PROBLEM SET 1, 18.155
BRIEF SOLUTIONS

(1) Prove (probably by induction) the multi-variable form of Leibniz formula for the derivatives of the product of two (sufficiently differentiable) functions:-

$$\partial^\alpha(fg) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} g. \tag{1}$$

Solution: One convenient way to see this is to think algebraically and separate out the variables to see that

$$\begin{aligned}\partial_{x_j}(f(x)g(x)) &= \big(\partial_{x_j} + \partial_{y_j}\big)f(x)g(y)\big)\big|_{x=y}\\ \partial^\alpha(f(x)g(x)) &= \big(\partial_{x_j} + \partial_{y_j}\big)^\alpha f(x)g(y)\big)\big|_{x=y}.\end{aligned}$$

Then one can expand out the sum of the two commuting variables and use the standard formula in terms of the combinatorial coefficients

$$\begin{aligned}(X+Y)^\alpha &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}X^\beta Y^{\alpha-\beta}\\ \binom{\alpha}{\beta} &= \left(\frac{\alpha!}{\beta!(\alpha-\beta)!}\right) = \prod_{i=1}^n\left(\frac{\alpha_i!}{\beta_i!(a_i-\beta_i)!}\right).\end{aligned} \tag{2}$$

To make this completely rigourous one can do it for $\mathcal{S}(\mathbb{R}^n)$ using the Fourier transform and then argue that it has to hold in general.

Another way is to argue by induction that there has to be a formula (1) for *some* coefficients and then apply it to monomials to see what they must be.

Or just hammer it out!

(2) Consider the norms, for each $N\in\mathbb{N}$, on $\mathcal{S}(\mathbb{R}^n)$

$$\|f\|_N = \sum_{|\beta|+|\alpha|\le N}\sup_{x\in\mathbb{R}^n}|x^\beta\partial^\alpha f(x)|.$$

Show that

$$\|f\|'_N = \sum_{|\beta|+|\alpha|\le N}\sup_{x\in\mathbb{R}^n}|\partial^\alpha(x^\beta f(x))|$$

are equivalent norms, $\|f\|_N \le C_N\|f\|'_N$ and $\|f\|'_N \le C'_N\|f\|_N$.

Solution: Use induction over N. We can apply Leibniz' formula to see that

$$\partial^\alpha(x^\beta f) = x^\beta \partial^\alpha f + \sum_{\beta'<\beta,\ \alpha'<\alpha} c_{\alpha',\beta'} x^{\beta'} \partial^{\alpha'} f$$

since as soon as one derivative falls on the coefficients the order drops in both senses. Each of the terms in the sum is bounded by the $N-1$ norm, so, using the inductive hypothesis

$$|\partial^\alpha(x^\beta f)| \le |x^\beta \partial^\alpha f| + C\|f\|_{N-1},\ |x^\beta \partial^\alpha f| \le |\partial^\alpha(x^\beta f)| + C\|f\|'_{N-1}.$$

Taking the supremum and summing over α and β gives the inductive step.

(3) Consider $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ which is an infinitely differentiable function of polynomial growth, in the sense that for each α there exists $N(\alpha) \in \mathbb{N}$ and $C(\alpha) > 0$ such that

$$|\partial^\alpha F(x)| \le C(\alpha)(1+|x|)^{N(\alpha)}.$$

Show that multiplication by F gives a map $\times F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$.

Solution: Again this is an application of Leibniz' formula. The product $F\phi$ with $\phi \in \mathcal{S}(\mathbb{R}^n)$ is infinitely differentiable with

$$|x^\beta \partial^\alpha (F\phi)| \le \sum_{\alpha' \le \alpha} c_{\alpha'} |x^\beta \partial^{\alpha'} F||\partial^{\alpha-\alpha'}\phi| \le C \sum_{\alpha'\le\alpha} C(1+|x|)^N \partial^{\alpha'}\phi| \le \|\phi|_M$$

where the power N is the max of $|\beta| + N(\alpha')$ and this determines M. Taking the supremum, $\times F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a continuous linear map.

(4) Show that if $s \in \mathbb{R}$ then $F_s(x) = (1+|x|^2)^{s/2}$ is a smooth function of polynomial growth in the sense discussed above and that multiplication by F_s is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$.

Solution: To see that this function is of polynomial growth, prove the stronger *symbol estimates*

$$|\partial^\alpha F_s(x)| \le C_{s,\alpha} F_{s-|\alpha|}. \tag{3}$$

These follow by using induction to check that

$$\partial^\alpha F_s(x) = p_\alpha(x) F_{s-2|\alpha|} \tag{4}$$

where p_α is a polynomial of degree at most $|\alpha|$ so $|p_\alpha| \le C_\alpha F_{|\alpha|}$.

