18.155 LECTURE 9, 3 OCTOBER, 2013

Let me go through the proof of the main theorem about homogeneous distributions. This is done in §3.2 of Hörmander's book although it is maybe not easy reading. Initially let me 'recall' the one-sided results in dimension one from Problem set 2.

There you considered the locally integrable function

$$x_+^z = \begin{cases} \exp(z\log x) & x > 0 \\ 0 & x \le 0 \end{cases}, \ \operatorname{Re} z > -1. \tag{1}$$

which therefore defines an element of $\mathcal{S}'(\mathbb{R})$ for which we use the same notation. For $\operatorname{Re} z > 1$ this is continously differentiable and

$$\frac{d}{dx}x_+^z = zx_+^{z-1} \tag{2}$$

holds as an equation between continous functions. In fact as an equation between distributions this makes sense for $\operatorname{Re} z > 0$ and actually holds then since the LHS is by definition

$$\begin{aligned}\frac{d}{dx}x_+^z(\phi) = x_+^z(-\frac{d}{dx}\phi) &= -\lim_{\epsilon\to 0}\int_\epsilon^\infty x^z\frac{d\phi}{dx}dx \\ &= z\lim_{\epsilon\to 0}\int_\epsilon^\infty x^{z-1}\phi(x)dx + \lim_{\epsilon\to 0}\epsilon^z\phi(\epsilon)dx = zx_+^{z-1}(\phi).\end{aligned}$$

Note that this can be interpreted as Euler's identity

$$(x\frac{d}{dx} - z)x_+^z = 0. \tag{3}$$

We can iterate (2) and (changing variable to $z+1$) conclude that

$$x_+^z = \frac{1}{(z+1)\cdots(z+k)}\frac{d^k}{dx^k}x_+^{z+k}, \ \operatorname{Re} z > -1. \tag{4}$$

Let me again emphasize that this is an identity we have proved between two distributions which depend on the parameter z.

The point of course is that the RHS in (4) makes sense for $\operatorname{Re} z > -k-1$ provided none of the factors $z+j$ vanishes. So we can use the identity to extend the definition of the LHS to this range of z.

Let me spell this out even more precisely:-

Lemma 1. *If $z \in \mathbb{C}\setminus -\mathbb{N}$ and $k \in \mathbb{N}$, $k > -\operatorname{Re} z - 1$, then*

$$x_+^z(\phi) = \frac{(-1)^k}{(z+1)\cdots(z+k)}\int_0^\infty x^{z+k}\frac{d^k\phi}{dx^k}dx, \ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{5}$$

defines an element of $\mathcal{S}'(\mathbb{R})$ which is independent of the choice of k.

Proof. If we use (5) as the definition for the smallest possible non-negative integer k then the integration-by-parts argument above shows that the same formula holds for larger integral k. $\square$

If z is near $-j$, $j \in \mathbb{N}$, and we look at (5) for some fixed $k \ge j$ then the limit

$$\lim_{z\to -j}(z+j)x_+^z(\phi) = \tag{6}$$

$$\lim_{z\to -j}\frac{(-1)^k}{(z+1)\cdots(z+j-1)(z+j+1)\cdots(z+k)}\int_0^\infty x^{z+k}\frac{d^k\phi}{dx^k}dx$$

exists and again defines a distribution. What distribution is it? The RHS converges to

$$\frac{(-1)^k}{(1-j)\cdots(-1)(1)\cdots(k-j)}\int_0^\infty x^{k-j}\frac{d^k\phi}{dx^k}dx.$$

We can take $k = j$ since we know the result is independent of $k \ge j$ and then integration by parts gives

$$\lim_{z\to -j}(z+j)x_+^z(\phi) = \frac{1}{(j-1)!}\frac{d^{j-1}\phi}{dx^{j-1}} = \frac{(-1)^{j-1}}{(j-1)!}(\frac{d^{j-1}\delta}{dx^{j-1}})(\phi). \tag{7}$$

