Recall part of the spectral theorem for a bounded self-adjoint operator $A$ that I did not prove in class. Namely the functional calculus extends from continuous functions on $[-\|A\|, \|A\|]$ so that $\chi(A) \in B(H)$ is defined for the characteristic function $\chi$ of any closed interval $[a, b]$, that $\chi(A)$ is a self-adjoint projection which commutes with $A$, it commutes with any bounded operator that commutes with $A$ and

\[ \text{spec}(\chi(A)A) \subset [a, b], \quad \text{spec}((\text{Id} - \chi(A))A) \cap (a, b) = \emptyset \]

(1) Show that if $U = A + iB$ is the decomposition of a unitary operator on a separable Hilbert space into its self-adjoint and anti-self-adjoint parts then $z \in \text{spec}(U)$ implies $\text{Re}(z) \in \text{spec}(A)$ and $\text{Im}(z) \in \text{spec}(B)$.

(2) Using a spectral projection as above, or otherwise (there are other methods), show that the unitary group on a separable Hilbert space is connected.

(3) Show that if $L$ is an unbounded self-adjoint operator (with domain $D \subset H$ as defined in the preceding problem set) which is such that $(L + i)^{-1} \in \mathcal{K}(H)$ then $H$ has an orthonormal basis consisting of eigenvectors of $L$, elements of $D$ such that $Le_j = \lambda_j e_j$.

(4) If $P(D)$ is a real elliptic polynomial of positive degree on $\mathbb{R}^n$ show that $P(D) : H^n(T^n) \rightarrow L^2(T^n)$ satisfies the conditions of the preceding problem. Here I am identifying the torus as the quotient $T^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n$, so functions on the torus are functions on $\mathbb{R}^n$ which are $2\pi$-periodic in each variable.

(5) Show that the eigenvalues appearing in the previous problem are the values $P(k)$ for $k \in \mathbb{Z}^n$. 

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