

PROBLEM SET 5 (FIRST ASSIGNMENT) 18.155
DUE FRIDAY 18 OCTOBER, 2013

This assignment replaces Problem set 5 and should represent about as much work, maybe a little more, as required for one of the Problem sets. If you have questions, including wanting more guidance etc, don't hesitate to ask. I will be traveling starting 11 October so there may be a little wait for a reply.

Here are the projects – some are easier than others, but in that case you are supposed to do better!

- (1) Paley-Wiener-Schwartz Theorem.
- (2) Atiyah's proof of Lojasiewicz-Hörmander Theorem.
- (3) Distributional deRham cohomology of Euclidean space.
- (4) Schwartz Kernel Theorem.
- (5) Gagliardo-Nirenberg Theorem.
- (6) Symbols and conormal distributions on Euclidean space.
- (7) Poisson summation formula.

1. PALEY-WIENER-SCHWARTZ THEOREM

This characterizes the Fourier transform of distributions of compact support in terms of entire functions on \mathbb{C}^n with prescribed growth properties. There are many variants of the theorem – I would like you to do it n dimensions – and you might like also to describe similar results for smooth functions of compact support and for tempered distributions, or L^2 functions, with support in a half-space (which is closer to what Wiener did for the half-line).

2. LOJASIEWICZ-HÖRMANDER

First get the accents correct on Lojasiewicz.

A fundamental solution of a differential operator with constant coefficients is a distribution – in this case $E \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$P(D)E(x) = \delta(x).$$

Taking the Fourier transform this reduces to the 'division problem' of finding $\hat{E} \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$P(\xi)\hat{E}(\xi) = 1$$

i.e. at least informally $\hat{E}(\xi) = 1/P(\xi)$ where the zeros will be the problem. This in fact has a solution for every non-zero polynomial, originally proved by Lojasiewicz then improved (at least the proof) by Hörmander and then in the late 60s Atiyah gave a conceptually simple (but much more sophisticated) proof using resolution of singularities following Hironaka.

Look at the paper by Atiyah and give a discuss the analytic aspects of his proof. Can you use his method to show solvability, that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ there exists $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$P(\xi)u(\xi) = f(\xi) \text{ in } \mathcal{S}'(\mathbb{R}^n)?$$

3. DERHAM THEOREM WITH DISTRIBUTIONAL COEFFICIENTS

Since we can differentiate distributions it is straight forward to define the deRham differential on distributional-forms

$$d \sum_I u_I dx^I = \sum_I \sum_j \partial_j u_I dx^j \wedge dx^I,$$

$$u_I \in \mathcal{S}'(\mathbb{R}^n) \text{ or } \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \text{ or } \mathcal{C}^{-\infty}(\mathbb{R}^n).$$

Discuss the cohomology of each of these three complexes.

4. SCHWARTZ KERNEL THEOREM FOR TEMPERED DISTRIBUTIONS

Show that there is a natural bijection between the space of linear operators

$$T : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

which are continuous in the sense that if $\phi_j \rightarrow \phi$ in $\mathcal{S}(\mathbb{R}^m)$ then $T\phi_j \rightarrow T\phi$ weakly in $\mathcal{S}'(\mathbb{R}^n)$, and the space $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$. This can be done using the Fourier transform and Schwartz Representation theorem. You might like to discuss what this means in terms of tensor products (or not). Note that a ‘kernel’ $A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ gives rise to an operator T_A (there is something to prove here of course) via the formula

$$(1) \quad (T_A\phi)(\psi) = A(\psi \boxtimes \phi)$$

where the ‘exterior tensor product’ here just means $\psi(x)\phi(y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$.

5. GAGLIARDO-NIRENBERG

The Sobolev spaces $H^s(\mathbb{R}^n)$ for $s > n/2$ are subspaces of $C_0^0(\mathbb{R}^n)$, the bounded continuous functions on \mathbb{R}^n with supremum norm. In fact they are also subalgebras, i.e. are closed under pointwise products:

$$H^s(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) \subset H^s(\mathbb{R}^n), \quad s > n/2$$

and indeed form a C^∞ -algebra. This result follows from the Gagliardo-Nirenberg estimates.

6. SYMBOLS AND CONORMAL DISTRIBUTIONS

The standard space of symbols (with bounds) usually written $S^m(\mathbb{R}^n)$ consists of those smooth functions satisfying the estimates

$$|\partial_x^\alpha a(x)| \leq C_\alpha (1 + |x|)^{m-|\alpha|}$$

for all multiindices α . Describe in detail the Fourier transform of these symbols (conormal distributions at the origin) and how these are related to Sobolev spaces. There is plenty more to do about these if you want (the beginning of microlocal analysis). It is difficult to get a precise characterization of the Fourier transform of S^m but you should characterize the union over m of the Fourier transforms as those \hat{a} that satisfy

$$\xi^\beta D_\xi^\alpha \hat{a} \in H^{s+|\beta|-|\alpha|}(\mathbb{R}^n)$$

for some s and conclude that they are smooth outside the origin and if one subtracts a $\psi \hat{a}$ where ψ is smooth, compactly supported and 1 near 0 the result is in $\mathcal{S}(\mathbb{R}^n)$.

7. POISSON SUMMATION FORMULA

This relates the Fourier transform and Fourier series. An element $u \in \mathcal{S}'(\mathbb{R}^n)$ is periodic (with respect to the lattice $2\pi\mathbb{Z}$) if $u(\phi(\cdot + t)) = u(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $t \in 2\pi\mathbb{Z}$. The Poisson summation formula characterizes the Fourier transforms of such periodic distributions.