BRIEF NOTES FOR 18.155 LECTURE 1
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Abstract.

- Basic aim of the course
- Outline of contents
  - Distributions and function spaces, Fourier transform
  - Constant coefficient differential operators
  - Operators on Hilbert space
  - Elliptic regularity (variable coefficients)
- Prerequisites, including $L^2(\mathbb{R}^n)$.
- Riesz' theorem as the 'weak' definition of $f \in L^2(\mathbb{R}^n)$.
- A biggish diagram of spaces to indicate where we are going in the immediate future (here is a smaller version which almost fits on the page):-

$$
\begin{tikzcd}
<x>^{-\infty}\, L^2(\mathbb{R}^n) \arrow[dr, hook] & & & \mathcal{S}(\mathbb{R}^n) \arrow[lll, hook'] \arrow[rrr, hook] \arrow[d, hook] & & & H^\infty(\mathbb{R}^n) \arrow[dl, hook'] \\
& \cdots \arrow[dr, hook] & & \cdots \arrow[d, hook] & & \cdots \arrow[dl, hook'] & \\
& & <x>^{-1}\, L^2(\mathbb{R}^n) \arrow[dr, hook] & H^1_{\mathrm{iso}}(\mathbb{R}^n) \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook] & H^1(\mathbb{R}^n) \arrow[dl, hook'] & & \\
& & & L^2(\mathbb{R}^n) \arrow[dl, hook'] \arrow[d, hook] \arrow[dr, hook] & & & \\
& & <x>^1\, L^2(\mathbb{R}^n) \arrow[r, hook] \arrow[dl, hook'] & H^{-1}_{\mathrm{iso}}(\mathbb{R}^n) \arrow[d, hook] & H^{-1}(\mathbb{R}^n) \arrow[l, hook'] \arrow[dr, hook] & & \\
& \cdots \arrow[dl, hook'] & & \cdots \arrow[d, hook] & & \cdots \arrow[dr, hook] & \\
<x>^{\infty}\, L^2(\mathbb{R}^n) \arrow[rrr, hook] & & & \mathcal{S}'(\mathbb{R}^n) & & & H^{-\infty}(\mathbb{R}^n) \arrow[lll, hook']
\end{tikzcd}
$$

There is a lot to this diagram – and a lot missing. Flipping around the central vertical axis is the Fourier transform (so $\mathcal{S}(\mathbb{R}^n)$ etc are invariant under it). Flipping around the central horizontal axis is duality, so $L^2(\mathbb{R}^n)$ is self-dual. Going up the three lines there is a general 'order k' line missing above (and corresponding order $-k$ below) where the dots are. The top spaces are all the intersections of the lines down to $L^2(\mathbb{R}^n)$ and the bottom spaces are all the unions of the lines from L^2 to them. The 'isotropic spaces' in the middle are the intersections of the sides above L^2 and the sums below.

- So we start with the space $\mathcal{S}(\mathbb{R}^n)$ which consists of the smooth functions with all derivatives decaying rapidly.
- Differentiability recalled, symmetry of higher derivatives, the notation $\partial^\alpha u$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ for derivatives and similiary x^β for powers.
- So explicitly

$$\mathcal{S}(\mathbb{R}^n) = \{u : \mathbb{R}^n \longrightarrow \mathbb{C}; \partial_x^\alpha u(x) \text{ exists } \forall\ \alpha \text{ and} \\ \|u\|_N = \sup_{|\alpha|+|\beta|\le N} |x^\beta \partial^\alpha u| < \infty\ \forall\ N \in \mathbb{N}_0^n\} \tag{1}$$

Here $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ for multiindices, despite the possible confusion.

- Each of the $\|u\|_N$ is a norm, so $\mathcal{S}(\mathbb{R}^n)$ is a *countably normed space.*
- A countably normed space has a metric topology given by

$$d(\phi, \psi) = \sum_N 2^{-N} \frac{\|\phi - \psi\|_N}{1 + \|\phi - \psi\|_N}. \tag{2}$$

You should check that this is a distance – each of the quotients is a distance and the sum is finite.

- In fact $\mathcal{S}(\mathbb{R}^n)$ is a complete metric space with respect to this distance, which is to say it is a Fréchet space. You should check that you follow the proof of this; it is a standard sort of completeness argument but needs to be done carefully (by you) at least once. In brief:-

If ϕ_n is Cauchy with respect to the distance then it is Cauchy with respect to each $\|\dot{|}\|_N$ (and conversely).

Each $\|\dot{|}\|_N$ is the supremum norm on the $x^\beta \partial^\alpha \phi_n$ for $|\alpha| + |\beta| \le N$. From the completeness of the bounded continuous functions it follows that each sequence $x^\beta \partial^\alpha \phi_n \to u_{\alpha,\beta}$ converges in supremum norm to a bounded continuous limit.

Finally by using standard theorems (in Rudin for instance, at least in 1-D) or better by integrating the derivatives and looking at convergence it follows that the limit of the sequence ϕ_n in supremum norm is $u \in \mathcal{S}(\mathbb{R}^n)$ since $x^\beta \partial^\alpha u = u_{\alpha,\beta}$.

The uniform convergence of each $x^\beta \partial^\alpha \phi_n$ to $x^\beta \partial^\alpha u$ now implies that $\phi_n \to u$ in the metric.

- Finally the space of tempered (also 'temperate') distributions is by definition

$$\mathcal{S}'(\mathbb{R}^n) = \{U : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}; U \text{ is linear and continuous.}\} \tag{3}$$

On Tuesday I will talk about the meaning of continuity here (in terms of the norms), the embedding of $L^2(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$, various operations on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and start to talk about the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$:

$$\hat{\phi}(\xi) = \int e^{-ix\cdot\xi} \phi(x) dx. \tag{4}$$

References

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