From this we can take two things. First, the limit of $(z+j)x_+^z$ as $z \to -j$ in $\mathcal{S}'(\mathbb{R})$ (in the weak topology) is $\frac{(-1)^{j-1}}{(j-1)!}\frac{d^{j-1}}{dx^{j-1}}\delta$. We can also go a little further. Suppose $\phi \in \mathcal{S}(\mathbb{R})$ satisfies

$$\frac{d^{j-1}\phi}{dx^{j-1}}(0) = 0. \tag{8}$$

Then

$$\lim_{z\to -j} x_+^z(\phi) \text{ exists.} \tag{9}$$

Certainly the limit in (7) vanishes as it should. In fact taking $k = j$ in (5)

$$x_+^z(\phi) = \frac{(-1)^j}{(z+1)\cdots(z+j)}\lim_{\epsilon\to 0}\int_\epsilon^\infty x^{z+j}\frac{d^j\phi}{dx^J}dx.$$

In the inner limit if we perform integration by parts then the boundary term vanishes in the limit because of (8) and the resulting integral makes sense:

$$\lim_{z\to -j} x_+^z(\phi) = \frac{1}{(j-1)!}\int_0^\infty x^{-1}\frac{d^{j-1}\phi}{dx^{j-1}}dx \tag{10}$$

because the integrand is actually smooth down to 0.

As I explained last time we can use this to construct homogenous distributions on $\mathbb{R}^n$. If $f \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree z and $\operatorname{Re} z > -n$ then it defines a locally integrable function on $\mathbb{R}^n$ and a tempered distibution:

$$f(\phi) = \int |x|^z f(\hat{x})\phi(x)dx = \int_0^\infty r^{z+n-1}\int_{\mathbb{S}^{n-1}} f(\omega)\phi(r\omega)d\omega dr. \tag{11}$$

Here I have introduced 'polar coordinates' in $\mathbb{R}^n$, identifying the compliment of the origin with $(0,\infty)_r \times \mathbb{S}^{n-1}$ and Lebesgue measure, which is homogeneous of degree n with $r^{n-1}drd\omega$ for the invariant metric on the sphere $\mathbb{S}^{n-1}$. Now $\operatorname{Re} z + n - 1 > -1$ if $\operatorname{Re} z > -n$ so we can write (11) in terms of a 1-dimensional integral

$$f(\phi) = \int_0^\infty r^{z+n-1}f(\omega)S_f(\phi)(r)dr,\ S_f\phi(r) = \int_{\mathbb{S}^{n-1}}\phi(r\omega)f(\omega)d\omega. \tag{12}$$

In fact S_f is a continous linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R})$. This allows us to think of (12) as a distributional pairing

$$f(\phi) = r_+^{z+n-1}(S_f\phi),\ \operatorname{Re} z > -n. \tag{13}$$

If we hold $f\big|_{\mathbb{S}^{n-1}} \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ fixed and think of z as a varying parameter then from the discussion above we know how to make sense of the right hand side in two cases. Without restriction it makes sense if $z+n-1 \neq -k$, $k \in \mathbb{N}$, which is to say $z \neq -n-p$, $p \geq 0$ integral. More generally when this condition fails, $z = -n-p$ we know how to make sense of (13) as a limit provided

$$\frac{d^p S_f \phi}{dr^p}(0) = 0. \tag{14}$$

From the definition of S_f in (12) it follows that

$$\frac{d^p S_f \phi}{dr^p}(0) = \int_{\mathbb{S}^{n-1}} \phi_p(\omega) f(\omega) d\omega \tag{15}$$

where $\phi_p(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$ is the part homogeneous of degree p is the Taylor series expansion of ϕ at the origin. This means we are in a position to prove the result from last time but this time written out in words:

Proposition 1. *For $z \in \mathbb{C}$ not of the form $-n-j$, j a non-negative integer, the space of distributions homogeneous of degree z and smooth outside the origin is mapped isomorphically to $\mathcal{C}^\infty(\mathbb{S}^{n-1})$ by restriction to the unit sphere. For the case $z = -n-j$ with j a non-negative integer the same space restricts to the subspace of $\mathcal{C}^\infty(\mathbb{S}^{n-1})$ consisting of the functions which are orthogonal to the monomials of degree j (restricted to the sphere) with the kernel of this map consisting of the linear combination of the derivatives of order j of the Dirac delta at* 0.

