PROBLEM SET 2, 18.155
SKETCHED SOLUTIONS

(1) Let $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ be the subspace of compactly supported smooth functions – those that vanish for $|x| > R$ for some R (depending on the element of course). Show that this is a dense inclusion.

Solution: In class we constructed $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\phi(x) = 1$ in $|x| \le \frac{1}{2}$ and $\phi(x) = 0$ in $|x| > 1$. Given $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $n \in \mathbb{N}$.

$$\psi_n(x) = \phi(\frac{x}{2n})\psi(x) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$$

vanishes in $|x| > 2n$ and is equal to ψ in $|x| < n$. The difference and its derivatives satisfy

$$|D^\alpha(\psi - \psi_n)| = |D^\alpha(1 - \phi(\frac{x}{2n})\psi| \le \sum_{|\beta| \le |\alpha|} C_\beta n^{-|\beta|}(1+|x|)^{-k}\chi_{|x| \ge n}. \tag{1}$$

Here I have expanded out the product, estimated the finite number of derivatives of ϕ involved by constants and the derivatives of ψ by multiples of $(1+|x|)^{-k}$ and noted that all terms vanish in $|x| < n$. It follows that $\psi_n \to \psi$ in the norms of $\mathcal{S}(\mathbb{R}^n)$ and hence in this as a metric space.

(2) Prove that $\mathcal{S}(\mathbb{R}^n)$ is a *Montel space* which means that it has an analogue of the Heine-Borel property. Namely, (you have to show that) if $D \subset \mathcal{S}(\mathbb{R}^n)$ is closed and 'bounded' in the sense that for each N there exists C_N such that $\|\phi\|_N \le C_N$ for all $\phi \in D$, then D is compact.

Solution: Use Ascoli-Arzela or the characterization of precompact sets in $L^2(\mathbb{R}^n)$ (equicontinuous-in-the means and equismall at infinity). In either case the boundednedd with respect to a 'higher norm' implies the precompactness of D with respect to a given norm. There are countably many norms so from a sequence in D one can extract successive subsequences Cauchy with respect to successive (increasing) norms, and then pass to a diagonal sequence in D which is Cauchy in $\mathcal{S}(\mathbb{R}^n)$ and hence converges. So D is compact.

(3) A) Show (as in remind yourself and the grader) that if $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is measurable and

$$(1+|x|)^{-N}u \in L^1(\mathbb{R}^n) \tag{2}$$

for some N then $I(u)(\phi) = \int u\phi$, for $\phi \in \mathcal{S}(\mathbb{R}^n)$ defines an element $I(u) \in \mathcal{S}'(\mathbb{R}^n)$.

B) Now, refute the idea that these are the 'most general' functions which define distributions – this is a dangerously vague statement anyway and I'm sure you would not say such a thing. NAMELY observe that

$$u(x) = \exp(i\exp(x))$$

defines an element of $\mathcal{S}'(\mathbb{R})$ and hence conclude that, in a sense you should make clear, so does

$$\exp(x)\exp(i\exp(x)) \tag{3}$$

but that this does NOT satisfy (2) above.

Solution. As a function

$$\exp(x)\exp(i\exp(x)) = D_x u(x).$$

Since $u(x)$ is bounded and continuous, it defines a distribution and its distributional derivative is defined by

(4)

$$(D_x I(u))(\phi) = -\int u(x) D_x\phi = \lim_{N\to\infty}\left(\int_{-N}^{N}\exp(x)\exp(i\exp(x))\phi + i\big[u(x)\phi(x)\big]_{-N}^{N}\right).$$

The second term here tends to zero as $N \to \infty$ since u is bounded. So in the sense that the *limiting* integral

$$\exp(\cdot)\exp(i\exp(\cdot))(\phi) = \int_{-N}^{N}\exp(x)\exp(i\exp(x))\phi \tag{5}$$

defines a distribution then this function 'is' a distribution. Note however that this integral is not absolutely convergent.

(4) (Riesz' regularization extended by Gel'fand and Shilov)

A) Using a problem above, show that for $z \in \mathbb{C}$, $\operatorname{Re} z > -1$, the function

$$\begin{cases} x^z = \exp(z\log x) & x > 0 \\ 0 & x \le 0 \end{cases} \tag{6}$$

defines an element, we denote as $x_+^z \in \mathcal{S}'(\mathbb{R})$.

Solution: The function x^z for $\operatorname{Re} z > -1$ is locally integrable and if $N > \operatorname{Re} z + 1$, $(1+|x|)^{-N}x_+^z$ is in L^1.

