Let $H$ be a separable Hilbert space throughout. An unbounded operator on $H$ is a linear map 
\[ A : D \rightarrow H \]
where $D \subset H$ is a dense subspace and 
\[ \text{Gr}(A) = \{(u, Au); u \in D\} \subset H \times H \]
is closed. Such an unbounded operator is said to be self-adjoint if 
\[ D^* = \{g \in H; D \ni v \rightarrow \langle g, Av \rangle \text{ extends to be continuous on } H\} = D \]
and (symmetry) 
\[ \langle u, Av \rangle = \langle Au, v \rangle \forall u, v \in D. \]

(1) Suppose $P(D)$ is an elliptic differential operator with constant coefficients of order $m$. Show that $P(D) : H^m(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is an unbounded operator on $H = L^2(\mathbb{R}^n)$.

(2) Suppose that $P(D)$ is an elliptic differential operator of order $m$ with $P$ having real coefficients, show that $P(D) : H^m(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is self-adjoint.

(3) Suppose $A : D \rightarrow H$ is an unbounded self-adjoint operator. Show if $z \in \mathbb{C}$ then $A+z \text{Id} : D \rightarrow H$ is an unbounded operator with domain $D$ and that if $\text{Im } z \neq 0$ then it is a bijection and that $(A + z \text{Id})^{-1}$ is a bounded operator.

(4) Still assuming that $A$ is unbounded self-adjoint, show that $(A+i \text{Id})^{-1}$ and $(A-i \text{Id})^{-1}$ commute, that their product is self-adjoint and that $E = ((A+i \text{Id})^{-1}(A-i \text{Id})^{-1})^{1/2}$ has range contained in $D$. Using this to see that $AE$ is a bounded self-adjoint operator (or otherwise) define $f(A) \in \mathcal{B}(H)$ (as $g(AE)$ for an appropriate $g$) for each bounded continuous function $f \in \mathcal{C}(\mathbb{R})$ with limits at infinity.

(5) Show that if $A = \Delta$ in the construction above, then 
\[ \overline{f(A)}u(\xi) = f(|\xi|^2)\hat{u}(\xi) \forall u \in L^2(\mathbb{R}^n). \]

Notes for P4: Having shown that $(A+i \text{Id})^{-1}(A-i \text{Id})^{-1}$ commutes with $(A \pm i \text{Id})^{-1}$ it follows that any polynomial in the former commutes with these two operators and hence that any continuous function on
the spectrum defines an operator which commutes with them. Thus $E$ commutes with $(A \pm i \text{Id})^{-1}$ from which it follows that $E : D \rightarrow D$. A similar argument shows that $[E, A]v = 0$ if $v \in D$. Now, compute the norm
\[\|AEv\|_H^2 = \langle EAv, EAv \rangle_H, \ v \in D.\]
The adjoint identity works here to show this is equal to
\[\langle v, (\text{Id} - E)v \rangle_H\]
and hence that $AE$ extends by continuity to a bounded operator on $H$. The definition of self-adjointness shows that $E : H \rightarrow D$.

Now you should define $g$ by $f(x) = g(\frac{x}{\sqrt{x^2 + 1}})$ and show that $g$ is defined and continuous on $[-1, 1]$ which contains the spectrum of $AE$. 