Now $F_0 = 1$ and $F_{t+s} = F_s F_t$ so the inverse of $\times F_s$ is $\times F_{-s}$ and it follows that multiplication is an isomorphism.

(5) Consider one-point, or stereographic, compactification of $\mathbb{R}^n$. This is the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ obtained by sending x first to the point $z = (1, x)$ in the hyperplane $z_0 = 1$ where $(z_0, \dots, z_n)$

are the coordinates in $\mathbb{R}^{n+1}$ and then mapping it to the point $Z \in \mathbb{R}^{n+1}$ with $|Z| = 1$ which is also on the line from the 'South Pole' $(-1, 0)$ to $(1, x)$.

Derive a formula for T and use it to find a formula for the inversion map $I : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ which satisfies $I(x) = x'$ if $T(x) = (z_0, z)$ and $T(x') = (-z_0, z)$. That is, it correspond to reflection across the equator in the unit sphere.

Solution: Since $Z = Tx$ has $|Z| = 1$ and $Z = (1-s)(-1, 0)+ s(1, x)$ for some s,

$$(2s-1)^2 + s^2|x|^2 = 1 \Longrightarrow s = \frac{4}{4+|x|^2},$$
$$Tx = \left(\frac{4-|x|^2}{4+|x|^2}, \frac{4x}{4+|x|^2}\right)$$

since the other root, $s = 0$, is the South Pole. Then

$$Tx' = \left(-\frac{4-|x|^2}{4+|x|^2}, \frac{4x}{4+|x|^2}\right) \Longrightarrow x' = Ix = \frac{4x}{|x|^2}.$$

Show that if $f \in \mathcal{S}(\mathbb{R}^n)$ then $I^*f(x) = f(x')$, defined for $x \neq 0$, extends by continuity with all its derivatives across the origin where they all vanish.

Solution: Certainly I is smooth and is a diffeomorphism of the region $\mathbb{R}^n \setminus \{0\}$ to itself, since it has inverse I – from the definition it is an involution so $I^*f(x) = f(4x/|x|^2)$ is smooth outside the origin. As $|x| \to 0$, $|Ix| \to \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)$, so I^*f extends continously up to 0 where it vanishes. The chain rule gives the formula for the first derivatives of I^*f :

$$\partial_{x_j} I^*f = \sum_{k=1}^{n} g_{jk}(x) I^*(\partial_{y_k} f),\ g_{jk} = \partial_{x_j}\frac{4x_k}{|x|^2}. \tag{5}$$

We can work out the coefficients, but the important thing is that they are smooth functions in $|x| > 0$ which are homogeneous of degree -2. Here the $\partial_{y_k} f$ are the derivatives of f. Now, one can get the higher derivatives by differentiating inductively and it follows that

$$\partial_x^\alpha I^*f = \sum_{|\beta|\le|\alpha|} g_{\alpha,\beta}(x) I^*(\partial_y^\beta f) \tag{6}$$

where the coefficients are smooth in $|x| > 0$ and homogeneous of degree $-2|\alpha| + |\beta|$. This means we get estimates near 0

$$|\partial_x^\alpha I^* f(x)| \le C \sum_{|\beta| \le |\alpha|} |x|^{-2|\alpha|+|\beta|} |I^*(\partial_y^\beta f)|.$$

However, f being Schwartz means that $|I^* f(x)| \le C_N |x|^N$ for any N and the same thing applies to the derivatives. So in fact we conclude that all the derivatives of $I^* f$ extends continuously across the origin and vanish there (to infinte order as follows anyway). This means $I^* f$ itself is smooth as claimed with all derivatives vanishing at 0.

Conversely show that if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a function with all derivatives continuous, then $f \in \mathcal{S}(\mathbb{R}^n)$ if $I^* f$ has this property, that all derivatives extend continuously across the origin and vanish there – i.e. they all have limit zero at the origin.

Solution: Just reverse the argument. All we need so show is that $x^\beta \partial_x^\alpha$ is bounded for all α, β. Since I is its own inverse, (5) gives

$$\partial_y^\alpha f(y) = \sum_{|\beta| \le |\alpha|} g_{\alpha,\beta}(y) \partial_x^\beta (I^* f) \tag{7}$$

where the same statements apply, except now we are interested in what is happening as $|y| \to \infty$. However, using homogeneity and now the fact that all derivatives of $I^* f$ have bounds $C_N |x|^N$ for any N, we find that in $|y| > 1$,

$$|\partial_y^\alpha f(y)| \le C_M |y|^{-M} \tag{8}$$

for any M. So indeed the converse is also true.