18.155 LECTURE 14
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- Spectral theorem for compact self-adjoint operators:- If $A = A^* \in \mathcal{K}(H)$ then $\operatorname{null}(A)^\perp$ has an orthonormal basis of eigenvectors for A with eigenvalues λ_i and $|\lambda_i| \to 0$.
- Polar decomposition – every bounded operator can be written uniquely as a product

$$\begin{aligned}(1)\quad A = PV,\ P = P^* \geq 0,\ \operatorname{null}(P) = \operatorname{Ran}(A)^\perp,\\ \operatorname{null}(V) = \operatorname{null}(A),\ V : \operatorname{null}(A)^\perp \longmapsto \overline{\operatorname{Ran}(A)} \text{ an isometry.}\end{aligned}$$

- Spectral projections:- If $A = A^* \in \mathcal{B}(H)$ and $a \in \mathbb{R}$, there is a projection $Q_a = Q_a^* = Q_a^2$ such that $f(A)Q_a = Q_a f(A)$, $\operatorname{spec}(Q_a A) \subset (-\infty, a]$, $\operatorname{spec}((\operatorname{Id} - Q_a)A) \subset [a, \infty)$ and

(2)
$$\langle Q_a u, u\rangle = \inf\{\langle g(A)u, u\rangle; 0 \leq g \in \mathcal{C}(\mathbb{R}),\ g(s) = 1 \text{ in } s \leq a,\ g(s) \geq 0\},\ \forall\ u \in H.$$

- Hilbert-Schmidt operators. If H is separable and infinite dimensional, $A \in \mathcal{B}(H)$ is 'Hilbert-Schmidt' if

$$\sum_i \|Ae_i\|^2 < \infty \tag{3}$$

  for some orthonormal bases e_i. This is a 2-sided *-ideal of compact operators which is a Hilbert space with respect to the norm

$$\|A\|^2_{\mathrm{HS}} = \sum_i \|Ae_i\|^2 \tag{4}$$

  which is independent of the orthonormal basis.
- Trace class $\mathcal{T}(H)$. The trace class ideal consists of the finite sums of products of two Hilbert-Schmidt operators. Each element is the actually the product of two Hilbert-Schmidt operators and it is a 2-sided *-ideal which is a Banach subspace of $\mathcal{K}(H)$ with respect to the norm

$$\|A\|_{\mathrm{Tr}} = \sup \sum_i |\langle Ae_i, f_i\rangle| \tag{5}$$

  with the supremum over all pairs of orthonormal bases.
- A bounded operator A is Hilbert-Schmidt/trace class if and only if $(AA^*)^{\frac{1}{2}}$ is Hilbert-Schmidt/trace class.
- The trace functional,

(6)
$$\operatorname{Tr} : \mathcal{T}(H) \longrightarrow \mathbb{C},\ \operatorname{Tr}(AB) = \langle B, A^*\rangle_{\mathrm{HS}} = \sum_i \langle Te_i, e_i\rangle,\ T = AB,\ A, B \in \mathrm{HS}(H)$$

  is a continuous linear functional vanishing on commutators.

- An operators $L \in \mathcal{B}(H)$ is Fredholm if and only if it has finite dimensional null space and closed range with finite-dimensional orthocomplement.
- If L is Fredholm then there exists a uniquely defined $Q \in \mathcal{B}(H)$ such that

$$QL = \operatorname{Id} - P_{\operatorname{null}(L)}, \ \ LQ = \operatorname{Id} - P_{\operatorname{null}(L^*)}. \tag{7}$$

- The index

$$\operatorname{ind}(L) = \dim \operatorname{null}(L) - \dim \operatorname{null}(L^*) \tag{8}$$

is constant on, and labels, the components of the Fredholm operators.

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