PROBLEM SET 9, 18.155
DUE FRIDAY 22 NOVEMBER, 2013

Let's think about the sphere. The ball is the smoothly bounded domain

$$\mathbb{B}^n = \{x \in \mathbb{R}^n; F(x) = 1 - |x|^2 \geq 0\},\ \partial\mathbb{B}^n = \mathbb{S}^{n-1} \tag{1}$$

with boundary the sphere.

(1) For the special point $S = (0, \dots, 0, 1)$ on the sphere find a local diffeomorphism taking S to 0 with F the pull back of one (say the last) of the coordinates. The remaining coordinate functions then pull back to give local coordinates on the sphere near S, namely any smooth function restricted to the sphere can be expressed as a function of these coordinates.

Find such local coordinates at any other point on the sphere using an orthogonal transformation of $\mathbb{R}^n$. Discuss how the different local coordinates on the sphere are related in overlaps. [You don't need to get explicit expressions, just try to describe what is going on.]

(2) Show that (for $m \geq 0$) the following spaces of functions on $\mathbb{S}^{n-1}$ are the same – we denote them $H^m(\mathbb{S}^{n-1})$:-

i) The functions which have extensions to homogeneous functions (of some complex degree) on $\mathbb{R}^n \setminus \{0\}$ which are in $H^m_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. [Maybe start with degree 0].

ii) For $m > 0$, the restriction to $\mathbb{S}^{n-1}$ of $H^{m+\frac{1}{2}}(\mathbb{R}^n)$.

iii) The functions in $H^m_{\mathrm{loc}}(O)$, $O \subset \mathbb{R}^{n-1}$ for each of the coordinate neighbourhoods on the sphere.

(3) Let $P(D)$ be a constant coefficient, elliptic and homogeneous differential operator on $\mathbb{R}^n$ of order $m > 0$. Show, using the first characterization above that $P(D)$ defines a differential operator $Q(z) : H^m(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1})$ by using the extension of functions homogeneous of degree $z \in \mathbb{C}$.

(4) Show that $Q(z)$ defines an unbounded operator on $L^2(\mathbb{S}^{n-1})$ for each z with domain $H^m(\mathbb{S}^{n-1})$.

(5) Show that $Q(z)$ is Fredholm for each z, as a map from $H^m(\mathbb{S}^{n-1})$ to $L^2(\mathbb{S}^{n-1})$.