Once one understands distributions on the sphere the assumption of smoothness outside the origin can be dropped.

Proof. As we have already seen the idea is to define

$$u(\phi) = x_+^{n+z-1}(S_f\phi). \tag{16}$$

Since (differentiate under the integral and estimate at infinity) $S_f : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R})$ is continuous and we know that $x_+^\tau \in \mathcal{S}'(\mathbb{R})$ provided $\tau \notin -\mathbb{N}$, this does define a distribution. If ϕ has support disjoint from the origin, then so does $S_f\phi$ and (16) reduces to the integral

$$u(\phi) = \int_0^\infty x^{n+z-1}(S_f\phi)(x) = \int_{\mathbb{R}^n} |x|^z f(\hat{x})\phi(x)dx.$$

So indeed it is an extension. Moreover, the rescaling operation passes through S_f :

$$(S_f\phi)(rx) = S_f(\phi(r\cdot)(x)$$

so homogeneity follows from the homogeneity of x_+^τ.

In the 'tricky' cases we can do just the same, because the assumption on f, that

$$\int_{\mathbb{S}^{n-1}} f(\omega)p(\omega)d\omega = 0 \tag{17}$$

for all polynomials of degree k implies that

$$\frac{d^{k-1}}{dx^{k-1}} S_f(\phi)(0) = 0 \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n); \tag{18}$$

the point being that the kth term in the Taylor series of ϕ at the origin is what contributes through (17) to this value. So even for these values the definition (16) still makes sense in that the limit

$$\lim_{z \to -n-k} x_+^{n+z-1}(S_f\phi) \tag{19}$$

exists and defines a distribution. The same arguments shows that it is an extension and homogeneous. □

I mentioned last time that even if the vanishing conditions on $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ are not satisfied, $|x|^{-n-k}f(\hat{x})$ *still* has an extension 'across the origin' as tempered distribution, just not homogenous. How can one see this explicitly? In fact one can just use (16), but now on form those $\phi \in \mathcal{S}(\mathbb{R}^n)$ which satisfy the corresponding vanishing condition

$$D_x^\alpha \phi(0) = 0 \ \forall \ |\alpha| = k. \tag{20}$$

What we get this way is not quite a distribution, it is just a continous linear functional defined on the *closed* subspace of $\mathcal{S}(\mathbb{R}^n)$ given by (20). One could apply Hahn-Banach (extended a bit) but since there are only a finite number of conditions we can instead use a 'projection' – choose some $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 near 0 and set

$$\phi = \sum_{|\alpha|=k} \frac{1}{\alpha!}\partial^\alpha\phi(0)\psi + P_k\phi. \tag{21}$$

Then $P_k : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a continuous linear map onto the subspace (20) so we can really define a distribution $u \in \mathcal{S}'(\mathbb{R}^)$ by

$$\lim_{z\to -n-k} x_+^{n+z-1}(S_f P_k\phi). \tag{22}$$

Now, this still has the correct restriction property, since any ϕ with support disjoint from 0 satisfies $P_k\phi = \phi$. You can check that it is homogeneous iff f satisfies the constraints above for that k.

I'm sure other questions have immediately occurred to you, such as:- What is the Fourier transform of a homogeneous distribution? Well, for a test function one can see that

$$\hat{\phi}(r\cdot)(\xi) = \int e^{-ix\cdot\xi}\phi(rx)dx = r^{-n}\hat{\phi}(\xi/r)\hat{u}(\phi(r\cdot)) = u(\hat{\phi}(r\cdot)) = r^{-n}u(\phi(\cdot/r)) \tag{23}$$

from which it follows that the Fourier transform of a distribution which is homogeneous of degree z is homogeneous of degree $-n-z$.

