18.155 LECTURE 11
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Theorem 1 (Schwartz' kernel theorem). *Any linear map*

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m) \tag{1}$$

which is strongly continuous, in the sense that $(A\phi)(\psi)$ is continouous in $\phi \in \mathcal{S}(\mathbb{R}^n)$ for each fixed $\psi \in \mathcal{S}(\mathbb{R}^m)$, determines and is determined by an element $K_A \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$ through

$$(A(\phi))(\psi) = K_A(\psi \boxtimes \phi), \ (\psi \boxtimes \phi)(x, y) = \psi(x)\phi(y) \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n). \tag{2}$$

The basic step is to use Baire's theorem – If a complete metric space

$$B = \bigcup_N C_N \tag{3}$$

is written as a countable union of closed sets then at least one of the C_N must have an interior point.

Look at the closed unit ball $B \subset \mathcal{S}(\mathbb{R}^n)$ around the origin and set

$$C_N = \{\phi \in B; |(A\phi)(\psi)| \le N\|\psi\|_N \tag{4}$$

where the $\|\cdot\|_N$ are the norms on $\mathcal{S}(\mathbb{R}^m)$. Each $A\phi \in \mathcal{S}'(\mathbb{R}^m)$ satisfies these estimates for some N so (3) holds. Moreover, each C_N is closed since if $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ then by the assumed strong continuity of A, $(A\phi_n)(\psi) \to (A\phi)(\psi)$ so the same inequality holds in the limit. Thus one of the C_N has an interior point, so it has one in the open unit ball.

It follows that there exists an element $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\epsilon > 0$ such that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $d(\phi, 0) < \epsilon$ then

$$|(A(\phi + \phi_0))(\psi)| \le N\|\psi\|_N \longrightarrow |(A(\phi))(\psi)| \le 2N\|\psi\|_N.$$

Now, for some integer M and constatn $\delta > 0$ $\|\phi\|_M < \delta$ implies $d(\phi, 0) < \epsilon$. This gives the main estimate we want, that for some constants $C, N.M$

$$|(A(\phi))(\psi)| \le C\|\phi\|_M\|\psi\|_N. \tag{5}$$

we can always increase N, M and C.

So, recall that we can choose as norms defining the topology on $\mathcal{S}$ the weighted Sobolev norms

$$\|\phi\|_{(2N)} = \|(1 + \Delta)^N(1 + |x|^2)^N\phi\|_{L^2}.$$

In fact, by adding extra powers of $1+\Delta$ we can replace the L^2 by a negative Sobolev norm. So, for some sufficiently large constants we can arrange from (5) that

$$|(A(\phi))(\psi)| \le C\|(1 + \Delta_x)^N(1 + |x|^2)^N\phi\|_{H^{-m}}\|(1 + \Delta_y)^N(1 + |y|^2)^N\psi\|_{H^{-n}}. \tag{6}$$

Shifting the powers this means that the operator

(7)
$$B = (1+\Delta_x)^{-N}(1+|x|^2)^{-N}A(1+\Delta_y)^{-N}(1+|y|^2)^{-N} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m) \text{ and}$$
$$B : H^{-n}(\mathbb{R}^n) \longrightarrow H^m(\mathbb{R}^m).$$

Now, by Sobolev embedding $H^m(\mathbb{R}^m) \hookrightarrow \mathcal{C}^0_\infty(\mathbb{R}^m)$ is contained in the bounded continuous functions and $\delta(x-x') \in H^{-n}(\mathbb{R}^n)$ has norm indenpendent of x' and depends continuously on it. It follows that the function

(8)
$$K_B : \mathbb{R}^n \ni x' \longmapsto B(\delta(\cdot - x')) \in \mathcal{C}^0_\infty(\mathbb{R}^m)$$

is a bounded continuous function with values in the bounded continuous functions. In short

(9)
$$K_B \in \mathcal{C}^0_\infty(\mathbb{R}^m \times \mathbb{R}^n).$$

Now we can check that $K_B \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$ is a Schwartz kernel for B. Finally then

(10)
$$K_A(x,y) = (1+\Delta_x)^N(1+|x|^2)^N(1+\Delta_y)^N(1+|y|^2)^N A(x,y) \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$$

is a (the of course) Schwartz kernel for A.

Now, we clearly have examples:-

(1) $\delta(x-y)$ is the Schwartz kernel of the identity
(2) $P(D_x)\delta(x-y)$ is the Schwartz kernel of $P(D)$.
(3) $b(x-y)$ is the Schwartz kernel of $b\,*$.

Theorem 2. *If $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are open subsets and*

(11)
$$A : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega')$$

is a linear map which is strongly continuous in the sense that $A(\phi)(\psi)$ is continous in $\phi \in \mathcal{C}^\infty_c(\Omega)$ for each fixed $\psi \in \mathcal{C}^\infty_c(\Omega')$ then there exists a uniquely defined $K_A \in \mathcal{C}^{-\infty}(\Omega' \times \Omega)$ such that

(12)
$$(A\phi)(\psi) = K_A(\psi \boxtimes \phi).$$

I leave it to you to check that you know what this means. It can be deduced by localization from the theorem for Schwartz spaces.

Now, let's turn to Hilbert spaces and the Spectral Theorem for self-adjpint (or normal) and compact operators. I will quickly review the basics of Hilbert spaces and operators and then go through the Spectral Theorem more carefully. That is, unless someone complains that they need more detail on the basic Hilbert space theory.

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