18.155 LECTURE 13
26 OCTOBER, 2017

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No lecture on 24th October.
Read: Notes Chapter 2.

- Compact operators as closure of finite rank operators.
- $\overline{\operatorname{Ran} A}^{\perp} = \operatorname{Nul}(A^*)$.
- Since there is no lecture Tuesday you can (have) use(d) it to check that Baire's Theorem can be used to show:
  Uniform Boundedness=Banach-Steinhaus (straightforward)
  Open Mapping Theorem (trickier)
  Closed Graph Theorem (follows easily from OMT) – look at the commutative diagram

(1)
$$\begin{tikzcd} & \operatorname{Gr}(A) \subset H \times H \arrow[dl, shift right, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ H \arrow[ur, shift right, "(\operatorname{Id}, A)"'] \arrow[rr, "A"'] & & H \end{tikzcd}$$

If $\operatorname{Gr}(A)$ is closed the continuity of π_1, and hence A follows from the OMT.
  Note that if $A : H \longrightarrow H$ is bounded and a bijection, the continuity of A^{-1} follows from either the OMT or the CGT.
- Spectrum and resolvent of a bounded operator.
- Spectrum of a compact operator is discrete outside $\{0\}$.
- Spectrum of a self-adjoint operator is contained in $\mathbb{R} \subset \mathbb{C}$.
- If $A = A^* \in \mathcal{B}(H)$ then $\|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle|$.
- If $A = A^*$ then
$$\{\alpha\} \cup \{\beta\} \subset \operatorname{spec}(A) \subset [\alpha, \beta],\ \alpha = \inf_{\|u\|=1} \langle Au, u\rangle,\ \beta = \sup_{\|u\|=1} \langle Au, u\rangle.$$
- If $A = A^*$ and p is a polynomial with real coefficients then

(2)
$$\|p(A)\| \le \sup_{[\alpha,\beta]} |p(z)|.$$
- (Functional Calculus) If $A = A^* \in \mathcal{B}(H)$ there is a continuous linear map of norm one

(3)
$$\mathcal{C}([\alpha, \beta]) \ni f \longmapsto f(A) \in \mathcal{B}(H) \text{ s.t. } f(A)g(A) = (fg)(A).$$
where $f(A) = A^j$ if $f(x) = x^j$.
- Existence of spectral projections.

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