Problem 1. Show that if u is homogenous and smooth outside the origin then $\hat{u}$ is also smooth outside the origin. What can you say about the relation between the restrictions to the unit sphere?

Now, what can we get out of this construction? Let's think about constant coefficient differential operators $P(D)$ where P is a polynomial. In fact, let's for the moment think about the case where P is a *homogeneous* polynomial of degree m and $P(\xi) \neq 0$ for $0 \neq \xi \in \mathbb{R}^n$. These are the *elliptic* homogeneous polynomials. In fact let me just note

Definition 1. A polynomial of degree m is said to be elliptic (technically one should say 'of degree m' since the polynomial might actually have lower degree) if the leading part, homogeneous of degree m is elliptic in the sense above:

$$p(\xi) = \sum_{|\alpha|\le m} c_\alpha\xi^\alpha,\ p_m(\xi) = \sum_{|\alpha|=m} c_\alpha\xi^\alpha \neq 0 \text{ on } \mathbb{R}^n\setminus\{0\}. \tag{24}$$

Examples include the Laplacian

$$\Delta = \sum_{j=1}^{n} D_j^2 \text{ on } \mathbb{R}^n \tag{25}$$

and the Cauchy-Riemann operator on $\mathbb{R}^2 = \mathbb{C}$:

$$\partial_x + i\partial_y. \tag{26}$$

Of course, by definition, you can add any lower order terms to an elliptic operator and it stays elliptic. For the moment we will only consider the homogeneous case.

Proposition 2. *If $m < n$ then a homogeneous elliptic differential operator $p_m(D)$ has a homogeneous fundamental solution which is smooth outside the origin; in case $m \geq n$ it has a tempered fundamental solution which is smooth outside the origin but not homogeneous.*

In the case of the Laplacian, $m = 2$ and for $n > 2$ there is a fundamental solution of the form $c|x|^{-n+2}$ (so locally integrable) for an appropriate constant c. For $n = 2$ there is a fundamental solution $c \log |x|$. For the Cauchy-Riemann operator there is a fundamental solution of the form $c(x + iy)^{-1}$.

Proof. A fundamental solution is by definition a distribution E satisfying

$$P(D)E = \delta_0. \tag{27}$$

Now, if E is tempered then we can look for its Fourier transform instead, which should satisfy

$$P(\xi)\hat{E}(\xi) = 1. \tag{28}$$

So, we need to make distributional sense of $1/P(\xi)$. In case $P = p_m$ is non vanishing outside the origin, $1/p_m(\xi)$ is homogeneous of degree $-m$ on $\mathbb{R}^n \setminus \{0\}$ and if $m < n$ we are in the locally integrable case and there is a unique homogeneous extension. Call this

$$\hat{E} \text{ homogeneous of degree } -m > -n, \ \hat{E} = 1/p_m(\xi) \text{ on } \mathbb{R}^n \setminus \{0\}. \tag{29}$$

Certainly $p_m(\xi)\hat{E}$ is homogeneous of degree 0 and is equal to 1 outside the origin, so it must actually be the distribution 1. Taking the inverse Fourier transform we have (27). Why is E smooth outside the origin?

Now, what about the non-integrable case for $1/p_m$, where $m \geq n$? As we saw above, we can find a tempered distribution $\hat{E}$ which extends $1/p_m$ across the origin. Now, is it the Fourier transform of a fundamental solution? For that we need to compute $p_m(D)E$, which is to say $p_m(\xi)\hat{E}$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$p_m(\xi)\hat{E}(\phi) = \hat{E}(p_m(\xi)\phi(\xi)). \tag{30}$$

Since p_m is a homogeneous polynomial of degree m, all the derivatives of $p_m\phi$ at the origin, up to order $m - 1$ must vanish. The specific choice of $\hat{E}$ made above is corresponds to $k = m - n$:

$$\hat{E}(\phi) = \lim_{z \to -n-(m-n)} x_+^{n+z-1}(S_{p_m} P_{m-n}\phi). \tag{31}$$

Thus $P_{m-n}(p_m\phi) = p_m\phi$ and it follows that $p_m\hat{E} = 1$ since $p_m\hat{E}$ is a *homogeneous* distribution equal to 1 outside the origin. □

I have passed over the issue of the smoothness of E outside the origin – not because it is minor but on the contrary it is extremely important!