B) Carry out the integration by parts necessary to check the formula for the distributional derivative

$$\frac{d}{dx}x_+^z = z x_+^{z-1} \text{ if } \operatorname{Re} z > 0. \tag{7}$$

Solution: By definition

(8)
$$\frac{d}{dx}x_+^z(\phi) = -\int_0^\infty x_+^z \frac{d\phi}{dx} = -\lim_{\delta\downarrow 0}\int_\delta^\infty x_+^z \frac{d\phi}{dx} = \lim_{\delta\downarrow 0}\left(\int_\delta^\infty z x_+^z \frac{d\phi}{dx} + \delta^z\phi(\delta)\right)$$

If $\operatorname{Re} z > 0$ the second term vanishes in the limit showing (7).

C) Writing this formula as

$$x_+^\tau = (\tau+1)^{-1}\frac{d}{dx}x_+^{\tau+1},\ \operatorname{Re}\tau > -1 \tag{9}$$

observe that the right side makes sense for $\operatorname{Re}\tau > -2$ provided $\tau \neq -1$ and this can be used to define x_+^τ for this range of τ.

Solution: Right!

D) Iterate this argument to show that one can define x_+^z for $z \in \mathbb{C} \setminus -\mathbb{N}$ this way.

For $\phi \in \mathcal{S}(\mathbb{R})$, what is the value of the limit

$$\lim_{z\to -1}(z+1)x_+^z(\phi)?$$

Solution: One gets

$$x_+^\tau = (\tau+k)^{-1}\cdots(\tau+1)^{-1}\frac{d^k}{dx^k}x_+^{\tau+k},\ \operatorname{Re}\tau > -k-1$$

and this is consistent with the previous definition inductive definition when $\operatorname{Re}\tau > -k$. Note that the nicest way to see this is to use the meromorphy of the function

$$\int_0^\infty x^z\phi(x). \tag{10}$$

What you are actually showing here is that this function, defined by the integral for $\operatorname{Re} z > -1$, actually has a meromorphic extension to the complex plain with poles only at the points $z \in -\mathbb{N}$.

You can compute all the 'residues' but the one at $z = -1$ follows directly by evaluation

$$\lim_{z\to -1}(z+1)\int_0^\infty x^z\phi = \lim_{z\to -1}(z+1)\int_1^\infty x^z\phi$$
$$+\lim_{z\to -1}(z+1)\int_0^1 x^z(\phi(x)-\phi(0)) + \lim_{z\to -1}(z+1)\phi(0)\int_0^1 x^z.$$

The first two integrand converge absolutely in L^1 as $z \to -1$ (because of an extra factor of x in the second case) so the limit is 1.

(5) The Dirac delta 'function' $\delta \in \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\delta(\phi) = \phi(0) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{11}$$

is amongst the most important distributions (it is a measure).

A) Find explicit formulae for the derivatives $\partial^\alpha \delta$ evaluated on test functions
Solution: $\partial^\alpha \delta(\phi) = (-1)^{|\alpha|}\delta(\partial^\alpha \phi) = (-1)^{|\alpha|}\partial^\alpha \phi(0)$.

B) Compute the Fourier transform of $\partial^\alpha \delta$. Since $\hat{\delta}(\phi) = \delta(\hat{\phi})$, $\hat{\delta}(\phi) = \hat{\phi}(0) = \int \phi$ so $\hat{\delta} = I(1) = 1$ as we now say. The derivatives then follow from the general formula that

$$\widehat{D^\alpha u} = \xi^\alpha \hat{u}, \ \widehat{D^\alpha \delta} = \xi^\alpha (= I(\xi^\alpha).$$

C) Show that

$$\partial^\alpha \delta \in H^{-|\alpha|-n/2-\epsilon}(\mathbb{R}^n) \tag{12}$$

for $\epsilon > 0$ but not for $\epsilon = 0$.
Solution: This is the statement that $(1+|\xi|)^{-n/2-|\alpha|-\epsilon}\xi^\alpha \in L^2$ if and only if $\epsilon > 0$.

Hints:

(1) Use a bump function, conventionally called χ, as constructed in Lecture 3 which is equal to 1 in $|x| < \frac{1}{2}$ and vanishes in $|x| > 1$ and then show that $\chi(\frac{x}{k})\phi(x) \to \phi(x)$ in $\mathcal{S}(\mathbb{R}^n)$.

(2) You may apply the Ascoli-Arzela theorem if you check that a set bounded with respect to the norm $\| \cdot \|_1$ is equicontinuous on $\mathbb{R}^n$!

(3) The function in (3) is a multiple of the derivative of the bounded function u in (3).