocal operators is also pseudolocal, essentially from the definition.

Theorem 1. *If P in (1) is elliptic at each point $x \in \mathbb{R}^n$ for each given $l \in \mathbb{N}$ there exists a pseudolocal operator, B_l, of Sobolev order $-m$ and a pseudolocal Sobolev operator, E_l, of order $-l$ such that*

$$B_l P(x, D_x) = \mathrm{Id} - E_l; \tag{23}$$

it follows that if $u \in H^{-\infty}$ and $U \subset \mathbb{R}^n$ is a bounded open set then

$$P(x, D_x)u \in H^s_{\mathrm{loc}}(U) \Longleftrightarrow u \in H^{m+s}_{\mathrm{loc}}(U). \tag{24}$$

Proof. First lets check the last part, given (23). Given that $f\big|_U \in H^s_{\mathrm{loc}}(U)$ all we need to show is that if $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $\mathrm{supp}(\phi) \subset U$ then $\phi u \in H^{s+m}(\mathbb{R}^n)$. If $u \in H^{-\infty}(\mathbb{R}^n)$ then $u \in H^k(\mathbb{R}^n)$ for some, possible very negative, k. We have fixed s so now choose $l > |s| + m + |k|$, so $k + l > s + m$. The Sobolev order means that $E_l u \in H^{s+m}(\mathbb{R}^n)$. If we apply (23) to u, as we can, we find

$$u = B_l f + E_l u \Longrightarrow \phi u = \phi B_l (1-\psi) f + \phi B_l \psi f + E_l u \in H^{s+m}(\mathbb{R}^n)$$

if $\mathrm{supp}(\phi) \subset U$ and $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(U)$ is equal to 1 on a neighbourhood of $\mathrm{supp}(\phi)$. In fact the first term on the right is smooth and the other two are in $H^{s+m}(\mathbb{R}^n)$.

So, to construct the 'pseudo-inverse' in (23) – I will not get to this until next Tuessay but it is just combining what we have above. □

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