Let me back up a bit and check something that I did mention earlier:-

Lemma 2. *The Fourier transform of a compactly supported distribution is in* $\mathcal{C}^\infty(\mathbb{R}^n)$ *(and has derivatives of slow growth).*

Much more is actually true here and you might like to do this as your 'Assignment 1'. The Fourier transform of a distribution of compact support extends to be an *entire* function on $\mathbb{C}^n$, of exponential type. The image of $\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ under Fourier transform can be characterized by some additional estimates, and this is a version of the Theorem of Paley-Weiner-Schwartz. First time I have mentioned a mathematician at MIT.

Proof. The basic idea we have established. If $u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ then u extends (from $\mathcal{S}(\mathbb{R}^n)$) to $\mathcal{C}^\infty(\mathbb{R}^n)$ by continuity. This means that

$$\hat{u}(\xi) = u(e^{-ix\cdot\xi}) \tag{32}$$

makes sense. In fact we can easily check that $\hat{u}(\xi)$ is smooth. Well, it is differentiable because we can compute the limit of the difference quotient

$$\lim_{t\to 0} \frac{\hat{u}(\xi + te_j) - \hat{(}\xi)}{t} = \lim_{t\to 0} u\Big(\frac{\exp(-ix\cdot(\xi+te_j)) - \exp(-ix\cdot\xi)}{t}\Big) \tag{33}$$

and the argument converges in $\mathcal{C}^\infty(\mathbb{R}^n)$ to $-ix_j \exp(-ix\cdot\xi$. Iterating this we find that $\hat{u}$ is infinitely differentiable, the continuity estimates for u show that it has slow growth.

The only little question is:- Is $\hat{u}$ in fact the Fourier transform of u? Once we have $\hat{u}$ it is easy to to the 'exchange of limits' which shows this to be so, namely

$$\hat{u}(\phi) = u(\hat{\phi}),\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

□

Going back to the question of the (inverse) Fourier transform of a homogeneous distribution which is smooth outside the origin, we can use a cut off $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\psi = 1$ near 0 to divide the distribution into a compactly supported part and a smooth part and apply the Lemma above to the compactly supported part. Now for the smooth part we use homogenity to check – as we really did before – that

$$|D^\alpha_\xi(f(\xi)(1-\psi(\xi))| \le C|\xi|^{M-|\alpha|} \tag{34}$$

where $M = -m$ is the inital homogeneity. In particular, if $|\alpha| > M + n + k$ then

$$(1 + |\xi|)^k D^\alpha_\xi(f(\xi)(1-\psi(\xi)) \in L^2(\mathbb{R}^n). \tag{35}$$

So the (inverse) Fourier transform of say $\Delta^N_\xi(1-\psi)f$ is in H^k provide N is large enough. If $\hat{v} = (1-\psi)f$ this means

$$|x|^{2N} v \in H^k\ \forall\ N > M + n + k. \tag{36}$$

By the Sobolev embedding theorem this shows that v is smooth outside the origin – where $|x|^2$ vanishes. Thus in fact E is smooth outside the origin.

The same argument applies in the case that $m \ge n$ since it is only homogeneity of the restriction to the complement of a ball around the origin that is used.

So what does this buy us? The answer is the first version of elliptic regularity:-

Proposition 3. *Suppose p_m is a homogeneous and elliptic polynomial and $u \in \mathcal{C}^{-\infty}(\Omega)$ satisfies $p_m(D)u \in \mathcal{C}^{\infty}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$, then $u \in \mathcal{C}^{\infty}(\Omega)$.*